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S AND g_λ^*-FUNCTIONS ON COMPACT LIE GROUPS

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Abstract

We characterize the Hardy spaces $H^p(G)$ of a compact Lie group G by means of S-functions in analogy with the theorem of Fefferman-Stein for $\mathbb{R}^n$. We also characterize $H^p(G)$ by means of the g_λ^*-functions.

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1. Introduction

The characterization of $H^p(\mathbb{R}^n)$ by means of S-functions is a well-known result of Fefferman-Stein [4, Theorem 8]. Using previously obtained atomic characterizations of $H^p(G)$ [1], we prove an analogous result for compact connected semisimple Lie groups G. As an application, we show that $\left\|g_\lambda^*(f)\right\|_p \le C \left\|f\right\|_{H^p(G)}$. This inequality gives us another characterization of $H^p(G)$ by means of the g_λ^*-function.

The Hardy space $H^p(G)$ of distributions on a connected simply-connected compact group G is defined to be $H^p(G) = \left\{f \in \mathscr{S}'(G) \mid u_f^* \in L^p(G)\right\}$ where $u_f^*(x) = \sup_{(y,t)\in\Gamma(x)} |P_t * f(y)|$, P_t is the Poisson kernel associated with the Casimir operator of G, and $\Gamma(x) = \left\{(y,t) \in G \times \mathbb{R}^+ \mid d(x,y) < t\right\}$ is the cone with vertex $x \in G$ defined by a bi-invariant metric d on G. For suitable radial functions ϕ on the Lie algebra $\mathfrak{t}$ of a maximal torus T of G (see (3.1) for a complete description), we define the S-function by

$$S_\phi f(x) = \left(\int_{\Gamma(x)} |(f * \phi_t)(y)|^2 t^{-(n+1)}\, dydt\right)^{1/2}.$$

Our main result concerning the S-function is:

THEOREM 3.3. *For $f \in \mathscr{S}'(G)$, $f \in H^p(G)$ if and only if $S_\phi(f) \in L^p(G)$. Moreover, $\|u_f^*\|_p \cong \|S_\phi(f)\|_p$.*

For f a distribution on G and $\lambda > 1$, we define the g_λ^*-function of f by

$$g_\lambda^*(f)(x) = \left(\int_0^\infty \int_G \left[\frac{t}{t + d(x, y)}\right]^{\lambda n} |(f * \phi_t)(y)|^2 t^{-(1+n)}\, dy\, dt\right)^{1/2}.$$

The g_λ^*-function characterization of $H^p(G)$ that we obtain in Section 4 is contained in these two theorems:

THEOREM 4.1. *Suppose that $f \in \mathscr{S}'(G)$. For $0 < p \leq 1$ and $\lambda > 2/p$, $f \in H^p(G)$ if and only if $g_\lambda^*(f) \in L^p(G)$. Moreover $\|g_\lambda^*(f)\|_p \simeq \|S_\Phi(f)\|_p \simeq \|u_f^*\|_p$.*

THEOREM 4.2. *For $p > 1$ and $\lambda > 2/p$, $\|g_\lambda^*(f)\|_p \leq C\|f\|_p$.*

In fact, in this paper we will show that these are characterizations of atomic Hardy space $H_a^p(G)$ as defined in Section 2. The authors have previously demonstrated the equivalence of atomic Hardy space $H_a^p(G)$ and $H^p(G)$.

2. Notation and definitions

Let G be a connected simply-connected compact Lie group of dimension n. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{t}$ be the Lie algebra of a fixed maximal torus T of G of dimension ℓ. Let A be a system of positive roots for the pair $(\mathfrak{g}, \mathfrak{t})$. Then $\mathrm{Card}(A) = (n - \ell)/2$. Let $\delta = \sum_{\alpha \in A} \alpha/2$.

If $|\cdot|$ is the norm on $\mathfrak{g}$ induced by the negative of the Killing form B on $\mathfrak{g}^{\mathbb{C}}$, the complexification of $\mathfrak{g}$, then $|\cdot|$ induces a bi-invariant metric d on G. Furthermore, since $B|_{\mathfrak{t}^{\mathbb{C}} \times \mathfrak{t}^{\mathbb{C}}}$ is non-degenerate, for each complex linear functional $\lambda \in \mathrm{hom}_{\mathbb{C}}(\mathfrak{t}^{\mathbb{C}}, \mathbb{C})$ there is a unique $H_\lambda \in \mathfrak{t}^{\mathbb{C}}$ such that $\lambda(H) = B(H, H_\lambda)$ for $H \in \mathfrak{t}^{\mathbb{C}}$. The inner product and norm on $\mathfrak{t}$ give rise to an inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ on $\mathrm{hom}_{\mathbb{C}}(\mathfrak{t}, i\mathbb{R})$ by means of this canonical isomorphism.

The weight lattice P is defined by $P = \{\lambda \in \mathrm{hom}_{\mathbb{C}}(\mathfrak{t}, i\mathbb{R}) : \lambda(X) \in 2\pi i\mathbb{Z}\}$. The set Λ of dominant weights is defined by $\Lambda = \{\lambda \in P : \langle \lambda, \alpha \rangle \geq 0 \text{ for } \alpha \in A\}$. The set $\widehat{G}$ of equivalence classes of irreducible unitary representations of G is parameterized by Λ : $\widehat{G} = \{[U_\lambda]\}_{\lambda \in \Lambda}$. The representation U_λ has dimension d_λ and character $\chi_\lambda(X)$ given by

$$d_\lambda = \prod_{\alpha \in A} \frac{\langle \lambda + \delta, \alpha \rangle}{\langle \delta, \alpha \rangle}; \qquad \chi_\lambda(X) = \frac{\sum_{w \in W} \varepsilon(w) e^{i\langle w(\lambda+\delta), X\rangle}}{\sum_{w \in W} \varepsilon(w) e^{i<w\delta, X>}}, \quad (X \in \mathfrak{t})$$

where W is the Weyl group and $\varepsilon(w)$ is the signature of w. Let $\{X_1, \ldots, X_n\}$ be an orthonormal basis of $\mathfrak{g}$. The Casimir operator

$$\Delta = \sum_{i=1}^n X_i^2$$

is an elliptic bi-invariant operator on G that is independent of the choice of basis. Let W_t and P_t be the Gauss-Weierstrass and Poisson kernels defined on $G^+ = G \times \mathbb{R}^+ = G \times (0, \infty)$ by

$$W_t(x) = \sum_{\lambda \in \Lambda} e^{-t(\|\lambda+\delta\|^2 - \|\delta\|^2)} d_\lambda \chi_\lambda(x) \qquad (x, t) \in G^+$$

and

$$P_t(x) = \sum_{\lambda \in \Lambda} e^{-t\sqrt{\|\lambda+\delta\|^2 - \|\delta\|^2}} d_\lambda \chi_\lambda(x) \qquad (x, t) \in G^+.$$

The solutions to the heat equation

$$\frac{\partial \varphi}{\partial t}(x, t) = \Delta\varphi(x, t) \quad \varphi(g, 0^+) = f(x)$$

and the Poisson equation

$$\frac{\partial^2 \varphi}{\partial t^2}(x, t) + \Delta\varphi(x, t) = 0 \quad \varphi(g, 0^+) = f(x)$$

for $f \in L^1(G)$ are given by $W_t * f$ and $P_t * f$ respectively. Here and elsewhere, Haar measures on compact groups are normalized to have total mass one. All Lebesgue spaces to be discussed will be with respect to such measures.

Let $\Gamma(x) = \{(y, t) \in G^+ \mid d(x, y) < t\}$. For a distribution f in $\mathscr{S}'(G)$, let

$$u_f(x, t) = P_t * f(x) \quad \text{and} \quad u_f^*(x) = \sup_{(y,t)\in\Gamma(x)} |u_f(y, t)|.$$

Then, for $0 < p < \infty$,

$$H^p(G) = \{f \in \mathscr{S}'(G) \mid u_f^* \in L^p(G)\}.$$

The 'norm' $\|f\|_{H^p(G)}$ of f in $H^p(G)$ is the Lebesgue norm $\|u_f^*\|_p$. Although $\|\cdot\|_{H^p(G)}$ is not a norm in general, it induces a complete metrizable topology on $H^p(G)$. Since $H^p(G) = L^p(G)$ for $p > 1$, we will restrict our attention to the case $0 < p \leq 1$.

We will also need the atomic Hardy spaces as originally defined by Coifman-Weiss [3] in the context of spaces of homogeneous type. We will actually use the modification for compact groups found in Clerc [2]. For each y in G, let L_y denote left translation by y in G. Let ε_1 and δ_1 be positive numbers such that $\exp^{-1} \circ L_{x^{-1}}$ is a diffeomorphism from the G-ball $B(x, \varepsilon_1)$ into the ball $B(0, \delta_1)$ of $\mathfrak{g}$ for all x in G. Let $T_x(G)$ be the tangent space of G at x. For a positive integer k and an element y of G, let

$$\mathscr{P}_k(y) = \left\{P : P = q \circ \exp^{-1} \circ L_{y^{-1}} \text{ for some polynomial } q \text{ on } \mathfrak{g} \text{ of degree } \leq k\right\}.$$

Let $0 < p \leq 1 \leq q \leq \infty$. Set $k(p) = [n\,(1/p - 1)]$. A *regular* (p, q) *atom* on G is a function $a(x)$ supported in some ball $B(y, \rho)$ $(0 < \rho < \varepsilon_1)$ such that

(i) $\|a\|_q \leq \rho^{n(1/q-1/p)}$ (size condition);

(ii) $\int_G a(x)P(x)\,dx = 0,\ P \in \mathscr{P}_{k(p)}(y)$ (cancellation condition).

An *exceptional* atom is a function bounded by 1. The atomic *Hardy space* $H_a^{p,q}(G)$ is the space of all $f \in \mathscr{S}'(G)$ of the form

$$f = \sum_k c_k a_k, \qquad \sum_k |c_k|^p < \infty,$$

the decomposition being in terms of regular (p, q) and exceptional atoms. The 'norm' $\|f\|_{p,q,a}$ of f in $H_a^{p,q}(G)$ is defined to be $\inf\left\{\left(\sum_k |c_k|^p\right)^{1/p}\right\}$ taken over all atomic decompositions of f. It is known in the more general context of spaces of homogeneous type that for fixed p, identical atomic Hardy spaces arise for all $q \in [1, \infty]$. We therefore need only consider the $q = \infty$ case. We denote $H_a^{p,\infty}(G)$ by $H_a^p(G)$. We will denote the norm of this space by $\|\cdot\|_{p,a}$.

3. The S-function characterization of $H^p(G)$

Let ϕ be a radial function in $\mathscr{S}(\mathbb{R}^\ell)$ which satisfies

$$\text{(3.1)} \qquad \begin{aligned} &\text{(i)} \quad \hat{\phi}(0) = 0 \\ &\text{(ii)} \quad \int_0^\infty \phi(s)^2\,ds/s = c(\phi) \neq 0. \end{aligned}$$

We define a central function in $C^\infty(G)$ by its restriction to T:

$$\text{(3.2)} \qquad \phi_t(x) = \sum_{\lambda \in \Lambda} \hat{\phi}\,(t\,\|\lambda + \delta\|)\,d_\lambda \chi_\lambda(x).$$

Let R be defined as in [2] and let μ^R denote the number of singular positive roots (as defined in [2, p. 87]). Let $D^R(H) = \prod_\alpha \sin\alpha(H)/2$, the product being over all positive non-singular roots. For a multi-index $J = (j_1, \dots, j_n)$, let $X^J = X_1^{j_1} X_2^{j_2} \cdots X_n^{j_n}$ and let $|J| = j_1 + \cdots + j_n$.

LEMMA 3.1. *Suppose that $x \in G$ is conjugate to* $\exp H$ *for $H \in \mathfrak{t}$. Then there is a constant C independent of x and t such that for any multi-index J and $m \in \mathbb{N}$*

$$(3.3)\quad \begin{aligned} &\text{(i)} && \left|X^J\phi_t(x)\right| \le Ct^{-m} \text{ if } t \ge \varepsilon_1,\\ &\text{(ii)} && \left|X^J\phi_t(x)\right| \le Ct^{-|J|-n},\\ &\text{(iii)} && \left|X^J\phi_t(x)\right| \le Ct^m\left(\|H\|^{-m-n-|J|} + t^{-\mu^R}D^R(H)^{-1}\right) \text{ if } \|H\| > t. \end{aligned}$$

The proof of this lemma is the same as the proof of [2, Theorem 5.4]. We will continue to denote unimportant constant by C, without distinguishing between different constants, if they have no crucial dependence on objects under consideration.

For any $f \in S'(G)$, we define the S-function of f by

$$(3.4)\qquad S_\phi f(x) = \left(\int_{\Gamma(x)} |(f*\phi_t)(y)|^2\, t^{-(n+1)}\, dy\, dt\right)^{1/2}$$

and the g-function of f by

$$(3.5)\qquad g(f)(x) = \left(\int_0^\infty |(f*\phi_t)(x)|^2\, dt/t\right)^{1/2}.$$

LEMMA 3.2. $\|g(f)\|_2 = c(\phi)\cdot\|f\|_2$.

PROOF. Since $\|g(f)\|_2^2 = c(\phi)\int_0^\infty\int_G |(f*\phi_t)(x)|^2\, dx dt/t$, the lemma follows from (3.1) and the Plancherel Theorem.

THEOREM 3.3. *For $f \in \mathcal{S}'(G)$, $u_f^* \in L^p(G)$ if and only if $S_\phi(f) \in L^p(G)$. Moreover, $\left\|u_f^*\right\|_p \cong \left\|S_\phi(f)\right\|_p$.*

PROOF. If $u_f^* \in L^p(G)$, then $f \in H_a^p(G)$ [1]. Therefore f has an atomic decomposition $f = \sum_j c_j a_j$ with $\sum\left|c_j\right|^p \le C\left\|u_f^*\right\|_p^p$. Now

$$\begin{aligned} \left\|S_\phi f(x)\right\|_p^p &= \int_G\left(\int_{\Gamma(x)}\left|\sum_j c_j\left(a_j*\phi_t\right)(y)t^{-(n+1)/2}\right|^2 dydt\right)^{p/2} dx\\ &\le \sum_j\left|c_j\right|^p\int_G\left(\int_{\Gamma(x)}\left|\left(a_j*\phi_t\right)(y)t^{-(n+1)/2}\right|^2 dydt\right)^{p/2} dx. \end{aligned}$$

If we show that

$$(3.6)\qquad \int_G\left(\int_{\Gamma(x)} |(a*\phi_t)(y)|^2\, t^{-(n+1)}\, dydt\right)^{p/2} dx \le C$$

for a constant C independent of the atom a, then

$$\left\|S_\phi f(x)\right\|_p^p \le C \sum_j \left|c_j\right|^p \le C\,\|f\|_{p,a}^p \le C\,\left\|u_f^*\right\|_p^p .$$

As the proof of (3.6) for $p \in (0, 1)$ is the same as for the case $p = 1$, we will assume that $p = 1$ for notational simplicity. The proof for exceptional atoms a is an easy consequence of Hölder's inequality and Lemma 3.2:

$$\begin{aligned}\left\|S_\phi(a)\right\|_1 &\le C\,\left\|S_\phi(a)\right\|_2 \\ &\le \left(\int_G \int_0^\infty \int_{d(x,y)<t} |(a * \phi_t)(y)|^2\, t^{-(n+1)}\, dy dt dx\right)^{1/2} \\ &\le C\,\|g(a)\|_2 \le C\,\|a\|_2 \le C.\end{aligned}$$

Now let $a(x)$ be a regular $(1, \infty)$ atom supported, without loss of generality, in $B(I, \rho)$. Using (i) of (3.3) we may assume that $t < \varepsilon_0$ for some fixed ε_0. We break up $\int_G \left|S_\phi(a)(x)\right|\, dx$ into two pieces according to whether $d(x, I) \ge 8L\rho$ or $d(x, I) < 8L\rho$ where L is the largest root length. Then

$$\int_{d(x,I)<8L\rho} \left|S_\phi(a)(x)\right|\, dx \le C\rho^{n/2}\,\|g(a)\|_2 \le C\rho^{n/2}\,\|a\|_2 \le C.$$

The remaining piece $\int_{d(x,I)\ge 8L\rho} \left|S_\phi(a)(x)\right|\, dx$ of $\left\|S_\phi(a)\right\|_1$ is itself broken into two pieces by partitioning each $\Gamma(x)$ into $\Gamma_1(x) = \{(y, t) : d(y, x) < t \le 2L\rho\}$ and $\Gamma_2(x) = \{(y, t) : d(y, x) < t, 2L\rho < t\}$. We will show that each

$$J_i = \int_{d(x,I)\ge 8L\rho} \left| \int_{\Gamma_i(x)} \left(\int_{B(I,\rho)} a(\xi)\phi_t(\xi^{-1}y)\, d\xi \right)^2 t^{-(n+1)}\, dy dt \right|^{1/2} dx \quad (i = 1, 2)$$

is bounded independently of a. For ξ, x and (y, t) in the integration in J_1,

$$d(\xi, I) \ge d(x, I) - d(y, \xi) - d(x, y) \ge d(x, I)/4 \ge 2L\rho > t.$$

Therefore, by (iii) of (3.3),

$$\begin{aligned}|J_1| \le C &\int_{d(x,I)\ge 2L\rho} \left\{ \int_{\Gamma_1(x)} \sup_{\xi\in B(y,\rho)} d(I, \xi)^{-2(n+1)} t^{1-n}\, dy\, dt \right\}^{1/2} dx \\ &+ C\,\|a\|_\infty \int_G \left\{ \int_0^{2L_\rho} \int_G \int_G D^R(\xi^{-1}y)^{-2}\, d\xi\, t^{2n-1}\, dy\, dt \right\}^{1/2} dx.\end{aligned}$$

The second summand is obviously bounded; for the first,

$$\int_{d(x,I)\ge 2L\rho} d(I, x)^{-(n+1)} \left\{ \int_{d(x,y)<t} \int_0^{2L\rho} t^{1-n}\, dy dt \right\}^{1/2} dx \le C\rho^{-1} \left\{ \int_0^{2L\rho} t\, dt \right\}^{1/2} \le C.$$

We estimate J_2 by partitioning each $\Gamma_2(x)$ into two pieces

$$\gamma_1(x) = \{(y,t) : d(y,x) < t,\ d(y, B(I,\rho)) \geq 4L\rho,\ t > 2L\rho\}$$

and

$$\gamma_2(x) = \{(y,t) : d(y,x) < t,\ d(y, B(I,\rho)) < 4L\rho,\ t > 2L\rho\}.$$

Write $J_2 \leq I_1 + I_2$ where

$$I_i = \int_{d(x,I)\geq 8L\rho} \left| \int_{\gamma_i(x)} \left(\int_{B(I,\rho)} a(\xi)\phi_t(\xi^{-1}y)\, d\xi \right)^2 t^{-(n+1)}\, dy\, dt \right|^{1/2} dx \quad (i = 1, 2).$$

Since a is a $(1,\infty)$-atom,

$$|I_2| \leq C\, \|a\|_\infty\, \rho^{n+1} \int_{B(I,8L\rho)^c} \left(\int_{\gamma_2(x)} \mathscr{M}\phi\,(y,t)\, t^{-(n+1)}\, dydt \right)^{1/2} dx$$

where

$$\mathscr{M}\phi\,(y,t) = \sup\left\{ \left|X_j\phi_t(\xi)\right|^2 : \xi \in B(y,\rho),\ 1 \leq j \leq n \right\}.$$

If $y \in \gamma_2(x)$, then $t > d(y,x) > d(x,I) - d(y,I) > d(x,I)/4$. Therefore, by (ii) and (iii) of (3.3), we have

$$\begin{aligned} |I_2| &\leq C\rho \int_{d(x,I)\geq 8L_\rho} \left| \int_{d(x,I)/4}^{\varepsilon_0} t^{-(2n+3)}\, dt \right|^{1/2} dx \\ &\quad + C\rho \int_G \left(\int_{2L\rho}^{\varepsilon_0} t^{2n} \int_{d(x,y)<t} \sup\left\{ t^{-2\mu^R} D^R(\xi)^{-2} : \xi \in B(y,\rho) \right\} dy\, dt \right)^{1/2} dx. \end{aligned}$$

The first summand is easily seen to be bounded and the second is bounded by $C\int_G D^R(y)^{-2}\, dy \leq C$ (cf. [2 , Lemma 6.4]).

In the first step in estimating I_1, we also use (ii) and (iii) of (3.3) as well as [2 , Lemma 6.4] to obtain

$$|I_1| \leq C\rho \int_{B(I,8L\rho)^c} \left[\int_{\gamma_1(x)} \sup_{\xi\in B(y,\rho)} (t + \|\xi\|)^{-2(n+1)}\, t^{-(n+1)}\, dydt \right]^{1/2} dx + C.$$

For any $y \in \gamma_1(x)$ and $\xi \in B(y,\rho)$, $d(\xi, I) \geq d(y, I)$ and $t + d(y,I) > (d(x,I) + t)/4$. Therefore

$$\begin{aligned} |I|_1 &\leq C\rho \int_{B(I,8L\rho)^c} \left[\int_{\gamma_1(x)} \left(t^{1/4}(t + d(y,I))^{-(n+1)}\right)^2 t^{-n-3/2}\, dydt \right]^{1/2} dx \\ &\leq C\rho \int_{B(I,8L\rho)^c} \left[\int_{2L\rho}^{\varepsilon_0} t^{-3/2}\, dt \right]^{1/2} d(x,I)^{-n-3/4}\, dx \leq C. \end{aligned}$$

This completes the proof that $\left\|S_\phi(a)\right\|_1 \leq C$ with C independent of the $(1, \infty)$-atom a.

We turn to the other direction of the equivalence, assuming that $S_\phi(f) \in L^p(G)$. Let Ψ be a radial function in $\mathscr{S}'(\mathbb{R}^\ell)$ that satisfies

$$\text{(3.7)}\qquad \begin{aligned} &\text{(i)} && \operatorname{supp}(\Psi) \subseteq \{\theta : |\theta| \leq 1\} \\ &\text{(ii)} && \int_{\mathbb{R}^\ell} \theta^I \Psi(\theta)\, d\theta = 0\ (I \in \mathbb{N}^\ell,\ |I| \leq 3n + 3 + 2n(1/p - 1/2)) \\ &\text{(iii)} && \int_0^\infty \hat{\phi}(t)\hat{\Psi}(t)\, dt/t = 1. \end{aligned}$$

By the Calderon reproducing formula on G, any $f \in \mathscr{S}'$ has a reproducing transformation

$$\text{(3.8)}\qquad f(x) = \int_{G^+} (f * \phi_t)(y)\, \Psi_t(xy^{-1})\, dy\, t^{-1}\, dt$$

which we break up as the sum of I_1 and I_2 where

$$\text{(3.9)}\qquad \begin{aligned} I_1(x) &= \int_0^\varepsilon \int_G (f * \phi_t)(y)\Psi_t(xy^{-1})\, dy\, t^{-1}\, dt, \quad \text{and} \\ I_2(x) &= \int_\varepsilon^\infty \int_G (f * \phi_t)(y)\Psi_t(xy^{-1})\, dy\, t^{-1}\, dt \end{aligned}$$

for a small ε that will be determined later. For this fixed ε, there is a constant C_ε such that

$$\|I_2\|_\infty \leq \left[\int_\varepsilon^\infty \int_G |(f * \phi_t)(y)|^2\, t^{-(n+1)}\, dy\, dt\right]^{1/2}.$$

Since G is compact, there are elements $x_1, \ldots, x_N$ ($N = N(G, \varepsilon)$) such that G is covered by the open $\varepsilon/4$-balls centered at these points. Let χ_i denote the characteristic function of $B(x_i, \varepsilon)$. Then

$$\begin{aligned} \|S_\Phi(f)\|_p^p &\geq N^{-1} \int_G \sum_{i=1}^N \chi_i(x) \left[\int_\varepsilon^\infty \int_{d(x,y)<t} |(f * \phi_t)(y)|^2\, t^{-(n+1)}\, dydt\right]^{p/2} dx \\ &\geq C_{N,p} \int_G \left[\sum_{i=1}^N \int_\varepsilon^\infty \int_{d(x,y)<t} \chi_i(x)\, |(f * \phi_t)(y)|^2\, t^{-(n+1)}\, dydt\right]^{p/2} dx \\ &\geq C_{N,p} \int_G \left[\sum_{i=1}^N \int_\varepsilon^\infty \int_{B(x_i,\varepsilon/4)} |(f * \phi_t)(y)|^2\, t^{-(n+1)}\, dydt\right]^{p/2} dx \\ &\geq C_{N,p} \left[\int_\varepsilon^\infty \int_G \chi_i(x)\, |(f * \phi_t)(y)|^2\, t^{-(n+1)}\, dydt\right]^{p/2} \\ &\geq C_{p,\varepsilon}\, \|I_2\|_\infty^p. \end{aligned}$$

Thus, we can find a constant $C_{p,\varepsilon}$ depending only on p and ε such that $I_2(x) = C_{p,\varepsilon}\, a(x)$ where $a(x)$ is an exceptional atom and $\left|C_{p,\varepsilon}\right| \leq C \left\|S_\phi(f)\right\|_p$.

To estimate $I_1(x)$, we let xy^{-1} be conjugate to $\exp\theta \in T$. Then $D(xy^{-1}) = \prod_{\alpha\in A} \sin\frac{1}{2}\alpha(\theta)$. There is a polynomial $P_{n(p)}$ of degree $2n+2+n(1/p-1/2)$ such that

$$\begin{aligned} I_1(x) = C\int_0^\varepsilon \int_G \Bigg\{ \Bigg(\prod_{\alpha\in A} \alpha(\theta) + \sum_{\alpha\in A} C_\alpha \alpha(\theta)^3 \prod_{\beta\in A} \beta(\theta) + \cdots + P_{n(p)}(\theta) \Bigg) \times \\ D^{-1}(\theta)\Psi_t(xy^{-1})\,(f*\phi_t)\,(y) \Bigg\}\, dy\, t^{-1}\, dt \\ + C\int_0^\varepsilon \int_G (f*\phi_t)\,(y) R(\theta) D^{-1}(\theta)\Psi_t(xy^{-1})\, dy\, t^{-1}\, dt \end{aligned} \tag{3.10}$$

where $R(\theta)$ is a $C^\infty(\mathbb{R}^\ell)$-function such that $R(\theta)D(\theta)^{-1} = O(\|\theta\|^{n+3+n(1/p-1/2)})$ and $X_i R(\theta) D(\theta)^{-1} = O(\|\theta\|^{n+2+n(1/p-1/2)})$. As a consequence of these estimates, we have $\left\|R(x)D^{-1}(x)\Psi_t(x)\right\|_\infty \leq Ct^{3+n(1/p-1/2)}$.

To complete the proof of Theorem 3.1, we will prove that each term in (3.10) has a suitable atomic decomposition. There are two types of terms that we must deal with. We will show that

$$I_{1,1}(x) = \int_0^\varepsilon \int_G (f*\phi_t)\,(y)\Psi_t(xy^{-1}) \prod_{\alpha\in A} \frac{\alpha(\theta)}{\sin\alpha(\theta)}\, dyt^{-1}\, dt$$

and

$$I_R(x) = \int_0^\varepsilon \int_G (f*\phi_t)\,(y)\Psi_t(xy^{-1}) R(\theta) D\,(\theta)^{-1}\; dyt^{-1}\, dt$$

have atomic decompositions

$$I_{1,1}(x) = \textstyle\sum_j \lambda_j a_j(x), \qquad I_R(x) = \sum_j \nu_j b_j(x) \tag{3.11}$$

with each a_j a $(p,2)$-atom, each b_j an exceptional atom and $\sum_j \left|\lambda_j\right|^p \leq \|S_\Phi(f)\|_p^p$ and $\sum_j \left|\nu_j\right|^p \leq \|S_\Phi(f)\|_p^p$. All other terms in (3.10) are handled in the same way as $I_{1,1}$.

Let ε_1 be as in the definition of atoms. For a choice of $\varepsilon \in (0, \varepsilon_1/32)$, the ball $B(x, 16\varepsilon)$ is contained in a local coordinate chart $\{V_x, \eta\}$ with $\operatorname{diam}(V_x) < \varepsilon_1$. Let $\{x_1, \ldots, x_N\}$ be such that $G = \cup_{j=1}^N B(x_j, \varepsilon)$. Let $U_j = B(x_j, \varepsilon)$, let $\chi_j(x) = \chi_{U_j}(x)$ and set $\xi_j(x) = \chi_j(x)/\sum_{i=1}^N \chi_j(x)$. Let $M(\theta) = M(xy^{-1}) = \prod_{\alpha\in A} \alpha(\theta)/\sin\alpha(\theta)$. Then $I_{1,1}(x) = \sum_{j=1}^N F_j(x)$ and $I_R(x) = \sum_{j=1}^N G_j(x)$ where

$$F_j(x) = C\int_0^\varepsilon \int_G \xi_j(y)\,(f*\phi_t)\,(y)\Psi_t(xy^{-1}) M(\theta)\, dyt^{-1}\, dt$$

and

$$G_j(x) = C \int_0^{\varepsilon} \int_G \xi_j(y)\,(f * \phi_t)\,(y)\Psi_t(xy^{-1})R(\theta)D(\theta)^{-1}\,dyt^{-1}\,dt.$$

It suffices to show that each F_j and G_j has an atomic decomposition of the indicated type. Henceforth we drop the index j.

Since $\{16 \cdot U, \eta\}$ is contained in a local coordinate chart, we may assume without loss of generality that $\eta(U)$ is the open cube of side length ε centered at $0 \in \mathbb{R}^n$ and that $d(x, y) = |\eta(x) - \eta(y)|$. We will write $\ell(B)$ for the sidelength of a dyadic cube in $\eta(U)$ and write $|B|$ for $\left|\eta^{-1}(B)\right|$. Let

$$\mathscr{B} = \{I_B : (y, t) \in I_B \text{ if and only if } y \in B \text{ and } \ell(B)/2 < t \leq \ell(B)\}.$$

For each $I_B \in \mathscr{B}$, we will write $\widetilde{I}_B$ for $(\ell(B)/2, \ell(B)) \times \eta^{-1}(B)$. If

$$f_B(x) = \int_{\widetilde{I}_B} \xi(y)\,(f * \phi_t)\,(y)\Psi_t(xy^{-1})M(\theta)\,dyt^{-1}\,dt \tag{3.12}$$

and

$$g_B(x) = \int_{\widetilde{I}_B} \xi(y)\,(f * \Phi_t)\,(y)\Psi_t(xy^{-1})R(\theta)D(\theta)^{-1}\,dyt^{-1}\,dt \tag{3.13}$$

then

$$F = \sum\nolimits_{I_B \in \mathscr{B}} f_B \quad \text{and} \quad G = \sum\nolimits_{I_B \in \mathscr{B}} g_B \tag{3.14}$$

in S'.

Observe first of all that f_B and g_B are C^∞-functions supported in $4\eta^{-1}(B)$ since Ψ is C^∞ and the integrands in (3.13) and (3.14) vanish unless $B \cap B(x, t)$ is not an empty set for some $t \in \widetilde{I}_B$. Also

$$\int_G f_B(x)\,P(x)\,dx = 0 \tag{3.15}$$

for all polynomials P with degree at most $n[1/p - 1] + n$. In fact, if log is the inverse of the local exponential map, then

$$\int_G f_B(x)P(x)dx = \int_{I_B} (f * \phi_t)\,(y)\xi(y) \int_G \Psi_t(xy^{-1})M(xy^{-1})P(\log(xy^{-1}y))dxdydt/t.$$

To prove (3.15), it therefore suffices to show that

$$\int_G \Psi_t(x)M(x)P(\log(xy))\,dx = 0$$

for any fixed $t < \varepsilon$ and $y \in G$. Since $\Psi_t \cdot M$ is a central function, we need only prove that

$$\int_G \int_G \Psi_t(x) M(x) P(\log(zxz^{-1}y))\, dx\, dz = 0$$

or

$$\int_G \Psi_t(x) M(x) \int_G P(\log(zxz^{-1}y))\, dz\, dx = 0.$$

But $\int_G P(\log(zxz^{-1}y))\, dz$ is a class function that is a polynomial of $\theta = \log(x)$ with degree at most $n[1/p - 1] + n$. Thus it suffices to show that

$$\int_G \Psi_t(x) M(x)\, (\log(x))^J\, dx = C \int_{\mathfrak{t}} \Psi_t(\exp\theta) M(\exp\theta) \theta^J D^2(\theta)\, d\theta = 0$$

for all multi-indices J with $|J| \le n[1/p - 1] + n$. This follows by Poisson summation in view of the choice of Ψ_t.

From the preceding observations, we know that each f_B is a constant multiple of a (p, ∞)-atom. It does not yet follow, however, that the first equation of (3.14) is an atomic decomposition of F since the norms of the f_B's do not sum properly. For each $I_B \in \mathscr{B}$ we define

$$S_B = \left(\int_{\tilde{I}_B} |(f * \phi_t)(y)|^2\, dy t^{-1}\, dt \right)^{1/2}.$$

We claim that for all $I_B \in \mathscr{B}$ and all multi-indices J,

$$\text{(3.16)} \qquad \begin{aligned} &\text{(i)} \quad \left\| X^J f_B \right\|_\infty \le C S_B\, |B|^{-1/2 - |J|/n}, \text{ and} \\ &\text{(ii)} \quad \|g_B\|_\infty \le C S_B\, |B|^{1/p + 2/n} \end{aligned}$$

where C depends on J but not on B.

By Schwarz's inequality,

$$\left| X^J f_B(x) \right| \le C S_B \left(\int_{\tilde{I}_B} \left| X^J(\Psi_t(y^{-1}x) M(yx^{-1})) \right|^2 dy\, t^{-1}\, dt \right)^{1/2}.$$

Therefore $\|\Psi_t M\|_\infty \le C t^{-n} \le C\, |B|^{-1}$, $\left\| X^J(\Psi_t M) \right\|_\infty \le C\, |B|^{-1-|J|/n}$, and $\|M\|_\infty \le C$; thus (i) of (3.16) follows. Similarly,

$$\begin{aligned} |g_B(x)| &\le C S_B \left(\int_{\tilde{I}_B} \left| \Psi_t(y^{-1}x) D(y^{-1}x)^{-1} R(y^{-1}x) \right|^2 dy\, t^{-1}\, dt \right)^{1/2} \\ &\le C S_B\, |B|^{1/2} \sup \left\{ \left\| \Psi_t D^{-1} R \right\|_\infty : \ell(B)/2 \le t \le \ell(B) \right\} \\ &\le C S_B\, |B|^{1/p + 2/n} \end{aligned}$$

which completes the proof of (3.16).

For each integer k, let $\Omega_k = \{x : S_\Phi f(x) > 2^k\}$ and let $\mathscr{B}_k$ be defined by

$$\mathscr{B}_k = \left\{I_B \in \mathscr{B} : \left|\eta^{-1}(B) \cap \Omega_k\right| > \left|\eta^{-1}(B)/4\right| \text{ and } \left|\eta^{-1}(B) \cap \Omega_{k+1}\right| \le \left|\eta^{-1}(B)/4\right|\right\}$$

where $B/2$ is any one of the 2^n subdyadic cubes of B. It is easy to see that $\Omega_{k+1} \subset \Omega_k$ and that each I_B must belong to precisely one $\mathscr{B}_k$. We claim that there is a C independent of k such that

$$\sum_{I_B \in \mathscr{B}_k} S_B^2 \le C 2^{2k} |\Omega_k| . \tag{3.17}$$

To see this, let M_{HL} denote the Hardy-Littlewood maximal function and let $\tilde{\Omega}_k = \{x : M_{HL}(\chi_{\Omega_k})(x) > 4^{-n}\}$. Observe that $\Omega_k \subset \tilde{\Omega}_k$ and that $\left|\tilde{\Omega}_k\right| \le C |\Omega_k|$ by the Hardy-Littlewood maximal theorem. These imply that

$$\int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} |S_\Phi f(x)|^2 \, dx \le 2^{2k+2} \left|\tilde{\Omega}_k\right| \le C 2^{2k} |\Omega_k| . \tag{3.18}$$

Let

$$v_k(y, t) = \left|\left\{x \in \tilde{\Omega}_k \setminus \Omega_{k+1} : d(x, y) < t\right\}\right| .$$

Notice that

$$\int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} |S_\Phi f(x)|^2 \, dx = \int_0^\infty \int_G |(f * \Phi_t)(y)|^2 \, v_k(y, t) \, dy \, t^{-1-n} \, dt$$

and therefore

$$\int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} |S_\Phi f(x)|^2 \, dx \ge \sum_{I_B \in \mathscr{B}_k} \int_{I_B} |(f * \Phi_t)(y)|^2 \, v_k(y, t) \, dy \, t^{-1-n} dt .$$

In view of this and (3.18), in order to obtain (3.17), it suffices to show that there is a constant C independent of k such that

$$v_k(y, t) \ge C t^n \quad \text{for all } I_B \in \mathscr{B}_k \text{ and } (y, t) \in \tilde{I}_B . \tag{3.19}$$

Let $I_B \in \mathscr{B}_k$ and $(y, t) \in \tilde{I}_B$. Since $\left|\eta^{-1}(B) \cap \Omega_k\right| > \left|\eta^{-1}(B)/4\right|$, it follows that $M_{HL}\left(\chi_{\Omega_k}\right)(x) > 4^{-n}$ for $x \in \eta^{-1}(B)$. Therefore $\eta^{-1}(B) \subset \tilde{\Omega}_k$. Also, since $\left|\eta^{-1}(B) \cap \Omega_{k+1}\right| \le \left|\eta^{-1}(B)/4\right|$, it follows that $\left|\eta^{-1}(B) \setminus \Omega_{k+1}\right| \ge \left|3\eta^{-1}(B)/4\right|$. Thus,

$$\begin{aligned} \left\{x \in \tilde{\Omega}_k \setminus \Omega_{k+1} : d(x, y) < t\right\} &\supseteq \left\{x \in \tilde{\Omega}_k \setminus \Omega_{k+1} : d(x, y) < \ell(B)/2\right\} \\ &\supseteq \eta^{-1}(B)/2 \cap \tilde{\Omega}_k \setminus \Omega_{k+1} \supseteq \left(\eta^{-1}(B)/2\right) \setminus \Omega_{k+1} . \end{aligned}$$

It follows that

$$\begin{aligned} v_k(y,t) &\geq \left|\left(\eta^{-1}(B)/2\right)\setminus\Omega_{k+1}\right| \\ &\geq \left|\eta^{-1}(B)/2\right| - \left|\eta^{-1}(B)\cap\Omega_{k+1}\right| \\ &\geq \left|\eta^{-1}(B)/4\right| \\ &\geq Ct^n \end{aligned}$$

proving (3.19) and, as noted, (3.17).

We can define a partial ordering on $\mathscr{B}_k$ by inclusion. Let $\left\{B^i\right\}$ be an enumeration of the maximal elements of $\mathscr{B}_k$. Each $B\in\mathscr{B}_k$ satisfies $B\subset B^i$ for some i; for every $B\in\mathscr{B}_k$ we choose such an $i=i(B)$. For every i, let $\mathscr{B}_k^i=\{B\in\mathscr{B}_k : i(B)=i\}$. Thus, $\mathscr{B}_k=\bigcup_i\mathscr{B}_k^i$, disjoint. Define

$$\varphi_k^i=\textstyle\sum_{B\in\mathscr{B}_k^i} f_B \quad\text{and}\quad \gamma_k^i=\textstyle\sum_{B\in\mathscr{B}_k^i} g_B.$$

We claim that there exists a C independent of k and i such that

$$\text{(3.20)}\qquad \begin{aligned} &\text{(i)} && \left\|\gamma_k^i\right\|_\infty \leq C\sum_{B\in\mathscr{B}_k^i} S_B\,|B|^{1/p+2/n} \\ &\text{(ii)} && \left\|\varphi_k^i\right\|_\infty \leq C\sum_{B\in\mathscr{B}_k^i} S_B^2. \end{aligned}$$

The first estimate of (3.20) is immediate from (3.16). For the second estimate, we let $B_1, B_2, \ldots$ be an enumeration of $\mathscr{B}_k^i$ ordered so that $|B_r|\geq|B_s|$ if $r\leq s$. In the proof of this estimate, we will write f_r and S_r for f_{B_r} and S_{B_r}. Then

$$\left\|\varphi_k^i\right\|_2^2=\textstyle\sum_r \|f_r\|_2^2+2\,\mathrm{Re}\sum_{r<s}\displaystyle\int_G f_r\overline{f_s}.$$

Now $\|f_r\|_2^2=\int_{4\eta^{-1}(B_r)}\left|f_j\right|^2\leq C\left|4B_j\right|\left|B_j\right|^{-1}S_j^2$. To estimate the cross terms, we need only consider r and s such that $4B_r\cap 4B_s\neq\phi$, for $f_r f_s$ vanishes identically otherwise. Therefore we suppose that $r<s$ and $4B_r\cap 4B_s\neq\phi$. We let x_s be the center of B_s and let $P_{r,s}$ be the Taylor polynomial of f_r at x_s of degree $a=n[1/p-1]+n/2$. Then, by (3.15) we have

$$\begin{aligned} \left|\int_G f_r(x)\overline{f_s}(x)\,dx\right| &= \left|\int_G (f_r(x)-P_{rs}(x))\,\overline{f_s}(x)\,dx\right| \\ &= \left|\int_G\left(\int_{\tilde{I}_{B_r}}\xi(y)\,(f*\Phi_t)\,(y)\Psi_t(y^{-1}x)M(y^{-1}x)\,dyt^{-1}\,dt-P_{rs}(x)\right)\overline{f_s}(x)\,dx\right| \\ &\leq C\int_{4\eta^{-1}(B_s)}\sum_{|J|\leq a+1}\left\|X^J f_r\right\|_\infty \|f_s\|_\infty\, d(x,x_s)^{a+1}\,dx \\ &\leq C\sum_{|J|\leq a+1}|B_s|^{1+(a+1)/n}\,|B_r|^{-1/2-(a+1)/n}\,|B_s|^{-1/2}\,S_rS_s \\ &\leq C\left(|B_s|\,/\,|B_r|\right)^{1/2+(a+1)/n}S_rS_s. \end{aligned}$$

For these indices we set $\beta_{rs} = (|B_s|/|B_r|)^{1/2+(a+1)/n}$ and we set $\beta_{rs} = 0$ otherwise. We must show that $\sum_{rs} \beta_{rs} S_r S_s \leq C \sum_s S_s^2$ for some C. To do this it suffices to show that there is a constant C such that

$$(3.21) \qquad \textstyle\sum_r \beta_{rs} < C \text{ for all } s \quad \text{and} \quad \sum_s \beta_{rs} < C \text{ for all } r.$$

If so, $\sum_r \left(\sum_s \beta_{rs} S_s\right)^2 \leq \sum_r \left(\sum_s \beta_{rs}\right)\left(\sum_s \beta_{rs} S_s^2\right) \leq C \sum_s S_s^2$ and therefore

$$\textstyle\sum_{rs} \beta_{rs} S_r S_s \leq \left(\sum_r S_r^2\right)^{1/2} \left(\sum_r \left(\sum_s \beta_{rs} S_s\right)^2\right)^{1/2} \leq C \sum_s S_s^2.$$

We turn to (3.21). For each $m \in \mathbb{N}$ there are at most $16^n 2^{mn}$ values of s such that $|B_s| = 2^{-mn}|B_r|$ and $4B_r \cap 4B_s \neq \phi$. For each s there are at most 16^n values of r such that $|B_r| = 2^{mn}|B_s|$ and $4B_r \cap 4B_s \neq \phi$. Therefore

$$\begin{gathered}\textstyle\sum_r \beta_{rs} \leq C \sum_{m=0}^{\infty} 2^{mn} 2^{-(mn/2)-m(a+1)} \leq C \sum_{m=0}^{\infty} 2^{m(n/2-a-1)} \leq C \\ \text{and} \qquad \textstyle\sum_s \beta_{rs} \leq C \sum_{m=0}^{\infty} 2^{-m(n/2+a+1)} \leq C.\end{gathered}$$

Recall that $F(x) = \sum_{I_B \in \mathscr{B}} f_B(x) = \sum_{ik} \varphi_k^i(x)$. Let $\lambda_k^i = \|\varphi_k^i\|_2 / |4B^i|^{1/2-1/p}$ and $a_k^i(x) = \varphi_k^i(x)/\lambda_k^i$. Then, by [5, p. 240], $F(x) = \sum_{ik} \lambda_k^i a_k^i(x)$ is an atomic decomposition in which each a_k^i is a $(p,2)$-atom and $\sum_{ik} |\lambda_k^i|^p \leq \|S(f)\|_p^p$. Thus, (3.11) is finally proved.

Now let $\nu_k^i = C \sum_{B \in \mathscr{B}_k^i} S_B |B|^{1/p+2/n}$ and let $b_k^i(x) = C\gamma_k^i(x)/\nu_k^i$. By (3.20), $b_k^i(x)$ is an exceptional atom. Moreover, for $\kappa = 2/(2-p) > 1$,

$$\sum_{ik} |\nu_k^i|^p \leq C \sum_{ik} \sum_{B \in \mathscr{B}_k^i} S_B^p |B|^{1+2p/n} \leq C \sum_k \left(\sum_{B \in \mathscr{B}_k} S_B^2\right)^{p/2} \left(\sum_i \sum_{B \in \mathscr{B}_k^i} |B|^\kappa\right)^{1/\kappa}.$$

Since there are, for each B^i, at most 2^{mn} cubes $B \in \mathscr{B}_k^i$ such that $|B| = 2^{-mn}|B^i|$ we conclude that

$$\sum_{B \in \mathscr{B}_k^i} |B|^\kappa = \sum_{m=1}^{\infty} 2^{mn} 2^{-mn\kappa} |B^i|^\kappa \leq C |B^i|.$$

Thus,

$$\sum_{ik} |\nu_k^i|^p \leq C \sum_k 2^{kp} |\Omega_k|^{p/2} |\Omega_k|^{1-p/2} = C \sum_k 2^{kp} |\Omega_k|$$

and the required atomic decomposition $I_R(x) = \sum_{ik} \nu_k^i b_k^i(x)$ has been proved.

4. The characterization of $H^p(G)$ by the g_λ^*-function

For $f \in \mathscr{S}'(G)$ and $\lambda > 1$, we define the g_λ^*-function of $f(x)$ by

$$g_\lambda^*(f)(x) = \left(\int_0^\infty \int_G \left[\frac{t}{t+d(x,y)}\right]^{\lambda n} |(f*\phi_t)(y)|^2 t^{-(1+n)}\,dy\,dt\right)^{1/2}.$$

THEOREM 4.1. *Suppose that $f \in \mathscr{S}'(G)$. For $0 < p \le 1$ and $\lambda > 2/p$, $f \in H^p(G)$ if and only if $g_\lambda^*(f) \in L^p(G)$. Moreover $\|g_\lambda^*(f)\|_p \simeq \|S_\phi(f)\|_p \simeq \|u_f^*\|_p$.*

PROOF. Suppose that $0 < p \le 1$ and $\lambda > 2/p$. By Theorem 3.1 we need only check that $\|S_\phi(f)\|_p \simeq \|u_f^*\|_p$. Since $S_\phi(f)(x) \le C g_\lambda^*(f)(x)$, only the estimate $\|g_\lambda^*(f)\|_p \le C\|S_\phi(f)\|_p$ requires further proof. As in the proof of Theorem 3.3, it suffices to show that there is a constant C such that for any atom $a(x)$,

$$\int_G\left(\int_0^\infty \int_G \left[\frac{t}{t+d(x,y)}\right]^{\lambda n} |(a*\phi_t)(y)|^2 t^{-(1+n)}\,dy\,dt\right)^{p/2} dx \le C. \tag{4.1}$$

We will supply details only for the case $p = 1$. If $a(x)$ is an exceptional atom, then

$$\begin{aligned}
\|g_\lambda^*(a)\|_1 &\le C\|g_\lambda^*(a)\|_2 \\
&= C\int_G\left(\int_0^\infty \int_G \left[\frac{t}{t+d(x,y)}\right]^{\lambda n} |(a*\phi_t)(y)|^2 t^{-(1+n)}\,dydt\right)dx \\
&= C\int_G\left(\left(\int_0^\infty\int_{d(x,y)<t} + \int_0^\infty\int_{d(x,y)>t}\right)\left[\frac{t}{t+d(x,y)}\right]^{\lambda n} \times\right. \\
&\qquad\qquad \left. |(a*\phi_t)(y)|^2 t^{-(1+n)}dydt\right)dx.
\end{aligned}$$

By Theorem 3.3, the first summand is bounded by $C\|a\|_2 \le C$. The second summand is bounded by

$$C\int_G\int_0^\infty |(a*\phi_t)(y)|^2 t^{-1}\,dt\,dy \le C\|g(a)\|_2 \le C\|a\|_2 \le C.$$

For a regular $(1,\infty)$-atom a, we may assume that the support of a is contained in $B(I,\rho)$ with ρ sufficiently small. Our analysis will be based on Lemmas 2.4 and 6.4 of [2].

We write

$$\|g_\lambda^*(a)\|_1 = \int_{d(x,I)\ge 8\rho} |g_\lambda^*(a)(x)|\,dx + \int_{d(x,I)<8\rho} |g_\lambda^*(a)(x)|\,dx = I_1 + I_2.$$

By Schwarz's inequality,

$$|I_2| \leq C\rho^{n/2}\left(\int_G\int_0^\infty\int_G\left[\frac{t}{t+d(x,y)}\right]^{\lambda n}|(a*\Phi_t)(y)|^2\,t^{-(1+n)}\,dy\,dt\,dx\right)^{1/2}$$

and therefore $|I_2| \leq C\rho^{n/2}\|a\|_2 \leq C$ as in the case of exceptional atoms. To estimate I_1, note that

$$\begin{aligned}|I_1| \leq &\int_{d(x,I)\geq 8\rho}\left(\int_{d(x,y)>t}\left[\frac{t}{t+d(x,y)}\right]^{\lambda n}|(a*\phi_t)(y)|^2\,t^{-(1+n)}\,dydt\right)^{1/2}dx\\ &+C\left\|S_\phi(a)\right\|_1.\end{aligned}$$

It suffices, then, to estimate the double integral above. We do this by breaking it into three pieces:

$$\begin{aligned}L_1 &= \int_{d(x,I)\geq 8\rho}\left(\int_{\Delta_{1,x}}\left[\frac{t}{t+d(x,y)}\right]^{\lambda n}|(a*\phi_t)(y)|^2\,t^{-(1+n)}\,dy\,dt\right)^{1/2}dx,\\ L_2 &= \int_{d(x,I)\geq 8\rho}\left(\int_{\Delta_{2,x}}\left[\frac{t}{t+d(x,y)}\right]^{\lambda n}|(a*\phi_t)(y)|^2\,t^{-(1+n)}\,dy\,dt\right)^{1/2}dx,\\ L_3 &= \int_{d(x,I)\geq 8\rho}\left(\int_\rho^\infty\int_{d(x,y)>t}\left[\frac{t}{t+d(x,y)}\right]^{\lambda n}|(a*\phi_t)(y)|^2\,t^{-(1+n)}\,dydt\right)^{1/2}dx\end{aligned}$$

where

$$\begin{aligned}\Delta_{1,x} &= \{(y,t) : d(y,x) > t, 0 < t < \rho, d(y,B(I,\rho)) \geq 2\rho\} \text{ and}\\ \Delta_{2,x} &= \{(y,t) : d(y,x) > t,\ 0 < t < \rho, d(y,B(I,\rho)) < 2\rho\}.\end{aligned}$$

We start with L_2. Notice that for any $(y,t) \in \Delta_{2,x}$ and $x \notin B(I,8\rho)$, $d(x,y) > d(x,I) - d(y,I) \geq d(x,I)/2$. Combine this with the estimate $\|a*\phi_t\|_\infty \leq \|a\|_\infty\|\phi_t\|_1 \leq C\|a\|_\infty$ and [2, Lemma 3.4] to get

$$\begin{aligned}L_2 &\leq C\rho^{-n}\int_{d(x,I)\geq 8\rho}d(x,I)^{-\lambda n/2}\left(\int_0^\rho t^{\lambda n-n-1}\int_{d(y,I)\leq 4\rho}dydt\right)^{1/2}dx + C\\ &\leq C\rho^{-n-\lambda n/2+n}\rho^{n/2}\rho^{-n/2+\lambda n/2} + C.\end{aligned}$$

To estimate L_1, note that for $(y,t) \in \Delta_{1,x}$ and $\xi \in B(y,\rho)$, $d(\xi,I) \geq d(y,I) - d(\xi,I) \geq d(y,I)/2$. Let

$$G(x,y,t) = \left(\frac{t}{t+d(x,y)}\right)^{\lambda n}\left(\frac{td(y,I)}{\left[t^2+d(y,I)^2\right]^{(n+3)/2}}\right)^2 t^{1-n}.$$

By Lemma 3.1, we conclude that

$$
\begin{aligned}
|L_1| \le & \int_{d(x,I)\ge 8\rho} \left(\int_0^\rho \int_{\Gamma(x)^c\cap B(I,\rho)^c} \left[\frac{t}{t+d(x,y)}\right]^{\lambda n} \left| \sup_{\xi\in B(y,\rho)} \phi_t(\xi)\right|^2 t^{-(1+n)}\, dydt \right)^{1/2} dx \\
& + C \\
= & L_{1,1} + L_{1,2} + C
\end{aligned}
$$

where

$$
L_{1,1} = \int_{d(x,I)\ge 8\rho} \left(\int_0^\rho \int_{\Gamma(x)^c\cap B(I,\rho)^c\cap\{y:d(y,I)>d(x,I)/2\}} G(x,y,t)\, dydt \right)^{1/2} dx
$$
$$
L_{1,2} = \int_{d(x,I)\ge 8\rho} \left(\int_0^\rho \int_{\Gamma(x)^c\cap B(I,\rho)^c\cap\{y:d(y,I)<d(x,I)/2\}} G(x,y,t)\, dydt \right)^{1/2} dx.
$$

Clearly

$$
L_{1,1} \le C \int_{d(x,I)\ge 8\rho} d(x,I)^{-(n+1)}\, dx \left(\int_0^\rho t\, dt \right)^{1/2} \le C.
$$

For $L_{1,2}$, note that in the region of integration $d(x,y) > d(x,I) - d(y,I) \ge d(x,I)/2$. Therefore,

$$
\begin{aligned}
L_{1,2} & \le C \int_{d(x,I)\ge 8\rho} d(x,I)^{-\lambda n/2} \left(\int_0^\rho t^{\lambda n - n+1} \int_{d(y,I)>\rho} d(y,I)^{-2n-2}\, dydt \right)^{1/2} dx \\
& \le C\rho^{-\lambda n/2+n} \rho^{\lambda n/2 - n/2+1} \rho^{-1-n/2} \le C.
\end{aligned}
$$

This completes the estimate of L_1.

It remains to estimate L_3. We divide the domain $\{(y,t) : d(x,y) > t, t > \rho\}$ into two pieces

$$
\Omega_{x,1} = \{(y,t) : d(y,x) > t, t > \rho, d(y, B(I,\rho)) \ge 2\rho\}
$$
$$
\Omega_{x,2} = \{(y,t) : d(y,x) > t, t > \rho, d(y, B(I,\rho)) < 2\rho\}
$$

and the integral L_3 into two terms $L_{3,1}$ and $L_{3,2}$ accordingly. For the latter, we argue as in Theorem 3.1 that

$$
\begin{aligned}
L_{3,2} & \le C\rho \int_{d(x,I)\ge 8\rho} d(x,I)^{-\lambda n/2} \left(\int_\rho^\infty t^{-2n-3-n} \int_{d(y,I)\le 3\rho} t^{\lambda n}\, dtdt \right)^{1/2} dx + C \\
& \le C\rho^{-\lambda n/2+n+1} \rho^{\lambda n/2 - n/2 - n-1} \rho^{n/2} \le C.
\end{aligned}
$$

Only $L_{3,1}$ remains. Let $\Theta_{x,1} = \{(y,t) \in \Omega_{x,1} : d(y,I) > \rho,\ d(y,I) > d(x,I)/2\}$ and $\Theta_{x,2} = \{(y,t) \in \Omega_{x,1} : d(y,I) > \rho,\ d(y,I) < d(x,I)/2\}$. Again using Lemmas

3.4 and [2, 6.4] we get

$$L_{3,1} \le C + C\rho \int_{B(I,8\rho)^c} \left(\int_{\Theta_{x,1}} \frac{d(y,I)}{t^2 + d(y,I)^2} G(x,y,t)\, dydt \right)^{1/2} dx$$
$$+ C\rho \int_{B(I,8\rho)^c} \left(\int_{\Theta_{x,2}} \frac{d(y,I)}{t^2 + d(y,I)^2} G(x,y,t)\, dydt \right)^{1/2} dx.$$

The first of these integral summands is bounded by

$$C\rho \int_{B(I,8\rho)^c} d(x,I)^{-n-1/2} \left(\int_\rho^\infty t^{\lambda n - n - 2} \int_{d(x,y)>t} d(x,y)^{-\lambda n}\, dydt \right)^{1/2} dx$$
$$\le C\rho^{1/2} \left(\int_\rho^\infty t^{-2} dt \right)^{1/2} \le C$$

and the second one is bounded by

$$C\rho \int_{B(I,8\rho)^c} d(x,I)^{-\lambda n/2} \left(\int_\rho^\infty t^{\lambda n - 2n - 2} \int_{B(I,\rho)^c} d(y,I)^{n+1}\, dy\, dt \right)^{1/2} dx \le C.$$

Therefore, $\left\| g_\lambda^*(a) \right\|_1 \le C$ for any atom $a(x)$, completing the proof of Theorem 4.1.

By an argument in [6], it is easy to prove that $\left\| g_\lambda^*(f) \right\|_p \le C \left\| f \right\|_p$ for $p \ge 2$ and $\lambda > 2/p$. Interpolation (Theorem E of [3]) then gives

THEOREM 4.2. *For $p > 1$ and $\lambda > 2/p$,* $\left\| g_\lambda^*(f) \right\|_p \le C \left\| f \right\|_p$.

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