

ON THE UNIFORM KADEC-KLEE PROPERTY WITH RESPECT TO CONVERGENCE IN MEASURE

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Abstract

Let $E(0, \infty)$ be a separable symmetric function space, let M be a semifinite von Neumann algebra with normal faithful semifinite trace μ , and let $E(M, \mu)$ be the symmetric operator space associated with $E(0, \infty)$. If $E(0, \infty)$ has the uniform Kadec-Klee property with respect to convergence in measure then $E(M, \mu)$ also has this property. In particular, if $L_\Phi(0, \infty)(\Lambda_\Phi(0, \infty))$ is a separable Orlicz (Lorentz) space then $L_\Phi(M, \mu)(\Lambda_\Phi(M, \mu))$ has the uniform Kadec-Klee property with respect to convergence in measure. It is established also that $E(0, \infty)$ has the uniform Kadec-Klee property with respect to convergence in measure on sets of finite measure if and only if the norm of $E(0, \infty)$ satisfies G. Birkhoff's condition of uniform monotonicity.

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0. Introduction

Let $(X, \|\cdot\|_X)$ be a Banach space, and let τ be a topological vector space topology on X that is weaker than the norm topology. The space $(X, \|\cdot\|_X)$ is said to have the uniform Kadec-Klee property with respect to τ (notation $X \in (\text{UH}_\tau)$) if for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for every sequence $(x_n) \subset X$ with $\|x_n\|_X = 1$, $\|x_n - x_m\|_X > \epsilon$ ($m \neq n$) and with limit x in the topology τ , we have $\|x\|_X < 1 - \delta(\epsilon)$.

We will consider the following cases:

- (1) X is a Banach space with a Schauder basis (e_n) , $\tau = \sigma(X, \Gamma)$ where $\Gamma = [e_n^*]$;
- (2) X is a symmetric function space $E(0, \infty)$, $\tau =$ convergence locally in measure and
- (3) $X = E(0, \infty)$ ($X = E(M, \mu)$), $\tau =$ convergence in Lebesgue measure m ($\tau =$ convergence in the measure topology on the set of all μ -measurable operators (see

[5])). We will denote the (UH_Γ) -property by (UH_Γ) in the first case; by $(UHlm)$ in the second and by $(UHm)((UH\mu))$ in the third.

See [8,10,11] for general information concerning Banach spaces and symmetric function spaces. For relevant terminology from the theory of the von Neumann algebras we refer to [17], and for the theory of non-commutative integration we refer to [5].

1. Property (UH_Γ) in spaces with Schauder basis

Let $(X, \|\cdot\|)$ be a Banach space with a Schauder basis (e_n) . Throughout this section (e_n^*) are the bi-orthogonal functionals associated with (e_n) , and P_n are the projections onto $[e_k]_{k=1}^n$ with kernel $[e_k]_{k=n+1}^\infty$; that is for every $x = \sum_{k=1}^\infty e_k^*(x)e_k$, we have $P_n x = \sum_{k=1}^n e_k^*(x)e_k$.

The basis (e_n) is said to satisfy the condition (C) if for every $c > 0$ there exists $\delta = \delta(c) > 0$ such that for every $x \in X$ and for each integer n it follows from the conditions $\|P_n x\| = 1$ and $\|(I - P_n)x\| \geq c$ that

$$\|x\| \geq 1 + \delta.$$

The Theorem below is due (in implicit form) to D. van Dulst and V. de Valk [4] for the case when X is an Orlicz sequence space; however, it is also true more generally. We omit the proof and refer to [4, Proposition 3].

THEOREM 1. *If a basis (e_n) satisfies the condition (C) then $X \in (UH_\Gamma)$, where $\Gamma = [e_n^*]_{n=1}^\infty$.*

It is well known that every Banach space with an unconditional basis (e_n) , whose unconditional constant is equal to 1, is a Banach lattice when the order is defined by $\sum_{n=1}^\infty a_n e_n \geq 0$ if and only if $a_n \geq 0$ for all n . A Banach lattice $(X, \|\cdot\|_x)$ is called a UMB-lattice (notation: $(X, \|\cdot\|_x) \in (UMB)$) if its norm satisfies G. Birkhoff's condition of uniform monotonicity; that is for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $f, g \in X$, $f \geq 0$, $g \geq 0$, $\|f\|_x = 1$ and $\|f + g\|_x \leq 1 + \delta(\epsilon)$ then $\|g\|_x \leq \epsilon$ (see [2]). In addition, if we suppose that $f \wedge g = 0$, then X is said to have the property (UMBd) (notation: $(X, \|\cdot\|_x) \in (UMBd)$). We remark that the property (UMBd) coincides with the property (C) which is considered in [3]. Evidently, if $(X, \|\cdot\|_x) \in (UMBd)$ and the order on X is defined by the unconditional basis (e_n) , then (e_n) satisfies condition (C). So, we have the following

COROLLARY 1. (See also [3]). *Let X be a Banach lattice whose order is induced by the 1-unconditional basis (e_n) . If $(X, \|\cdot\|_x) \in (UMBd)$ then $X \in (UH_\Gamma)$.*

The next theorem shows that in the preceding corollary the condition $(X, \|\cdot\|_x) \in (\text{UMBd})$ is also necessary if basis (e_n) is symmetric.

THEOREM 2. *Let X be a Banach lattice whose order is induced by the 1-symmetric basis (e_n) . Then the following conditions are equivalent:*

- (i) $(X, \|\cdot\|_x) \in (\text{UMB})$;
- (ii) $(X, \|\cdot\|_x) \in (\text{UMBd})$;
- (iii) $(X, \|\cdot\|_x) \in (\text{UH}_\Gamma)$.

PROOF. The implication (ii) \rightarrow (iii) follows from Corollary 1. The implication (i) \rightarrow (ii) is obvious.

Before proving the implication (iii) \rightarrow (i) of the theorem, we recall the following facts. Let $a = (a_i), b = (b_i) \in c_0$, let $a^* = (a_i^*), b^* = (b_i^*)$ be the sequences $(|a_i|), (|b_i|)$ arranged in non-increasing order. We say that a is weakly submajorized by b and write $a \prec_w b$ if $\sum_{i=1}^k a_i^* \leq \sum_{i=1}^k b_i^*$, for $k = 1, 2, 3, \dots$. It is well known that if $a \prec_w b, b = (b_i) \in X$ (we identify $b = \sum_i b_i e_i$ with (b_i)) then $a \in X$ and $\|a\|_x \leq \|b\|_x$.

Divide the set N into two disjoint subsets A and B with $\text{card } A = \text{card } B = \infty$. Let $\pi_1 : N \rightarrow A, \pi_2 : N \rightarrow B$ be arbitrary injections. Put $a' = \sum_k e_k^*(a) e_{\pi_1(k)}, b' = \sum_k e_k^*(b) e_{\pi_2(k)}$. It is clear that $a^* = (a')^*, b^* = (b')^*$.

The proof of the following lemma is straightforward. The details are therefore omitted. It should be pointed out that the lemma which follows is a special case of Lemma 3 below.

LEMMA 1. *Let $a, b \in X, a \geq 0, b \geq 0$. Then $a' + b' \prec_w a + b$ and therefore $\|a' + b'\|_x \leq \|a + b\|_x$.*

We remark that it follows immediately from Lemma 1 that (ii) implies (i).

Let us continue the proof of Theorem 2. Assume that $X \in (\text{UH}_\Gamma)$ but that the condition of uniform monotonicity fails to hold for X . Then there exist $\epsilon > 0; (x_n), (y_n) \subset X, x_n, y_n \geq 0$, such that $\|x_n\|_x = 1, \|y_n\|_x \geq \epsilon$, and $\|x_n + y_n\|_x < 1 + n^{-1}$. Fix the integer n and divide the set N into an infinite family of disjoint subsets C_k with $\text{card } C_k = \infty, k = 0, 1, 2, \dots$. Let $\pi_k : N \rightarrow C_k$ be arbitrary bijections. Put $a = \sum_i e_i^*(x_n) e_{\pi_0(i)}, b_k = \sum_i e_i^*(y_n) e_{\pi_k(i)}$. It is clear that $(a + b_k)^* = (a + b_m)^*$ for $k, m = 1, 2, \dots$, and $\sigma(X, \Gamma) - \lim_k b_k = 0$. Using Lemma 1, we have

$$(1) \quad 1 \leq \|a + b_k\|_x \leq \|x_n + y_n\|_x < 1 + n^{-1}.$$

Therefore $\|\alpha^{-1}(a + b_k)\|_x = 1, \|\alpha^{-1}(b_k - b_m)\|_x \geq \alpha^{-1}\|b_k\|_x = \alpha^{-1}\|y_n\|_x \geq \alpha^{-1}\epsilon \geq 2^{-1}\epsilon$ and $\sigma(X, \Gamma) - \lim_k (\alpha^{-1}(a + b_k)) = \alpha^{-1}a$, where $\alpha = \|a + b_k\|_x$. Since $X \in (\text{UH}_\Gamma)$ we have $\|\alpha^{-1}a\|_x < 1 - \delta(2^{-1}\epsilon)$. Using (1), it now follows that

$$1 = \|x_n\|_x = \|a\| < (1 + n^{-1})(1 - \delta(2^{-1}\epsilon))$$

for all $n = 1, 2, \dots$. This contradicts the fact that $\delta(2^{-1}\epsilon) > 0$ and thus completes the proof of Theorem 2.

REMARK. The equivalence of (ii) and (iii) has been established also in [6].

2. (UHlm)-property for the symmetric function spaces $E(0, \infty)$

Throughout this section $E(0, \infty)$ is a separable symmetric function space. It is well known (see [11]), that every space X with a symmetric basis (e_n) is a symmetric function space on an infinite discrete measure space Ω in which the mass of every point is one. In this context $\sigma(X, \Gamma)$ -convergence coincides with convergence in measure on sets of finite measure on the unit sphere of X . The following question therefore naturally arises from Theorem 2: are the properties (UHlm) and (UMB) equivalent in $E(0, \infty)$? The theorem below gives a positive answer to this question.

THEOREM 3. *For a separable symmetric space $(E(0, \infty), \|\cdot\|)$ the following conditions are equivalent:*

- (i) $(E(0, \infty), \|\cdot\|) \in (\text{UMB})$;
- (ii) $(E(0, \infty), \|\cdot\|) \in (\text{UMBd})$;
- (iii) $(E(0, \infty), \|\cdot\|) \in (\text{UHlm})$.

PROOF. Assertion (ii) is a consequence of assertion (i). It follows from [3], Theorem 3.3, that assertion (iii) is a consequence of (ii). The implication (iii) \rightarrow (i) is proved exactly in the same way as in Theorem 2. Instead of relation $a \prec_w b$ between sequences $a = (a_n)$ and $b = (b_n)$ we consider the relation $f \prec g$ between functions $f, g \in L_1(0, \infty) + L_\infty(0, \infty)$, where $f \prec g$ means that for every $0 < s < \infty$

$$\int_0^s f^*(t) dt \leq \int_0^s g^*(t) dt.$$

Here $f^*(t)$ is the decreasing rearrangement of $|f(t)|$. Lemma 1 is reformulated for this situation in the obvious way (see also Lemma 3 and Remark 2 below). Instead of a partition of N into disjoint subsets C_k and bijections π_k we use a partition of $(0, \infty)$ into an infinite family of disjoint subsets of infinite measure and measure preserving transformations.

COROLLARY 2. *Let Φ be an Orlicz function, and let $L_\Phi(0, \infty)$ be the corresponding Orlicz space equipped with the Luxemburg norm. Then the following conditions are equivalent:*

- (1) Φ satisfies the Δ_2 -condition;

(2) $L_\Phi(0, \infty) \in (\text{UHlm})$.

PROOF. By [1] the Δ_2 condition for Φ is equivalent to (UMB) property for $L_\Phi(0, \infty)$.

The Banach lattice E is said to satisfy a *lower q -estimate* if there exists a constant $C > 0$ such that for all finite sequence (x_n) of mutually disjoint elements in E

$$\left(\sum_n \|x_n\|_E^q \right)^{1/q} \leq C \left\| \sum_n x_n \right\|_E.$$

Combining Theorem 3 and Corollary 2.11 [3] we obtain the following:

COROLLARY 3. *If E is a symmetric function space on $(0, \infty)$, then E satisfies a lower q -estimate for some $1 < q < \infty$ if and only if there is an equivalent symmetric norm $\|\cdot\|_0$ on E such that $(E, \|\cdot\|_0) \in (\text{UMB})$.*

3. $(\text{UH}\mu)$ -property for the symmetric operator spaces $E(M, \mu)$

In this section (M, μ) will be a semifinite von Neumann algebra M with a faithful semifinite normal trace μ on M . Let $K(M, \mu)$ denote the space of all μ -measurable operators affiliated with M (see [5]). $K(M, \mu)$ is the closure of M with respect to the measure topology generated by the trace μ with fundamental system of neighbourhoods around 0 given by $V(\epsilon, \delta) = \{T \in K(M, \mu); \text{there exists a projection } P \text{ in } M \text{ such that } \|TP\|_\infty \leq \epsilon \text{ and } \mu(1 - P) \leq \delta\}$ for $\epsilon, \delta > 0$. Here 1 is the unit of M and $\|\cdot\|_\infty$ is the C^* -norm on M . We shall denote by $x_n \xrightarrow{\mu} x$ the convergence of the sequence (x_n) to x in the measure topology generated by the trace μ . Let $A \in K(M, \mu)$. The t -th singular number of A $\mu_t(A)$ is

$$\mu_t(A) = \inf\{\|AP\|_\infty : P \text{ is a projection in } M \text{ with } \mu(1 - P) \leq t\}, t > 0$$

(see, for example [5, Definition 2.1]). It is known [5] that $\mu_t(A) = \mu_t(A^*) = \mu_t(|A|)$ where $|A| = (A^*A)^{1/2}$.

Let $E(0, \mu(1))$ be a separable symmetric function space. The symmetric operator space $E(M, \mu)$ is the space of operators $A \in K(M, \mu)$ such that $\mu_t(A)$ belongs to $E(0, \mu(1))$ and

$$\|A\|_{E(M, \mu)} = \|\mu(A)\|_{E(0, \mu(1))}.$$

Before formulating the main result of this section which concerns the $(\text{UH}\mu)$ property, we note that if $E(M, \mu) \in (\text{UH}\mu)$ then $E(M, \mu)$ possesses the Kadec-Klee property with respect to measure convergence (notation: $E(M, \mu) \in (\text{H}\mu)$). In

the setting of symmetric function spaces, the property (Hm) has been investigated in [13, 14]. It has been shown ([5, 7]) that the non-commutative L^p -spaces have property (H μ); subsequently, it was proved in [16] that $E(0, \mu(1)) \in (\text{Hm})$ implies $E(M, \mu) \in (\text{Hm})$. Further, it has been established in [16] that if $E(0, \mu(1))$ is an arbitrary separable symmetric space then $E(M, \mu)$ can be renormed equivalently so that $E(M, \mu)$ endowed with the new norm $\|\cdot\|'$ is a symmetric operator space and $(E(M, \mu), \|\cdot\|') \in (\text{H}\mu)$.

The main result of this section shows the uniform Kadec-Klee property with respect to convergence in measure extends from the symmetric function space $E(0, \mu(1))$ to $E(M, \mu)$.

THEOREM 4. *If $E(0, \mu(1)) \in (\text{UHm})$, then $E(M, \mu) \in (\text{UH}\mu)$.*

The proof of Theorem 4 is based mainly on the following result (cf. [9, Theorem 2.1]).

LEMMA 2. *Let $M, \mu, E(0, \mu(1)), E(M, \mu)$ be as above, and let $(x_n) \subseteq E(M, \mu)$, satisfy $x_n \xrightarrow{\mu} 0$. There exist two sequences of pairwise orthogonal projections $(p_k), (q_k) \subseteq M$ and subsequence (x_{n_k}) such that*

$$\|x_{n_k} - q_k x_{n_k} p_k\|_{E(M, \mu)} \rightarrow 0.$$

We also need the following non-commutative analog of Lemma 1 (see also proof of the implication (iii) \rightarrow (i) of Theorem 3).

LEMMA 3. *Let $a, b, c, d \in E(M, \mu)$ be positive operators such that $\mu_t(b) = \mu_t(c)$, $\mu_t(a) \leq \mu_t(d)$ for all $t \geq 0$ and $ac = 0$. Then*

$$\|a + c\|_{E(M, \mu)} \leq \|d + b\|_{E(M, \mu)}.$$

PROOF OF LEMMA 3. Let $x, y \in K(M, \mu)$. The notation $x < y$ means

$$\int_0^t \mu_\tau(x) d\tau \leq \int_0^t \mu_\tau(y) d\tau$$

for all $t > 0$. Since $E(0, \mu(1))$ is a separable symmetric space the relation $x < y$, $y \in E(M, \mu)$ implies $x \in E(M, \mu)$ and $\|x\|_{E(M, \mu)} \leq \|y\|_{E(M, \mu)}$ (see, for example [16]). So, it suffices to prove that $a + c < d + b$. Fix $t > 0$. Without loss of generality we can assume that M has no minimal projections and therefore there are projections $P_1, P_2 \in M$ such that $P_1 P_2 = 0$, $\mu(P_1 + P_2) = t$ and $\int_0^t \mu_\tau(a + c) d\tau = \mu(a P_1 + c P_2)$ (see [5, Lemma 4.1 and subsequent remarks]). Since $\mu_t(c) = \mu_t(b)$, $\mu_t(a) \leq \mu_t(d)$ we can find the projections $P_3, P_4 \in M$ such that $\mu(P_3) = \mu(P_2)$, $\mu(P_4) = \mu(P_1)$

and $\mu(bP_3) = \mu(cP_2)$, $\mu(aP_1) \leq \mu(dP_4)$. Put $P_5 = P_4 \vee P_3$. Then $\mu(P_5) \leq t$ and it follows that

$$\begin{aligned} \int_0^t \mu_\tau(d+b)d\tau &\geq \mu((d+b)P_5) \geq \mu(dP_4) + \mu(bP_3) \\ &\geq \mu(aP_1 + cP_2) = \int_0^t \mu_\tau(a+c)d\tau. \end{aligned}$$

REMARK. Lemma 1, and its continuous analog in the implication (iii) \rightarrow (i) of Theorem 3, follow from Lemma 3 as particular cases.

PROOF OF THEOREM 4. Suppose that $x_n, x \in E(M, \mu)$, $\|x_n\|_{E(M, \mu)} = 1$, $\|x_n - x_m\|_{E(M, \mu)} \geq \epsilon$ ($m \neq n$), and $x_n \xrightarrow{\mu} x$. We can assume that $x \neq 0$. Put $x_n = x + y_n$. It is clear that $y_n \xrightarrow{\mu} 0$ and $\|y_n - y_m\|_{E(M, \mu)} \geq \epsilon$. By Lemma 2 we may assume by passing to a subsequence, if necessary, that there exist $(p_n), (q_n) \subset M$ such that $p_n = p_n^* = p_n^2$, $q_n = q_n^* = q_n^2$, for all $n = 1, 2, \dots$, $p_n p_m = q_n q_m = 0$ ($n \neq m$) and $\|y_n - p_n y_n q_n\|_{E(M, \mu)} \rightarrow 0$. Put $P_n = \bigvee_{i=n}^\infty p_i$, $Q_n = \bigvee_{i=n}^\infty q_i$. It is evident that $P_n \downarrow 0$, $Q_n \downarrow 0$ and hence $P_n^\perp \uparrow 1$, $Q_n^\perp \uparrow 1$. Our first objective is to show that

$$(2) \quad \|x - P_n^\perp x Q_n^\perp\|_{E(M, \mu)} \rightarrow 0$$

for all $x \in E(M, \mu)$. Indeed, without loss of generality we can assume that $x \geq 0$. Since $x = P_n^\perp x Q_n^\perp + P_n x Q_n^\perp + P_n^\perp x Q_n + P_n x Q_n$ it is sufficient to prove that $\|P_n^\perp x Q_n\|_{E(M, \mu)}, \|P_n^\perp x Q_n\|_{E(M, \mu)}, \|P_n x Q_n\|_{E(M, \mu)} \rightarrow 0$.

By ([16, Lemma 3]), we have

$$\|P_n^\perp x Q_n\|_{E(M, \mu)} = \|P_n^\perp \sqrt{x} \sqrt{x} Q_n\|_{E(M, \mu)} \leq \|\sqrt{x} P_n^\perp \sqrt{x}\|_{E(M, \mu)}^{1/2} \|Q_n x Q_n\|_{E(M, \mu)}^{1/2}.$$

Since

$$\|\sqrt{x} P_n^\perp \sqrt{x}\|_{E(M, \mu)} = \|P_n^\perp x P_n^\perp\|_{E(M, \mu)} \leq \|x\|_{E(M, \mu)} \quad \text{and}$$

$$\|Q_n x Q_n\|_{E(M, \mu)} \rightarrow 0$$

([16, Proposition 4]) we have $\|P_n^\perp x Q_n\|_{E(M, \mu)} \rightarrow 0$. Similarly, $\|P_n x Q_n^\perp\|_{E(M, \mu)}, \|P_n x Q_n\|_{E(M, \mu)} \rightarrow 0$.

Now, using (2) and the fact that $\|y_n - p_n y_n q_n\|_{E(M, \mu)} \rightarrow 0$, we may assume by passing to a subsequence and relabelling if necessary, that

$$(3) \quad x_n = P_n^\perp x Q_n^\perp + p_n y_n q_n + z_n$$

where $\|z_n\|_{E(M, \mu)} \leq \epsilon 2^{-n}$.

Fix $\epsilon > 0$. Now let $\delta = \delta(2^{-1}\epsilon)$ be chosen as in the definition of (UHm) for $E(0, \mu(1))$, and let the integer N simultaneously satisfy the inequalities

$$(4) \quad \|x - P_N^\perp x Q_N^\perp\|_{E(M, \mu)} < \delta 2^{-N},$$

$$(5) \quad \|P_N^\perp x Q_N^\perp\|_{E(M, \mu)} \geq 2^{-1} \|x\|_{E(M, \mu)}.$$

Divide $(0, \mu(1))$ into an infinite family of disjoint subsets $(A_i)_{i=N}^\infty$ such that $m(A_N) = \mu(Q_N^\perp)$, $m(A_i) = \mu(q_i)$, $i = N+1, \dots$ and choose sequence $(f_i(t))_{i=N}^\infty \subset E(0, \mu(1))$ such that $f_i(t)\chi_{A_i}(t) = f_i(t)$ for all $i \geq N$ and $f_N^*(t) = \mu_t(P_N^\perp x Q_N^\perp)$, $f_i^*(t) = \mu_t(p_i y_i q_i)$, $i \geq N$. Notice that for $n \geq N$

$$\begin{aligned} |P_N^\perp x Q_N^\perp + p_n y_n q_n|^2 &= (Q_N^\perp x^* P_N^\perp + q_n y_n^* p_n) (P_N^\perp x Q_N^\perp + p_n y_n q_n) \\ &= |P_N^\perp x Q_N^\perp|^2 + |p_n y_n q_n|^2. \end{aligned}$$

Hence $|P_N^\perp x Q_N^\perp + p_n y_n q_n| = |P_N^\perp x Q_N^\perp| + |p_n y_n q_n|$, and therefore

$$(6) \quad \mu_t(P_N^\perp x Q_N^\perp + p_n y_n q_n) = (f_N + f_n)^*(t)$$

for all $t > 0$ and for all $n \geq N$. Using (6) it then follows that for $n \geq N$,

$$(7) \quad \|f_N + f_n\|_{E(0, \mu(1))} = \|P_N^\perp x Q_N^\perp + p_n y_n q_n\|_{E(M, \mu)}.$$

Since $P_n^\perp \geq P_N^\perp$, $Q_n^\perp \geq Q_N^\perp$ for $n \geq N$ we have $\mu_t(P_n^\perp x Q_n^\perp) = \mu_t(P_N^\perp P_n^\perp x Q_n^\perp Q_N^\perp) \leq \mu_t(P_N^\perp x Q_N^\perp)$ for all $t \geq 0$ (see [5]) and hence by Lemma 3

$$\begin{aligned} |P_N^\perp x Q_N^\perp + p_n y_n q_n| &= |P_N^\perp x Q_N^\perp| + |p_n y_n q_n| < |P_n^\perp x Q_n^\perp| + |p_n y_n q_n| \\ &= |P_n^\perp x Q_n^\perp + p_n y_n q_n|. \end{aligned}$$

It follows that $|P_N^\perp x Q_N^\perp + p_n y_n q_n|_{E(M, \mu)} \leq \|P_n^\perp x Q_n^\perp + p_n y_n q_n\|_{E(M, \mu)}$ and so, by (7) and (3)

$$\|f_N + f_n\|_{E(0, \mu(1))} \leq 1 + \epsilon 2^{-n} \text{ for } n > N.$$

Using (5) and passing to a subsequence if necessary we may assume that $\|f_N + f_n\|_{E(0, \mu(1))} \rightarrow \alpha$, where $2^{-1} \|x\|_{E(M, \mu)} \leq \alpha \leq 1$. It follows that $\|\beta_n^{-1}(f_N + f_n)\|_{E(0, \mu(1))} = 1$ and

$$\|\beta_n^{-1}(f_N + f_n) - \beta_m^{-1}(f_N + f_m)\|_{E(0, \mu(1))} \geq 2^{-1}\epsilon$$

for sufficiently large m, n , where $\beta_n = \|f_N + f_n\|_{E(0, \mu(1))}$ (the last inequality is the simple consequence of the following correlations

$$\begin{aligned} \|p_n y_n q_n - y_n\|_{E(M, \mu)} &\rightarrow 0, \quad \|p_n y_n q_n - p_m y_m q_m\|_{E(M, \mu)} = \|f_n - f_m\|_{E(0, \mu(1))}, \\ \beta_n &\rightarrow \alpha \quad \text{and} \quad \|y_n - y_m\|_{E(M, \mu)} \geq \epsilon. \end{aligned}$$

Observe that $p_n y_n q_n \xrightarrow{\mu} 0$ implies $f_n \xrightarrow{m} 0$ and therefore $\beta_n^{-1}(f_N + f_n) \xrightarrow{m} \alpha^{-1} f_N$. So, $E(0, \mu(1)) \in (\text{UHm})$ implies $\|\alpha^{-1} f_N\|_{E(0, \mu(1))} < 1 - \delta$. It follows that $\|P_N^\perp x Q_N^\perp\|_{E(M, \mu)} = \|f_N\|_{E(0, \mu(1))} \leq 1 - \delta$. Then, by (4), $\|x\| \leq 1 - \delta + \delta 2^{-N}$. This completes the proof of Theorem 4.

The following corollary extends results of [5, 7] which assert $L_p(M, \mu) \in (\text{H}\mu)$ for all $p \geq 1$.

COROLLARY 4. *If $\Phi \in \Delta_2$, then $L_\Phi(M, \mu) \in (\text{UH}\mu)$.*

The proof immediately follows from Corollary 2 and Theorem 4.

COROLLARY 5. *If $\phi(t)$, $\phi(0) = 0$, is a concave, increasing function on $(0, \mu(1))$ such that $\phi(\infty) = \infty$ if $\mu(1) = \infty$, then the Lorentz space $\Lambda_\phi(M, \mu)$ has the uniform Kadec-Klee property with respect to convergence in measure.*

PROOF. By Theorem 4 it is sufficient to prove that $\Lambda_\phi(0, \mu(1)) \in (\text{UHm})$. But this is easy follows from the results [14, 15] which assert that if $y \xrightarrow{m} 0$, $y_n \in \Lambda_\phi(0, \mu(1))$ then

$$\|x_n + y_n\|_{\Lambda_\phi(0, \mu(1))} = \|x\|_{\Lambda_\phi(0, \mu(1))} + \|y_n\|_{\Lambda_\phi(0, \mu(1))} + o(1).$$

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