

# CHARACTERIZATION OF LIPSCHITZ SPACES ON COMPACT LIE GROUPS

DASHAN FAN and ZENGFU XU

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## Abstract

Lipschitz spaces are important function spaces with relations to  $H^p$  spaces and Campanato spaces, the other two important function spaces in harmonic analysis. In this paper we give some characterizations for Lipschitz spaces on compact Lie groups, which are analogues of results in Euclidean spaces.

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## 1. Introduction and Notation

Lipschitz spaces are very important function spaces with close relations to  $H^p$  spaces and Campanato spaces, the other two important function spaces in harmonic analysis. The theory of these spaces on Euclidean spaces has been fruitfully developed by several authors (see [2, 3, 4]). In this paper we consider Lipschitz spaces on compact Lie groups and give some characterizations for these spaces, which are analogues of the results in the case of Euclidean spaces.

Let  $G$  be a compact connected semisimple Lie group of dimension  $n$ . Choose  $T \subset G$  to be a fixed maximal torus of  $G$  with  $\dim T = \text{rank } G = l$ . Let  $\mathfrak{g}$  and  $\mathfrak{t}$  denote the Lie algebras of  $G$  and  $T$  respectively, and  $\mathfrak{g}_c$  and  $\mathfrak{t}_c$  their complexification. If  $\Delta$  is the set of roots of  $(\mathfrak{g}_c, \mathfrak{t}_c)$  we choose in  $\Delta$  a system  $\Delta^+$  of positive roots. Write  $m = \text{card } \Delta^+$  and  $n = 2m + l$ . We choose inner product  $(\cdot, \cdot)$  on  $\mathfrak{g}$  which is invariant under the adjoint action of  $G$  on  $\mathfrak{g}$ . Put  $(\cdot, \cdot)^{1/2} = |\cdot|$ , so that  $|\cdot|$  is a norm on  $\mathfrak{g}$ . Let  $d$  be the geodesic metric on  $G$  associated with  $(\cdot, \cdot)$ , so that for small  $t$ ,  $d(\exp tX, e) = |t| |X|$ , for  $X \in \mathfrak{g}$ , where  $e$  is the identity of  $G$ . Let  $\hat{G}$  be the set of equivalence classes of irreducible unitary representations of  $G$  and denote by  $\chi_\lambda$  and  $d_\lambda$  the character and dimension of the unitary representation  $U_\lambda$  corresponding to  $\lambda \in \hat{G}$ .

Fix once and for all an orthonormal basis  $\{X_1, \dots, X_n\}$  of  $\mathfrak{g}$ . Set  $X^\gamma = X_1^{\gamma_1} \dots X_n^{\gamma_n}$  for each positive integral  $n$ -tuple  $\gamma = (\gamma_1, \dots, \gamma_n)$ .

In the sequel we denote by  $c$  a positive constant which may change from line to line.

### 2. Definitions and main results

Suppose  $\alpha \geq 0$ . We introduce Lipschitz spaces on  $G$ .

DEFINITION 1. Denote by  $\dot{\Lambda}_\alpha$  the homogeneous Lipschitz spaces and  $\Lambda_\alpha$  the (inhomogeneous) Lipschitz spaces.

- (i) For  $0 < \alpha < 1$ ,  $\dot{\Lambda}_\alpha = \{f : |f(x) - f(y)| \leq cd(x, y)^\alpha, x, y \in G\}$  and  $\|f\|_{\dot{\Lambda}_\alpha}$  is the infimum of all  $c$  for which the above estimate holds.
- (ii) For  $\alpha = 1$ ,  $\dot{\Lambda}_1 = \{f : \|f\|_{\dot{\Lambda}_1} < \infty\}$ , where

$$\|f\|_{\dot{\Lambda}_1} = \sup \left\{ \frac{|f(xy) - 2f(x) + f(xy^{-1})|}{d(y, e)} : x, y \in G, y \neq e \right\}.$$

- (iii) For  $\alpha > 1$ , let  $k$  be the positive integer such that  $k < \alpha \leq k + 1$ . Then  $\dot{\Lambda}_\alpha = \{f : \|f\|_{\dot{\Lambda}_\alpha} < \infty\}$  where

$$\|f\|_{\dot{\Lambda}_\alpha} = \sum_{|\gamma|=k} \|X^\gamma f\|_{\dot{\Lambda}_{\alpha-k}}.$$

- (iv) For any  $\alpha > 0$  we define  $\|f\|_{\Lambda_\alpha} = \|f\|_1 + \|f\|_{\dot{\Lambda}_\alpha}$  and  $\Lambda_\alpha = \{f \in L^1(G) : \|f\|_{\Lambda_\alpha} < \infty\}$ .

For an integer  $k \geq 0$  we define  $A_k$  and  $\tilde{A}_k$  by

$$A_k = \left\{ \varphi \in S(\mathfrak{t}) : \begin{array}{l} \varphi \text{ radial, } \text{supp } \varphi \subset \{H : |H| \leq 1\} \\ \text{and } \int_t \varphi(H)H^\gamma dH = 0 \text{ for all } |\gamma| \leq k \end{array} \right\},$$

$$\tilde{A}_k = \left\{ \varphi \in A_k : \int_0^\infty \hat{\varphi}(tH)^2 t^{-1} dt = 1, H \neq 0 \right\}$$

where  $|\gamma| = \gamma_1 + \dots + \gamma_n$  for a positive integral  $n$ -tuple  $\gamma = (\gamma_1, \dots, \gamma_n)$ . By [3] we know  $\tilde{A}_k$ , and therefore  $A_k$ , are not empty.

For  $t > 0$  let  $\varphi_t(x) = \sum_{\lambda \in \hat{G}} \hat{\varphi}(t|\lambda + \beta|)d_\lambda \chi_\lambda(x)$ , where  $\beta = \sum_{\alpha \in \Delta^+} \alpha/2$ .

DEFINITION 2. For  $0 < \alpha < 1$  and  $\varphi \in \tilde{A}_\alpha$  we set  $\|f\|_{\dot{B}_\alpha} = \sup\{(t^{-\alpha} |\varphi_t * f(x)| : x \in G, t > 0)\}$ ,  $\|f\|_{B_\alpha} = \|f\|_1 + \|f\|_{\dot{B}_\alpha}$ , and define  $\dot{B}_\alpha = \{f : \|f\|_{\dot{B}_\alpha} < \infty\}$  and  $B_\alpha = \{f : \|f\|_{B_\alpha} < \infty\}$ .

A standard argument gives

**THEOREM 1.** *For  $0 < \alpha < 1$  we have  $\dot{\Lambda}_\alpha = \dot{B}_\alpha$  and  $\Lambda_\alpha = B_\alpha$ .*

The following theorem gives a characterization of  $\Lambda_\alpha$ , for all  $\alpha > 0$ .

**THEOREM 2.** *Let  $\alpha > 0$ , and  $k \in \mathbb{Z}, k \geq [\alpha]$  where  $[\alpha]$  denotes the greatest integer part of  $\alpha$ . Then the following are equivalent.*

- (i)  $f \in \Lambda_\alpha$ .
- (ii) For all  $\varphi \in A_{2k+n}$ ,  $\sup\{t^{-\alpha} |\varphi_t * f(x)| : x \in G, t > 0\} \leq c_{\varphi, f}$ .
- (iii) There exists  $\varphi \in \dot{A}_{2k+n}$  such that  $\sup\{t^{-\alpha} |\varphi_t * f(x)| : x \in G, t > 0\} \leq c_{\varphi, f}$ .  
 Moreover, if we define  $\|f\|_{B_\alpha} = \|f\|_1 + \sup_{(x,t)} t^{-\alpha} |\varphi_t * f(x)|$  for some  $\varphi \in A_{2k+n}$  then  $\|f\|_{\Lambda_\alpha} \sim \|f\|_{B_\alpha}$ .

Let  $\exp$  be the exponential map of  $\mathfrak{g}$  to  $G$ . Then  $\exp$  is an analytic diffeomorphism on an open neighbourhood of the origin of  $\mathfrak{g}$ . Choose  $\epsilon_0$  and  $\epsilon'_0$  to be the maximal positive numbers so that  $\exp$  is such that a diffeomorphism of  $B(0, \epsilon'_0)$  onto  $B(0, \epsilon_0)$ . For a positive integer  $k$  we set

$$P_k = \left\{ p : \begin{array}{l} p(x) = q(\exp^{-1} x) \text{ for } x \in B(e, \epsilon_0), \\ q \text{ is a polynomial on } \mathfrak{g} \text{ with degree } \leq k \end{array} \right\}.$$

**DEFINITION 3.** Let  $1 \leq q < \infty, \lambda > 0$  and  $k \geq 0, k \in \mathbb{Z}$ . The Campanato spaces are defined as  $L_k^{q,\lambda}(G) = \{f : \|f\|_{L_k^{q,\lambda}} < \infty\}$  where

$$\|f\|_{L_k^{q,\lambda}} = \|f\|_1 + \sup_{x \in G, 0 < r < \epsilon_0} \inf_{p \in P_k} \left\{ |B(x, r)|^{-\lambda/n} \int_{B(x,r)} |f(y) - p(x^{-1}y)|^q dy \right\}$$

and  $|B(x, r)|$  is the Haar measure of the ball  $B(x, r) = \{y : d(y, x) < r\}$ .

We now state a theorem which gives the relationship between the Lipschitz spaces, the Hardy spaces  $H^p(G)$  and the Campanato spaces.

**THEOREM 3.** *Let  $1 \leq q < \infty, \alpha > 0, k = [\alpha]$  and  $p = n/(n + \alpha)$ . Then the following are equivalent.*

- (a)  $f \in \Lambda_\alpha$ .
- (b) There exists  $c > 0$  such that for each  $x \in G$  and  $0 < r < \epsilon_0$ , there is a  $p_k = p_{x,r,k} \in P_k$  satisfying

$$\sup_{y \in B(x,r)} |f(y) - p_k(x^{-1}y)| \leq cr^\alpha.$$

- (c)  $f \in L_k^{q,n+\alpha q}(G)$ .
- (d)  $f \in (H^p)^*$ , where  $(H^p)^*$  denotes the dual of  $H^p(G)$ .

### 3. Proof of theorems

To prove Theorem 2 we first establish a lemma.

LEMMA. *Suppose  $\alpha, k$  are as in Theorem 2. Then the following statements are equivalent:*

- (a) *For all  $\varphi \in A_{2k}, \sup_{x,t} t^{-\alpha} |\varphi_t * f(x)| \leq c_{\varphi,f}$ .*
- (b) *There exists some  $\varphi \in \tilde{A}_{2k}$  such that  $\sup_{x,t} t^{-\alpha} |\varphi_t * f(x)| \leq c_{\varphi,f}$ .*

PROOF. We need only prove (b) implies (a). Suppose that  $\varphi \in \tilde{A}_{2k}$  such that  $\sup_{x,t} t^{-\alpha} |\varphi_t * f(x)| \leq c_{\varphi,f}$ . Then  $f(x) = \int_0^\infty \varphi_t * \varphi_t * f(x) t^{-1} dt$ . For any  $\psi \in A_{2k}$  we have

$$\begin{aligned} \psi_t * f(x) &= \int_0^\infty \psi_t * \varphi_s * \varphi_s * f(x) s^{-1} ds \\ &= \int_0^t \psi_t * \varphi_s * \varphi_s * f(x) s^{-1} ds + \int_t^\infty \psi_t * \varphi_s * \varphi_s * f(x) s^{-1} ds \\ &= I_1(x, t) + I_2(x, t). \end{aligned}$$

It is easy to check that for any  $\varphi \in A_k, \varphi_t \in L^1(G), \text{supp } \varphi_t \subset B(e, t)$ , the Poisson formula gives

$$(1) \quad \varphi_t(x) = \varphi_t(\exp H) = ct^{-1} D(\exp H)^{-1} \prod_{\alpha \in \Delta_+} \frac{\partial}{\partial \alpha} \left( \varphi \left( \frac{H}{t} \right) \right),$$

where  $\exp H \in T$  is the only element conjugate to  $x$  and  $D(\exp H)$  is the Weyl function

$$D(x) = D(\exp H) = (-2i)^m \prod_{\alpha \in \Delta_+} \sin \left( \frac{i\alpha(H)}{2} \right).$$

So we can easily deduce that for all  $x \in G, t > 0, |I_1(x, t)| \leq c_{\varphi,\psi} t^\alpha$ .

For  $I_2$ , we can assume  $t < \epsilon_0$  since  $|I_2(x, t)| \leq c_{\varphi,\psi,f} \leq c_{\varphi,\psi,f} t^\alpha$  otherwise. Denote by  $\alpha_1, \dots, \alpha_m$  all the positive roots. Then

$$(2) \quad D(x) = D(\exp H) = G_k(H) + F_k(H),$$

where

$$G_k(H) = c_m \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j 2^{m+2j} \sum_{J \in A_j} \prod_{i=1}^m \frac{1}{j_i!} \alpha_i(H)^{j_i},$$

with  $J = (j_1, \dots, j_m)$  and

$$A_j = \{(j_1, \dots, j_m) : j_1 + \dots + j_m = 2j + m, j_1, \dots, j_m \text{ odd}\},$$

and where  $F_k$  has the following estimate:

$$(3) \quad F_k(H) = O(|H|^{m+2[\frac{k}{2}]+2}), \quad |H| < \epsilon_0.$$

Write

$$\begin{aligned} I_2(x, t) &= \int_t^\infty \Phi_{k,t}^{(1)} * \varphi_s * \varphi_s * f(x) s^{-1} ds + \int_t^\infty \Phi_{k,t}^{(2)} * \varphi_s * \varphi_s * f(x) s^{-1} ds \\ &= J_1(x, t) + J_2(x, t) \end{aligned}$$

where

$$(4) \quad \begin{aligned} \Phi_{k,t}^{(1)}(x) &= \Phi_{k,t}^{(1)}(\exp H) = \Psi_t(\exp H) G_k(H) D(\exp H)^{-1}, \\ \Phi_{k,t}^{(2)}(x) &= \Phi_{k,t}^{(2)}(\exp H) = \Psi_t(\exp H) F_k(H) D(\exp H)^{-1}. \end{aligned}$$

Let  $h_s(x) = \varphi_s * \varphi_s * f(x)$ . Observe that there exists  $q \in \mathbb{Z}^+$  such that  $\|\hat{f}(\lambda)\| \leq c_f |\lambda + B|^q$  and  $\hat{\varphi} \in S(t)$ . We then have

$$\|h_s\|_\infty \leq \begin{cases} c_{\varphi,f} s^\alpha & \text{if } s < \epsilon_0, \\ c_{\varphi,f} s^{-n-q} & \text{if } s > \epsilon_0. \end{cases}$$

From this and (1), (3) we have  $|J_2(x, t)| \leq c_{\varphi,\psi,f} t^\alpha$ .

Let  $Y_1, Y_2, \dots, Y_l$  be an orthonormal basis of  $\mathfrak{t}$ . Then for  $y \in B(e, \epsilon_0)$  there exist  $\tilde{y} \in G$  and  $H \in \mathfrak{t}$ ,  $|H| < \epsilon_0$  such that

$$y^{-1} = \exp(\text{Ad}(\tilde{y})H) = \exp\left(\sum_{j=1}^l H_j \text{Ad}(\tilde{y})Y_j\right).$$

Thus, by the Taylor series expansion (see [5])

$$(5) \quad h_s(xy^{-1}) = \sum_\gamma \frac{\tilde{X}(\gamma)h_s(x)}{\gamma_1! \dots \gamma_l!} H_1^{\gamma_1} H_2^{\gamma_2} \dots H_l^{\gamma_l}, \quad y \in B(e, \epsilon_0), \quad x \in G,$$

where  $\tilde{X}(\gamma)$  is the coefficient of  $H_1^{\gamma_1} H_2^{\gamma_2} \dots H_l^{\gamma_l}$  in the formal polynomial

$$\frac{\gamma_1! \dots \gamma_l!}{|\gamma|!} (H_1 \text{Ad}(\tilde{y})Y_1 + \dots + H_l \text{Ad}(\tilde{y})Y_l)^{|\gamma|}.$$

Observe that  $\text{Ad}(\tilde{y})Y_j f(x) = \sum_{i=1}^n c_{ij}(\tilde{y})X_i f(x)$ , where the  $c_{ij}(\tilde{y})$  satisfy

$$\sum_{i=1}^n |c_{ij}(\tilde{y})|^2 = 1,$$

and

$$|X^\gamma h_s(w)| \leq \begin{cases} c_{\varphi, f} s^{\alpha-|\gamma|} & \text{if } s < \epsilon_0, \\ c_{\varphi, f} s^{-(q+n)} & \text{for some } q \in \mathbb{Z}^+ \text{ otherwise.} \end{cases}$$

By (1) and (4) we then can deduce that

$$|J_1(x, t)| \leq \int_t^\infty \int_G |\Phi_{k,t}^{(1)}(y)| \sup_{|\gamma| \leq k+1, w \in B(x,t)} |X^\gamma h_s(w)| dy \leq c_{\varphi, \psi, f} t^\alpha$$

which completes the proof of the Lemma.

**PROOF OF THEOREM 2.** In view of Theorem 1 and the lemma we are reduced to proving the equivalence of (i) and (ii) for  $\alpha \geq 1$ . We begin with the implication (i) implies (ii) for  $\alpha = 1$ . Let  $f \in \Lambda_1$ . For any  $\varphi \in A_{2k+n}$  it is easy to check that  $|\varphi_t * f(x)| \leq c \|f\|_{\Lambda_1} t$  for  $t > \epsilon_0$ , and  $\varphi_t * f(x) = \int_G \varphi(y) f(xy) dy$ . By (2) and (4) we have, for  $t < \epsilon_0$ ,

$$|\varphi_t * f(x)| \leq c \|f\|_{\Lambda_1} t + \int_G (|f(xy^{-1})| + |f(xy)|) |\Phi_{2k+n,t}^{(2)}(y)| dy;$$

then we use (1) and (3) to obtain the desired estimate.

Now we prove (ii) implies (i) for  $\alpha = 1$ . We can assume  $\|f\|_{B_1} < \infty$  for  $\varphi \in \tilde{A}_{2k+n}$ . Then for any  $y \in G$  the Calderon formula gives

$$\Delta_y^2 f(x) = \int_0^{d(e,y)} \Delta_y^2 \varphi_s * \varphi_s * f(x) s^{-1} ds + \int_{d(e,y)}^\infty \Delta_y^2 \varphi_s * \varphi_s * f(x) s^{-1} ds$$

where

$$\Delta_y^2 f(x) = f(xy) - 2f(x) + f(xy^{-1}).$$

Observe that  $\|X^\gamma \varphi_t\|_1 \leq c_\varphi t^{-|\gamma|}$  for  $t < \epsilon_0$ , and that  $\|X^\gamma \varphi_t\|_1 = O(1)$  otherwise. We then have

$$\begin{aligned} |\Delta_y^2 f(x)| &\leq c_\varphi \|f\|_{B_1} d(y, e) + \int_{d(y,e)}^\infty d(y, e) \sum_{|\gamma|=2} \|X^\gamma \varphi_s * \varphi_s * f\|_\infty s^{-1} ds \\ &\leq c_\varphi \|f\|_{B_1} d(y, e) \end{aligned}$$

which shows  $\|f\|_{\Lambda_1} \leq c_\varphi \|f\|_{B_1}$ . Thus the theorem follows for  $\alpha = 1$ .

For  $\alpha > 1$  choose  $j \in \mathbb{Z}$ ,  $0 \leq j \leq k$  such that  $k - \alpha < j \leq k - \alpha + 1$  and  $\alpha - 1 \leq |\gamma| = k - j < \alpha$ . A similar argument as for the case  $\alpha = 1$  gives  $X^\gamma f \in \dot{\Lambda}_{\alpha-|\gamma|}$  if  $f \in B_\alpha$  and  $\|f\|_{\Lambda_\alpha} \leq c \|f\|_{B_\alpha}$ . All that remains now is (i) implies

(ii). Suppose  $f \in \Lambda_\alpha, \alpha > 1$ . Then  $X^\gamma f \in \dot{B}_{\alpha-|\gamma|}$  by Theorem 1. Choose  $\varphi \in \tilde{A}_2$  to obtain

$$\begin{aligned} \varphi_t * f(x) &= \int_0^t \Phi_{k+n,t}^{(1)} \varphi_s * \varphi_t * f(x) s^{-1} ds + \int_t^\infty \Phi_{k+n,t}^{(1)} \varphi_s * \varphi_s * f(x) s^{-1} ds \\ &\quad + \int_0^t \Phi_{k+n,t}^{(2)} \varphi_s * \varphi_s * f(x) s^{-1} ds + \int_t^\infty \Phi_{k+n,t}^{(2)} \varphi_s * \varphi_t * f(x) s^{-1} ds \\ &= \sum_{i=1}^4 \sigma_i(x, t), \end{aligned}$$

where  $\Phi_{k+n,t}^{(1)}, \Phi_{k+n,t}^{(2)}$  are as in (4). We may, and do, assume that  $t < \epsilon_0$ . Observe that  $\|\varphi_s\|_\infty \leq c_\varphi s^{-n}$ . Thus  $|\sigma_i(x, t)| \leq c_\varphi \|f\|_{\Lambda_\alpha} t^\alpha$  for  $i = 3, 4$ . Let  $h_1(x) = \varphi_s * \varphi_s * f(x)$  and  $h_2(x) = \varphi_s * \varphi_t * f(x)$ . Then by (1), (2) and (4)

$$|\sigma_2(x, t)| \leq ct^{k+1} \int_t^\infty \int_{B(e,t)} |\Phi_{k+n,t}^{(1)}(y)| \sup_{|v|=k+1, w \in B(x,t)} |X^v h_1(w)| dy s^{-1} ds \leq c \|f\|_{\Lambda_\alpha} t^\alpha$$

and

$$|\sigma_1(x, t)| \leq c_\varphi \int_0^t s^{|\gamma|} \sup_{|\gamma|=k-j, w \in B(x,r)} |X^\gamma h_2(w)| s^{-1} ds \leq c \|f\|_{\Lambda_\alpha} t^\alpha.$$

This proves (i) implies (ii) for  $\alpha > 1$  and completes the proof of Theorem 2.

PROOF OF THEOREM 3. The equivalence of (c) and (d) is proved in [1]. Now we prove (b) implies (c) and (d) implies (a) implies (b).

Suppose that (b) is true. Then  $f \in L^\infty(G)$  due to the compactness of  $G$ . Let  $p_k$  be the polynomial as given in (b) for a fixed  $x \in G$  and  $0 < r < \epsilon_0$ . We then have

$$|B(x, r)|^{-(n+\alpha q)/n} \int_{B(x,r)} |f(y) - p_k(x^{-1}y)|^q dy \leq c$$

which implies (c).

To prove (d) implies (a) we need to show that, for some  $\varphi \in \tilde{A}_{4k+n}, \|\varphi_t * f\|_\infty \leq ct^\alpha$  if  $f \in (H^p)^*$ . Observe that  $\|\varphi_t\|_{H^p} \leq c \sum_{\lambda \in \hat{G}} |\hat{\varphi}(t|\lambda + \beta)|^2 d_\lambda^2$ . Hence, for  $t \geq \epsilon_0, \|\varphi_t * f\|_\infty \leq c \|f\|_{(H^p)^*} t^\alpha$ .

We now assume  $t < \epsilon_0$  and write

$$\varphi_t * f(x) = \Phi_{2k+n,t}^{(1)} * f(x) + \Phi_{2k+n,t}^{(2)} * f(x)$$

where  $\Phi_{2k+n,t}^{(1)}, \Phi_{2k+n,t}^{(2)}$  are as in (4). By (3),  $\|\Phi_{2k+n,t}^{(2)} * f\|_\infty \leq c \|f\|_{(H^p)^*} t^\alpha$ . Observe that for any  $p_{2k+1} \in \mathbf{P}_{2k+1}$ ,

$$\int_G t^{-\alpha} \Phi_{2k+n,t}^{(1)}(x) p_{2k+1}(x) dx = ct^{-\alpha-1} \int_{G/T} \left\{ \int_{B(0,t)} G_{2k+n}(H)_{q_{2k+1}} \left( \sum_{j=1}^n \left( \sum_{i=1}^n c_{ij}(y) H_i \right) X_j \right) \times \right. \\ \left. \times \prod_{\alpha \in \Delta^+} \frac{\partial}{\partial \alpha} \left( \varphi \left( \frac{H}{t} \right) \right) dH \right\} dy = 0.$$

Moreover  $\text{supp} \{t^{-\alpha} \Phi_{2k+n,t}^{(1)}\} \subset B(e, t)$ , and  $\|t^{-\alpha} \Phi_{2k+n,t}^{(1)}\|_\infty \leq ct^{-n/p}$ . Thus  $t^{-\alpha} \Phi_{2k+n,t}^{(1)}$  is a  $(p, \infty)$  atom which implies  $\|\Phi_{2k+n,t}^{(1)} * f\|_\infty \leq c \|f\|_{(H^p)}^* t^\alpha$ . The implication (d) implies (a) is therefore proved.

It remains to prove (a) implies (b). Suppose first that  $\alpha$  is not an integer. Then  $k < \alpha < k + 1$ . For any  $x \in G$  and  $0 < r < \epsilon_0$ , if  $y \in B(x, r)$  then there exists a  $z_y \in B(x, r)$  such that

$$f(y) = \sum_{|\gamma| \leq k-1} X(\gamma) f(x) P_\gamma(x^{-1}y) + \sum_{|\gamma|=k} X(\gamma) f(z_y) P_\gamma(x^{-1}y)$$

where  $P_\gamma \in P_{|\gamma|}$ . Let  $p_k(x^{-1}y) = \sum_{|\gamma| \leq k} X(\gamma) f(x) P_\gamma(x^{-1}y)$  for  $y \in B(x, r)$ . Then

$$|f(y) - p_k(x^{-1}y)| \leq c \sum_{|\gamma|=k} |X(\gamma) f(x) - X(\gamma) f(z_y)| |P_\gamma(x^{-1}y)| \leq c \|f\|_{\Lambda_\alpha} r^\alpha$$

which gives  $\|f\|_{L_k^{q,\lambda}} \leq c \|f\|_{\Lambda_\alpha}$  for  $k < \alpha < k + 1$ . Now we consider the case when  $\alpha$  is an integer. Firstly we suppose  $\alpha = 1$ . Taking a radial function  $\eta \in C^\infty(\mathfrak{t})$  satisfying  $\text{supp} \eta \subset B(O, \frac{1}{2})$  and  $\int_t \eta(H) dH = 1$  we then define  $\eta_r(x) = \hat{\eta}(r|\beta|) \sum_{\lambda \in \hat{G}} \hat{\eta}(r|\lambda + \beta|) d_\lambda \chi_\lambda(x)$  and  $N_r = \eta_r * \eta_r$ . It is easy to verify that  $N_r$  has the following properties.

- (i)  $\int_G N_r(x) dx = 1$ ,
- (ii)  $N_r(x^{-1}) = N_r(x)$ ,
- (iii)  $N_r$  is central, and
- (iv)  $\int_G X^\nu N_r(x) dx = 0$  for  $|\nu| = 2$ .

Thus

$$|f(y) - N_r * f(y)| = \frac{1}{2} \left| \int_G [2f(y) - f(yz^{-1}) - f(yz)] N_r(z) dz \right| \\ \leq \frac{1}{2} \|f\|_{\Lambda_1} \int_G \int_G [d(zw^{-1}, e) + d(w, e)] |\eta_r(w) \eta_r(w^{-1}z)| dw dz \\ \leq c \|f\|_{\Lambda_1} r.$$

Given  $x \in G$  and  $0 < r < \epsilon_0$ , for any  $y \in B(x, r)$  we write

$$N_r * f(y) = \sum_{|\gamma| \leq 1} X(\gamma) (N_r * f)(x) P_\gamma(x^{-1}y) + \sum_{|\gamma|=2} X(\gamma) (N_r * f)(z_y) P_\gamma(x^{-1}y)$$

where  $z_y \in B(x, r)$  as before. Thus we are reduced to proving that, for  $f \in \Lambda_1$ ,

$$(6) \quad \sup_{w \in B(x,r), 1 \leq i, j \leq n} |X_i X_j (N_r * f)(w)| \leq cr^{-1} \|f\|_{\Lambda_1},$$

where  $c$  is a constant independent of  $x$  and  $r$ . To see this let  $\mathbf{D}(G)$  be the algebra of all left-invariant differential operators on  $G$ . We recall that  $\mathfrak{g}$  can be regarded as a Lie subalgebra of  $\mathbf{D}(G)$  with  $[X, Y] = XY - YX$ , where  $[X, Y]$  is the bracket of  $X$  and  $Y$ . Then we have

$$X_i X_j N_r (y^{-1}) = X_i X_j N_r (y) + \sum_{k=1}^n c_k^{ij} X_k N_r (y)$$

and

$$(7) \quad \begin{aligned} X_i X_j N_r * f(y) &= \frac{1}{2} \int_G X_i X_j N_r (y) (f(xy) + f(xy^{-1}) - 2f(x)) \\ &\quad - \sum_{s=1}^n c_s^{ij} X_s N_r * f(y). \end{aligned}$$

Observe that  $\Lambda_1 \subset \Lambda_{1/2}$  and we have

$$|X_s N_r * f(y)| \leq \|f\|_{\Lambda_1} \int_G |X_s N_r (yz^{-1})| d(y, z)^{1/2} dz \leq c \|f\|_{\Lambda_1} r^{-1}.$$

Thus (6) follows from (8).

Now we consider the case when  $\alpha = k \in \mathbb{Z}^+$  and  $k \geq 2$ . Suppose  $f \in \Lambda_k$ . Then  $f \in L^1$  and

$$(8) \quad \|X^\gamma (f - N_r * f)\|_\infty \leq c \|X^\gamma f\|_{\Lambda_1} r = c \|f\|_{\Lambda_\alpha} r$$

for all  $\gamma, |\gamma| = k - 1$ . On the other hand, Taylor’s formula gives

$$f(y) - N_r * f(y) = q_{k-2}(x^{-1}y) + \sum_{|\gamma|=k-1} X(\gamma)(f - N_r * f)(z_y) P_\gamma(x^{-1}y)$$

for each  $y \in B(x, r)$  and some  $z_y \in B(x, r)$ , where  $q_{k-2} \in \mathbf{P}_{k-2}$  and  $P_\gamma \in \mathbf{P}_{|\gamma|}$ . By (8),

$$(9) \quad |f(y) - N_r * f(y) - q_{k-2}(x^{-1}y)| \leq c \|f\|_{\Lambda_\alpha} r^\alpha.$$

For each  $y \in B(x, r)$ , there is also a  $z'_y \in B(x, r)$  such that

$$N_r * f(y) = \tilde{q}_k(x^{-1}y) + \sum_{|\gamma|=k+1} X(\gamma)(N_r * f)(z'_y) P_\gamma(x^{-1}y)$$

where

$$\tilde{q}_k(x^{-1}y) = \sum_{|\gamma| \leq k} X(\gamma)(N_r * f)(x) P_\gamma(x^{-1}y).$$

Write  $|\gamma| = |\gamma_1| + |\gamma_2|$  where  $|\gamma_1| = 2$ . Then

$$|N_r * f(y) - \tilde{q}_k(x^{-1}y)| \leq c \sum_{|\gamma_1|=2, |\gamma_2|=k-1} \|X^{\gamma_1} N_r * X^{\gamma_2} f\|_\infty r^{\alpha+1} \leq c \|f\|_{\Lambda_\alpha} r^\alpha$$

which together with (9) gives the desired result. The theorem is therefore proved.

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Department of Mathematical Sciences  
University of Wisconsin  
PO Box 413  
Milwaukee, WI 53201 USA

Mathematics - MPS  
Murdoch University  
Murdoch, WA 6150  
Australia