

INVARIANT SUBSPACES IN THE BIDISC AND COMMUTATORS

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Abstract

Let M be an invariant subspace of $L^2(T^2)$ on the bidisc. V_1 and V_2 denote the multiplication operators on M by coordinate functions z and w , respectively. In this paper we study the relation between M and the commutator of V_1 and V_2^* . For example, M is studied when the commutator is self-adjoint or of finite rank.

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1. Introduction

We let T^2 be the torus that is the cartesian product of 2 unit circles in \mathbb{C} . The usual Lebesgue spaces, with respect to the Haar measure m of T^2 , are denoted by $L^p = L^p(T^2)$, and $H^p = H^p(T^2)$ is the space of all f in L^p whose Fourier coefficients

$$\hat{f}(j, \ell) = \int_{T^2} f(z, w) \bar{z}^j \bar{w}^\ell dm(z, w)$$

are 0 as soon as at least one component of (j, ℓ) is negative.

A closed subspace M of L^2 is said to be invariant if

$$zM \subset M \quad \text{and} \quad wM \subset M.$$

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One can ask for a classification or an explicit description (in some sense) of all invariant subspaces of L^2 , but this seems out of reach. In a previous paper [8], the author studied the relation between M and the structure of $M \ominus wM$. An important role was played by invariant subspaces of the form FN , where F is unimodular and $H^2 \subseteq N \subseteq$ the closure of $\bigcup_{n=0}^{\infty} \bar{z}^n H^2$.

Such invariant subspaces are related to those invariant subspaces studied previously in [2, 4, 3, 1]. However the condition on $M \ominus wM$ in [8] is a little unnatural. In this paper we will find natural conditions on M which imply that M is of the form FN .

Given an invariant subspace M of L^2 , V_1 and V_2 denote the restriction of multiplications by z and w on M , respectively. Put

$$A_n = V_1^n V_2^* - V_2^* V_1^n \quad (n \geq 1)$$

and write $A = A_1$. In this paper we will describe invariant subspaces of L^2 when $A = 0$ or $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\subseteq wM$. Mandrekar [6] described M when $A = 0$ and M is in H^2 . In fact, Theorem 2 in [6] is a corollary of (2) of Theorem 5 in [8] that was proved independently. In Section 2 we study invariant subspaces under several hypotheses on the restriction of V_1 to the kernel of V_2^* . All but one are previously known results [8]. In Section 3, A_n ($n \geq 1$) is studied using results of Section 2. In Section 4 invariant subspaces are studied when A is of finite rank. In Section 5 we try to prove that if A is selfadjoint then $A = 0$. In Section 6 we give several examples to which the results of the previous sections can be applied.

We define several subspaces of L^2 which will be used later. Let $C(T^2)$ be the spaces of complex-valued continuous functions.

- (i) \mathcal{H}_1 or \mathcal{H}_2 is the set of f (in L^2) with Fourier series:

$$\sum_{\substack{j \geq 0 \\ k = 0}} a_{jk} z^j w^k \quad \text{or} \quad \sum_{\substack{k \geq 0 \\ j = 0}} a_{jk} z^j w^k,$$

respectively. Put $\mathcal{A}_j = \mathcal{H}_j \cap C(T^2)$ for $j = 1, 2$.

- (ii) \mathcal{L}_1 or \mathcal{L}_2 is the set of f (in L^2) with Fourier series:

$$\sum_{k=0} a_{jk} z^j w^k \quad (\text{no restriction on } j)$$

or

$$\sum_{j=0} a_{jk} z^j w^k \quad (\text{no restriction on } k),$$

respectively. Put $\mathcal{C}_j = \mathcal{L}_j \cap C(T^2)$ for $j = 1, 2$.

(iii) \mathbb{H}_1 or \mathbb{H}_2 is the set of f (in \mathcal{L}^2) with Fourier series:

$$\sum_{k \geq 0} a_{jk} z^j w^k \quad (\text{no restriction on } j)$$

or

$$\sum_{j \geq 0} a_{jk} z^j w^k \quad (\text{no restriction on } k)$$

respectively. Put $\mathcal{B}_j = \mathbb{H}_j \cap C(T^2)$ for $j = 1, 2$.

2. The restriction of V_1 to $\text{Ker } V_2^*$

Let M be an invariant subspace of L^2 . Put

$$S_1 = M \ominus wM \quad \text{and} \quad S_2 = M \ominus zM.$$

$S_1 = \text{Ker } V_2^*$ and $S_2 = \text{Ker } V_1^*$. In this section we derive a new result and results of the previous paper [8], which will be used in this paper.

PROPOSITION 1. *Let M be an invariant subspace of L^2 . V_2^* is a one-to-one operator if and only if $M = \chi_{E_1} F \mathbb{H}_2 + \chi_{E_2} L^2$ where F is unimodular, and χ_{E_j} is a characteristic function of Borel set E_j on T^2 , $\chi_{E_1} \in \mathcal{L}_2$ and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e.*

The proof is in [3] and [7, page 164-165] since V_2^* is one-to-one if and only if $S_1 = \{0\}$.

PROPOSITION 2. *Let M be an invariant subspace of L^2 in which V_2^* has a nontrivial kernel.*

(1) $V_1(\text{Ker } V_2^*) = \text{Ker } V_2^*$ if and only if $M = \chi_{E_1} F \mathbb{H}_1 + \chi_{E_2} L^2$ where F is unimodular, χ_{E_1} is a nonzero function in \mathcal{L}_1 and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e.

(2) $V_1(\text{Ker } V_2^*) \subsetneq \text{Ker } V_2^*$ if and only if $M = FH^2$ for some unimodular function F .

PROOF. Theorem 5 in [8] shows that $zS_1 = S_1$ (or $zS_1 \subsetneq S_1$) if and only if M has the form: $M = \chi_{E_1} F \mathbb{H}_1 + \chi_{E_2} L^2$ in (1) (or the form: $M = FH^2$ in (2), respectively). This implies the proposition because $\text{Ker } V_2^* = S_1$.

By Propositions 1 and 2, we are interested in an invariant subspace such that $\text{Ker } V_2^*$ is not invariant under V_1 .

PROPOSITION 3. *Let M be an invariant subspace of L^2 in which V_2^* has a nontrivial kernel.*

(1) *There exists a nonzero function f in $\text{Ker } V_2^*$ such that $V_1^n f$ belongs to $\text{Ker } V_2^*$ for any integer n ($V_1^n = V_1^{*(-n)}$ when $n < 0$) if and only if $M = \chi_{E_1} F \mathbb{H}_1 + \chi_{E_2} L^2$ where F is unimodular, and χ_{E_1} is a non-zero function in \mathcal{L}_1 and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e.*

(2) *There exists a function f in $\text{Ker } V_2^*$ such that $V_1^n f$ belongs to $\text{Ker } V_2^*$ for any $n \geq 0$ and $V_1^\ell f$ is not in $\text{Ker } V_2^*$ for some $\ell < 0$ if and only if $M = FN$ where N is an invariant subspace which contains H^2 and is contained properly in \mathbb{H}_1 , and F is unimodular.*

PROOF. Part (1), under the hypothesis that $|f| > 0$ a.e., and (2), were proved in [8, Theorem 6]. We will prove (1) in general. Put $M_1 = \bigcap_{n \geq 0} w^n M$ then $M = (\sum_{n \geq 0} \oplus w^n S_1) \oplus M_1$. Let D be the largest closed subspace of S_1 with $zD \subseteq D$. If we let $D_3 = S_1 \ominus D$, $D_2 = \bigcap_{n \geq 0} z^n D$ and $D_1 = D \ominus D_2$ then

$$M = \left(\sum_{n \geq 0} \oplus w^n D_1 \right) \oplus \left(\sum_{n \geq 0} \oplus w^n D_2 \right) \oplus \left(\sum_{n \geq 0} \oplus w^n D_3 \right) \oplus M_1.$$

Since $zD_2 = D_2$, by [3] and [7, pp. 164-165]

$$\left(\sum_{n \geq 0} \oplus D_2 w^n \right) \oplus M_1 = \chi_{E_1} F_1 \mathbb{H}_1 \oplus \chi_{E_2} F_2 \mathbb{H}_2 \oplus \chi_{E_3} L^2$$

where $\chi_{E_j} \in \mathcal{L}_j$ ($j = 1, 2$), $\chi_{E_1} + \chi_{E_3} \leq 1$ a.e. and $\chi_{E_2} + \chi_{E_3} \geq 1$ a.e. If there exists a nonzero function f in $\text{Ker } V_2^*$ such that $z^n f$ belongs to $\text{Ker } V_2^*$ for any integer n , then χ_{E_1} is nonzero. Since $\chi_{E_1}(\bar{F}_1 M \ominus \mathbb{H}_1)$ is invariant under multiplication of w , $\chi_{E_1}(\bar{F}_1 M \ominus \mathbb{H}_1) = \{0\}$ and hence $\chi_{E_1} M = \chi_{E_1} F_1 \mathbb{H}_1$. Since $\chi_{E_1} M \subset M$, $(1 - \chi_{E_1})M \subset M$ and $M = \chi_{E_1} M \oplus (1 - \chi_{E_1})M$. We can prove $z(1 - \chi_{E_1})M = (1 - \chi_{E_1})M$. For $z\mathcal{A}_1(1 - \chi_{E_1})M \subset (1 - \chi_{E_1})M$ and

$$[z\mathcal{A}_1(1 - \chi_{E_1})] = (1 - \chi_{E_1})\mathcal{L}_1 \ni \bar{z}(1 - \chi_{E_1}).$$

Therefore $\bar{z}(1 - \chi_{E_1})M \subset (1 - \chi_{E_1})M$ and hence $z(1 - \chi_{E_1})M = (1 - \chi_{E_1})M$. Hence

$$(1 - \chi_{E_1})M = \chi_{E_4} F_4 \mathbb{H}_1 + \chi_{E_3} L^2$$

where $\chi_{E_4} \in \mathcal{L}_1$, $\chi_{E_4} + \chi_{E_3} \leq 1$ a.e. and F_4 is unimodular. Thus M has the form $\chi_{E'} F \mathbb{H}_1 + \chi_{E''} L^2$ for some unimodular F where $\chi_{E'} = \chi_{E_1} + \chi_{E_4}$ and $\chi_{E''} = \chi_{E_3}$.

3. Commutator of V_1^n and V_2^*

Put $A_n = V_1^n V_2^* - V_2^* V_1^n$ ($n \geq 1$) and write $A = A_1$. The following trivial lemmas are important.

LEMMA 1. $A_n V_2 = 0$ for any $n \geq 1$ and hence $\bigcap_{n=1}^{\infty} \text{Ker } A_n \supseteq wM$.

LEMMA 2. For any f in $\text{Ker } V_2^*$, $f \in \text{Ker } A_n$ if and only if $z^n f \in \text{Ker } V_2^*$.

PROOF. When $f \in \text{Ker } V_2^*$, if $f \in \text{Ker } A_n$ then $V_2^* V_1^n f = 0$ and hence $z^n f \in \text{Ker } V_2^*$. Conversely if $z^n f \in \text{Ker } V_2^*$ then $A_n f = V_1^n V_2^* f = 0$.

In general A_n is a nonzero operator. The structure of M is simple when $A = 0$ or $\bigcap_{n=1}^{\infty} \text{Ker } A_n \supsetneq wM$. The following theorems, which make this precise, are corollaries of [8, Theorem 5].

THEOREM 4. Let M be an invariant subspace of L^2 with $A = 0$. Then one and only one of the following occurs.

(1) $M = \chi_{E_1} F \mathbb{H}_1 + \chi_{E_2} L^2$ where χ_{E_1} is in \mathcal{L}_1 , $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e. and F is unimodular.

(2) $M = \chi_{E_1} F \mathbb{H}_2 + \chi_{E_2} L^2$ where χ_{E_1} is in \mathcal{L}_2 , $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e. and F is unimodular.

(3) $M = FH^2$ for some unimodular function.

Conversely, if (1), (2) or (3) holds for an invariant subspace M of L^2 , then $A = 0$.

PROOF. If M has the form (1) then $S_2 = \{0\}$ and hence $A^* = 0$ because $A^* V_1 = 0$. Therefore $A = 0$. If M has the form (2) then $S_1 = \{0\}$ and hence $A = 0$ because $AV_2 = 0$ by Lemma 1. If M has the form (3) then $zS_1 \subseteq S_1$ and hence $V_1 \text{Ker } V_2^* \subseteq \text{Ker } V_2^*$. Therefore $A = 0$ on $\text{Ker } V_2^*$ and $A = 0$ because $AV_2 = 0$. Conversely suppose $A = 0$. Then $zS_1 \subseteq S_1$. If $S_1 = 0$, then by Proposition 1, M has the form (2). If $S_1 \neq 0$, then (since $zS_1 \subseteq S_1$) Proposition 2 implies that M has either the form (1) or the form (3).

Mandrekar [6] considered Theorem 4 when M is in H^2 . Then since $\bigcap_{n=1}^{\infty} z^n H^2 = \bigcap_{n=1}^{\infty} w^n H^2 = \{0\}$, M has the form (3). Now we wish to consider invariant subspaces with $A \neq 0$.

THEOREM 5. *Let M be an invariant subspace of L^2 such that $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$. Then one and only one of the following holds.*

(1) $M = \chi_{E_1} F \mathbb{H}_1 + \chi_{E_2} L^2$ where χ_{E_1} is a nonzero function in \mathcal{L}_1 , $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e. and F is unimodular.

(2) $M = FN$ where N is an invariant subspace which contains H^2 and is contained properly in \mathbb{H}_1 , and F is unimodular.

Conversely, if (1), (2) or (3) holds for an invariant subspace M of L^2 , then $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$.

PROOF. Suppose $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$. Then there exists $f \neq 0$ in $(\bigcap_{n=1}^{\infty} \text{Ker } A_n) \ominus wM$. In particular, $f \in \text{Ker } V_2^*$. By Lemma 2, $z^n f \in \text{Ker } V_2^*$ for $n \geq 0$, so by Proposition 3, M is of either the form (1) or the form (2). Conversely if M has the form (1), then $A = 0$ by Theorem 4. Then V_2^* commutes with V_1 , so it commutes with every power of V_1 ; hence $A_n = 0$ for $n \geq 1$. Thus $\bigcap_{n=1}^{\infty} \text{Ker } A_n = M$. Since χ_{E_1} is a nonzero function, $M \neq wM$ because $M \ominus wM = \chi_{E_1} F \mathcal{L}_1$. Therefore $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$. If M has the form (2) by (2) of Proposition 3 there exists a function f in $\text{Ker } V_2^*$ such that $z^n f \in \text{Ker } V_2^*$ for any $n \geq 0$. By Lemma 2, $f \in \text{Ker } A_n$ while f is orthogonal to wM because $f \in \text{Ker } V_2^*$. Therefore $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$.

4. Finite rank commutators

Theorem 4 describes those invariant subspaces with $A = 0$. Now we are interested in invariant subspaces in which A has finite rank. The following lemma was pointed out to the author by Professor K. Takahashi. It implies that if $A_n = 0$, then $A = 0$.

LEMMA 3. $V_1^* A_n = A_{n-1}$ for $n > 1$ and hence $\text{Ker } A_n \subset \text{Ker } A_{n-1}$.

The proof is clear.

PROPOSITION 6. *Let M be an invariant subspace of L^2 .*

(1) *If $\dim \text{Ker } V_2^*$ is finite then A_n is finite rank r_n , $\sup_n r_n < \infty$, and $\bigcap_{n=1}^{\infty} \text{Ker } A_n = wM$.*

(2) *Suppose $\dim \text{Ker } V_2^*$ is infinite. If A_n is finite rank r_n and $\sup_n r_n < \infty$ then $\bigcap_{n=1}^{\infty} \text{Ker } A_n \not\supseteq wM$.*

PROOF. (1) By Lemma 1, A_n is finite rank r_n and $r_n \leq \dim \operatorname{Ker} V_2^*$. Using Lemma 2, if $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$, then there exists a nonzero function f such that $f \in M \ominus wM$ and $z^n f \in M \ominus wM$ for any $n \geq 0$, and this implies that $\dim \operatorname{Ker} V_2^*$ is infinite. (2) By Lemma 3 and hypothesis, $r_n \leq r_{n+1}$, so ultimately, r_n is constant. Also $r_n = \dim(\operatorname{Ker} A_n)^\perp$, while $(\operatorname{Ker} A_{n+1})^\perp$ contains $(\operatorname{Ker} A_n)^\perp$, so ultimately $(\operatorname{Ker} A_n)^\perp$ does not change with n . But then neither does $\operatorname{Ker} A_n$. In other words, $\operatorname{Ker} A_n = \operatorname{Ker} A_{n_0}$ if $n \geq n_0$, and hence $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n = \operatorname{Ker} A_{n_0}$, for some $n_0 \geq 1$. Therefore if $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n = wM$ then $\operatorname{Ker} A_{n_0} = wM$ and hence $(\operatorname{Ker} A_{n_0})^\perp = \operatorname{Ker} V_2^*$. Since $\dim \operatorname{Ker} V_2^*$ is infinite, this contradicts the hypothesis that A_{n_0} is of finite rank, and hence $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$.

COROLLARY 1. *Let M be an invariant subspace of H^2 . If A_n has finite rank r_n and $\sup_n r_n < \infty$ then $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$.*

PROOF. If $\dim \operatorname{Ker} V_2^*$ is finite, by [8, Theorem 3] there exists a nonzero function g in L^∞ such that $gM \subset M$ and $g \notin H^\infty$. By [1, Proposition 3] this implies that $M \not\subset H^2$, so $\dim \operatorname{Ker} V_2^*$ is infinite. Part (2) of Proposition 6 implies the corollary.

COROLLARY 2. *Let M be an invariant subspace with $\bigcap_{n=1}^{\infty} z^n M = \{0\}$. Then the following are equivalent:*

- (1) $\dim \operatorname{Ker} V_2^* = \infty$, A_n is finite rank r_n and $\sup_n r_n = r < \infty$;
- (2) $M = FN$ for some unimodular F and some invariant subspace with $N = K \oplus H^2 \subsetneq \mathbb{H}_1$. Moreover $N \ominus wN = (K \ominus wK) \oplus \mathcal{H}_1$ and if S is the largest closed subspace of finite codimension r of $N \ominus wN$ such that $zS \subset S$, then $S \supseteq \mathcal{H}_1$.

PROOF. (1) implies (2). By (2) of Proposition 6, $\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n \supsetneq wM$ and hence by hypothesis and Theorem 5, M has the form FN . If we put $S' = [\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n] \ominus wM$, then by Lemma 2, S' is the largest closed subspace of $S_1 = M \ominus wM$ with $zS' \subset S'$, and by hypothesis $\dim(S_1 \ominus S') = r < \infty$. Put $S = \bar{F}S'$; then S is the desired subspace and (2) follows.

We prove (2) implies (1). By hypothesis of (2), FS is the largest closed subspace of S_1 with $zFS \subset FS$ and hence $FS = [\bigcap_{n=1}^{\infty} \operatorname{Ker} A_n] \ominus wM$. This implies (1).

5. Selfadjoint commutators

In this section we will study the following conjecture: if $A = A^*$ then $A = 0$. Unfortunately we have not been able to resolve it. However we will give some partial answers.

LEMMA 4. $A_n = \sum_{j=0}^{n-1} V_1^{n-1-j} A V_1^j$. If $A = A^*$ then $A_n = V_1^{n-1} A$ and hence $\text{Ker } A_n = \text{Ker } A$.

PROOF. For any $n > 1$

$$\begin{aligned} A_n &= V_1^{n-1}(V_1 V_2^*) - (V_2^* V_1^{n-1}) V_1 \\ &= V_1^{n-1}(V_1 V_2^*) - (V_1^{n-1} V_2^* - A_{n-1}) V_1 \\ &= V_1^{n-1} A + A_{n-1} V_1 \end{aligned}$$

and hence $A_n = \sum_{j=0}^{n-1} V_1^{n-1-j} A V_1^j$. If $A = A^*$ then $A V_1 = 0$ and hence $A_n = V_1^{n-1} A$.

PROPOSITION 7. Suppose M is an invariant subspace with $A = A^*$. Then

- (1) $A_n^2 = 0$ for any $n > 1$;
- (2) if A has finite rank r , then A_n is also of finite rank r for any $n > 1$;
- (3) if $\text{Ker } V_1^* \cap \text{Ker } V_2^* = \{0\}$, then $A = 0$;
- (4) if $\text{Ker } V_1^* \cap \text{Ker } V_2^* \neq \{0\}$, then $\bigcap_{n=1}^{\infty} \text{Ker } A_n \supsetneq wM$.

PROOF. (1) If $A = A^*$ then $A_n^2 = V_1^{n-1} A V_1^{n-1} A = 0$ for $n > 1$ by Lemma 4 because $A^* V_1 = 0$. Part (2) is clear by Lemma 4. Since $A V_1 = A V_2 = 0$, $A = 0$ on $[zM + wM]$ (the closed linear span of $zM + wM$). If $\text{Ker } V_1^* \cap \text{Ker } V_2^* = \{0\}$, then $M = [zM + wM]$ and hence $A = 0$. Suppose $\text{Ker } V_1^* \cap \text{Ker } V_2^* \neq \{0\}$. If $zM \not\subseteq wM$ then $[zM + wM] \supsetneq wM$ and hence $\text{Ker } A \supsetneq wM$. By Lemma 4, $\bigcap_{n=1}^{\infty} A_n \supsetneq wM$. The case $zM \subseteq wM$ does not occur (as pointed out to me privately by Professor K. Takahashi). For if $f \in \text{Ker } V_1^* \cap \text{Ker } V_2^*$, then $A = A^*$ implies $V_2^* V_1 f = V_1^* V_2 f$. If $zM \subseteq wM$, then $V_2^* V_1 f = \bar{w} z f$, and hence

$$\|f\| = \|V_2^* V_1 f\| = \|V_1^* V_2 f\|.$$

Thus $V_1^* V_2 f = \bar{w} z f$, and so $\bar{w} z f = \bar{w} z f$; hence $f = 0$. (Since otherwise $z^2 = w^2$ in a set of positive measure.)

We do not know whether $A = A^*$ implies $A = 0$. However, there exist many invariant subspaces such that A is unitarily equivalent to A^* and $A \neq 0$ (see Example 2). Put $Uf(z, w) = f(w, z)$ for any f in L^2 . Then U is a unitary operator on L^2 and U^2 is the identity operator I on L^2 . Let M be an invariant subspace which is invariant under U . Then U is an isometry on M and $U^2 = I$ on M . Hence U can be assumed to be a unitary operator on M .

PROPOSITION 8. *Let M be an invariant subspace of L^2 which is invariant under U . Then $V_2U = UV_1$ and $UA^*U = A$.*

6. Examples

In the previous sections, invariant subspaces M , satisfying $\bigcap_{n=0}^{\infty} \text{Ker } A_n \supsetneq wM$, were important. In this section, we will give several examples of such invariant subspaces.

EXAMPLE 1. Suppose M is a non-trivial invariant subspace in H^2 . Let R be the orthogonal projection in L^2 with range $H^2 \ominus M$, and let the operator J_z on $H^2 \ominus M$ be defined by $J_z f = R(zf)$. If J_z is of finite rank n then there exists an analytic polynomial p of z of degree n such that $p(S_z) = 0$, and hence $p(z)H^2 \subset M$. The inner part of $p(z)$ is a finite Blaschke product $F = F(z)$ of degree m , and $m \neq 0$ because $M \neq H^2$. Since $\bar{F}H^2$ is in \mathbb{H}_1 , $N = \bar{F}M$ lies between H^2 and \mathbb{H}_1 . Then $M = FN$, $N = K \oplus H^2$ and $\dim K \ominus wK \leq m$. For

$$K \subset \sum_{j=0}^{\infty} \oplus (\bar{F}\mathcal{H}_1 \ominus \mathcal{H}_1)w^j$$

and hence $\dim K \ominus wK \leq \dim(\bar{F}\mathcal{H}_1 \ominus \mathcal{H}_1)$. By Theorem 5, $\bigcap_{n=1}^{\infty} \text{Ker } A_n \supsetneq wM$ and by Corollary 2, A_n is of finite rank r_n , and $\sup_n r_n \leq m$. Since $\dim(H^2 \ominus M) < \infty$ and $\dim(H^2 \ominus FH^2) = \infty$, $M \neq FH^2$ and hence $\dim K \ominus wK \neq 0$. By Corollary 2, $0 < \sup_n r_n \leq m$. For if $\sup_n r_n = 0$ then $(K \ominus wK) \oplus \mathcal{H}_1$ is an invariant subspace under the multiplication of z and $(K \ominus wK) \oplus \mathcal{H}_1 \subseteq \bar{F}\mathcal{H}_1$. By Beurling's theorem (cf [4, page 4])

$$(K \ominus wK) \oplus \mathcal{H}_1 = \bar{q}\mathcal{H}_1$$

and q is a nonconstant finite Blaschke product of degree $\leq m$ because $K \ominus wK \neq \{0\}$. Therefore $N = \bar{q}H^2$. If $q \neq F$ then $M = GH^2$ and $m > \ell$ (the degree of

G) where $G = F\bar{q}$. Hence J_z is of finite rank ℓ . This contradiction implies that $0 \neq \sup_n r_n$. By Theorem 4, M does not have the form qH^2 for any unimodular q . This is given in [1, Corollary 2].

EXAMPLE 2. If M is an invariant subspace in H^2 , of finite codimension n , then by Example 1, $M = F_1N_1 = F_2N_2$ where F_1 and F_2 are finite Blaschke products of z and w , respectively, and $H^2 \subset N_j \subset \mathbb{H}_j$ ($j = 1, 2$). Then both A and A^* are finite rank of degree $m \leq n$ and $m \neq 0$. By [8, Theorem 3], $\dim \text{Ker } V_2^* = \dim \text{Ker } V_1^* = \infty$. By Example 1, M does not have the form qH^2 for any unimodular q . Put $M = [zH^2 + wH^2]$: then M is of finite codimension 1. Moreover M is invariant under U . Hence A has rank one and $UA^*U = A$.

EXAMPLE 3. Let M be an invariant subspace of L^2 . Invariant subspaces M satisfying $w^n M \supset zM$ for any $n \geq 1$, or $z^n M \supset wM$ for any $n \geq 1$, were studied in [3, 4, 7]. In general, if $wM \supset zM$, then $AV_1 = 0$, since $AV_2 = 0$ by Lemma 1, and because $V_2M \supset V_1M$. Hence by the first part of Lemma 4, $A_n = V_1^{n-1}A$. Thus $\text{Ker } A_n = \text{Ker } A$ for any $n \geq 1$. If $w^n M \supset zM$ for any $n \geq 1$, it is known (see [7]) that

$$M = q(\mathcal{H}_2 \oplus z\mathbb{H}_2)$$

or

$$M = \chi_{E_1}\mathbb{H}_2 \oplus \chi_{E_2}L^2$$

where q is unimodular, $\chi_{E_1} \in \mathcal{L}_2$ and $\chi_{E_1} + \chi_{E_2} \leq 1$ a.e. Hence if $wM \neq M$ then $M \ominus wM = \{q\}$, $\dim \text{Ker } V_2^* = 1$, $\text{Ker } A_n = wM$ and A_n is of rank 1 for any $n \geq 1$. If $z^n M \supset wM$ for any $n \geq 1$, by Proposition 6 we have $\bigcap_{n=1}^{\infty} \text{Ker } A_n \supsetneq wM$ and $M \ominus wM = q(\mathcal{H}_1 \oplus \bar{z}\mathcal{H}_1)$ for some unimodular q . In [3], the authors considered the following generalizations of the above invariant subspaces: for any fixed $\ell \geq 1$ M satisfies $w^n M \supset z^\ell M$ for any $n \geq 1$; or $z^n M \supset w^\ell M$ for any $n \geq 1$. They described completely such invariant subspaces and showed that if $zM \neq M$ or $wM \neq M$, then $M = FN$, where F is unimodular, and $\mathbb{H}_1 \supset N \supset z^\ell \mathbb{H}_1$, or $\mathbb{H}_2 \supset N \supset w^\ell \mathbb{H}_2$. Hence if $zM \neq M$ and $z^n M \supset w^\ell M$, then $\bigcap_{n=1}^{\infty} \text{Ker } A_n \supsetneq wM$.

References

- [1] O. P. Agrawal, D. N. Clark and R. G. Douglas, 'Invariant subspaces in the polydisk', *Pacific J. Math.* **121** (1986), 1–11.
- [2] P. R. Ahern and D. N. Clark, 'Invariant subspaces and analytic continuation in several variables', *J. Math. Mech.* **19** (1970), 963–969.
- [3] R. E. Curto, P. S. Muhly, T. Nakazi and T. Yamamoto, 'On superalgebras of the polydisc algebra', *Acta Sci. Math. (Szeged)* **51** (1987), 413–421.
- [4] H. Helson, 'Analyticity on compact abelian groups', in: *Algebras in Analysis - Proceedings of the instructional conference and NATO advanced study institute, Birmingham, 1973* (Academic Press, London, 1975) pp. 1–62.
- [5] K. Izuchi, 'Unitarily equivalence of invariant subspaces in the polydisk', *Pacific J. Math.* **130** (1987), 351–358.
- [6] V. Mandrekar, 'The validity of Beurling theorems in polydiscs', *Proc. Amer. Math. Soc.* **103** (1988), 145–148.
- [7] T. Nakazi, 'Invariant subspaces of weak* Dirichlet algebras', *Pacific J. Math.* **69** (1977), 151–167.
- [8] ———, 'Certain invariant subspaces of H^2 and L^2 on a bidisc', *Canad. J. Math.* **XL** (1988), 1272–1280.
- [9] W. Rudin, 'Invariant subspaces of H^2 on a torus', *J. Funct. Anal.* **61** (1985), 378–384.

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