

MULTIPLIERS FOR HARDY SPACES ON LOCALLY COMPACT VILENKIN GROUPS

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Abstract

In a recent paper the authors proved a multiplier theorem for Hardy spaces $H^p(G)$, $0 < p \leq 1$, defined on a locally compact Vilenkin group G . The assumptions on the multiplier were expressed in terms of the “norms” of certain Herz spaces $K(1/p - 1/r, r, p)$ with r restricted to $1 \leq r < \infty$ and $p < r$. In the present paper we show how this restriction on r may be weakened to $p \leq r < \infty$. Furthermore, we present two modifications of our main theorem and compare these with certain results for multipliers on $L^p(\mathbb{R}^n)$ -spaces, $1 < p < \infty$, due to Seeger and to Cowling, Fendler and Fournier. We also discuss the sharpness of some of our results.

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1. Introduction

Throughout this paper G will denote a locally compact Vilenkin group, that is to say, G is a locally compact Abelian topological group containing a strictly decreasing sequence of open compact subgroups $(G_n)_{n=-\infty}^{\infty}$ such that

- (i) $\sup\{\text{order}(G_n/G_{n+1}) : n \in \mathbb{Z}\} < \infty$,
- (ii) $\bigcup_{n=-\infty}^{\infty} G_n = G$ and $\bigcap_{n=-\infty}^{\infty} G_n = \{0\}$.

The dual group of G is denoted by Γ and for each $n \in \mathbb{Z}$ we set

$$\Gamma_n = \{\gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in G_n\}.$$

We choose Haar measures μ on G and λ on Γ such that $\mu(G_0) = \lambda(\Gamma_0) = 1$. Then $(\mu(G_n))^{-1} = \lambda(\Gamma_n) = m_n$ for each $n \in \mathbb{Z}$. It is an easy consequence of condition (i) for G that for every $\alpha > 0$ there exists a constant $C > 0$, C depending only on α , such that for every $k \in \mathbb{Z}$, both

$$(1.1) \quad \sum_{j=k}^{\infty} (m_j)^{-\alpha} \leq C(m_k)^{-\alpha},$$

and

$$(1.2) \quad \sum_{j=-\infty}^k (m_j)^{\alpha} \leq C(m_k)^{\alpha}.$$

The metric d on $G \times G$ defined by $d(x, x) = 0$ and $d(x, y) = (m_n)^{-1}$ if $x - y \in G_n \setminus G_{n+1}$ generates the original topology on G . For $x \in G$ we set $|x| = d(x, 0)$. If A is any set then χ_A will denote the characteristic function of A . Also, for each $n \in \mathbb{Z}$ we set $\Delta_n = m_n \chi_{G_n}$. It is easy to see that the Fourier transform of Δ_n is given by $(\Delta_n)^{\wedge} = \chi_{\Gamma_n}$. In [5] the definition and a brief summary of the basic properties of the spaces of test functions $\mathcal{S}(G)$ and distributions $\mathcal{S}'(G)$ are given. We now present the definition of the Herz spaces and the Hardy spaces on G .

DEFINITION 1.1. Let $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$. A measurable function $f : G \rightarrow \mathbb{C}$ belongs to the Herz space $K(\alpha, p, q)$ if

$$\|f\|_{K(\alpha, p, q)} := \left(\sum_{l=-\infty}^{\infty} ((m_l)^{-\alpha} \|f \chi_{G_l \setminus G_{l+1}}\|_p)^q \right)^{1/q} < \infty,$$

with the usual modification if $q = \infty$.

DEFINITION 1.2. Let $0 < p \leq 1$. A distribution $f \in \mathcal{S}'(G)$ belongs to the Hardy space $H^p(G)$ if the function $f^* : G \rightarrow \mathbb{C}$ defined by $f^*(x) = \sup_l |f * \Delta_l(x)|$ belongs to $L^p(G)$. We set $\|f\|_{H^p} = \|f^*\|_p$.

DEFINITION 1.3. A function $a : G \rightarrow \mathbb{C}$ is a (p, ∞) atom, $0 < p \leq 1$, if there exists a set I of the form $x + G_n$ such that (i) $\text{supp } a \subset I$, (ii) $\|a\|_{\infty} \leq (\mu(I))^{-1/p} = (m_n)^{1/p}$, and (iii) $\int_G a(x) d\mu(x) = 0$.

In [5] it was shown that the Hardy spaces $H^p(G)$ can also be characterized in the usual way in terms of (p, ∞) atoms on G .

The space of (Fourier) multipliers of $H^p(G)$ will be denoted by $\mathcal{M}(H^p)$; thus $\phi \in \mathcal{M}(H^p)$ if $\phi \in L^\infty(\Gamma)$ and if there exists a constant $C > 0$ such that for all $f \in H^p(G)$ we have $\|(\phi \hat{f})^\vee\|_{H^p} \leq C \|f\|_{H^p}$. We mention here that in order to show that a function $\phi \in L^\infty(\Gamma)$ belongs to $\mathcal{M}(H^p)$ it is sufficient to show the existence of a constant $C > 0$ such that for all $k \in \mathbb{Z}$ and every (p, ∞) atom a with $\text{supp } a \subset G_n$ for some $n \in \mathbb{Z}$ and $\|a\|_\infty \leq (m_n)^{1/p}$ we have $\|(\phi_k \hat{a})^\vee\|_{H^p} = \|(\phi_k)^\vee * a\|_{H^p} \leq C$, where $\phi_k = \phi \chi_{\Gamma_k}$; see Remark (4.2) in [5] for further details.

2. Multipliers on Hardy spaces $H^p(G)$

Throughout this section we shall use the notation $\phi_k = \phi \chi_{\Gamma_k}$ and $\phi^k = \phi \chi_{\Gamma_{k+1} \setminus \Gamma_k}$, where $\phi \in L^\infty(\Gamma)$ and $k \in \mathbb{Z}$.

Before stating the main result of the paper, Theorem 2.1, we first prove two simple lemmas.

LEMMA 2.1. *Let $0 < p \leq 1$. Let f, g be measurable functions on G such that $\text{supp } g \subset G_n$ for some $n \in \mathbb{Z}$ and both f and g are constant on the cosets of G_k in G for some $k \geq n$. Then we have for every $x \in G$, $|f * g(x)|^p \leq (m_k)^{1-p} |f|^p * |g|^p(x)$.*

PROOF. Let $\{z_\alpha + G_k\}$ denote the collection of different cosets of G_k in G_n ; thus $G_n = \bigcup_\alpha z_\alpha + G_k$. For every $x \in G$ we have

$$\begin{aligned} f * g(x) &= \sum_\alpha \int_{z_\alpha + G_k} f(x-t)g(t) d\mu(t) \\ &= \sum_\alpha f(x-z_\alpha)g(z_\alpha)(m_k)^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} |f * g(x)|^p &\leq \sum_\alpha (m_k)^{-p} |f(x-z_\alpha)|^p |g(z_\alpha)|^p \\ &= (m_k)^{1-p} \int_G |f(x-t)|^p |g(t)|^p d\mu(t) \\ &= (m_k)^{1-p} |f|^p * |g|^p(x). \end{aligned}$$

LEMMA 2.2. Let $\alpha > 0$, let $p, r > 0$ and let $(a_j)_{-\infty}^{\infty}$ be any sequence of real numbers. Consider the following conditions:

$$(2.1) \quad \sup_k (m_k)^{1-p/r} \left(\sum_{j=k}^{\infty} ((m_j)^{p/r-p} |a_j|^p)^{\alpha} \right)^{1/\alpha} < \infty,$$

$$(2.2) \quad \sup_k (m_k)^{1-p} \left(\sum_{j=k}^{\infty} (|a_j|^p)^{\alpha} \right)^{1/\alpha} < \infty,$$

$$(2.3) \quad \sup_k (m_k)^{1/p-1} |a_k| < \infty.$$

Then

- (i) For $0 < p < r$, (2.1) is equivalent to (2.3).
- (ii) For $0 < p < 1$, (2.2) is equivalent to (2.3).
- (iii) For $p = 1$ and $1 < r$, (2.2) implies (2.1) and, hence, (2.3).

PROOF. (i) Clearly, (2.1) implies (2.3) for all $p, r > 0$. Conversely, if (2.3) holds then there exists $C > 0$ so that for all $j \in \mathbb{Z}$, $|a_j|^p < C(m_j)^{p-1}$. Therefore, for $0 < p < r$,

$$(m_k)^{1-p/r} \left(\sum_{j=k}^{\infty} ((m_j)^{p/r-p} |a_j|^p)^{\alpha} \right)^{1/\alpha} \leq C(m_k)^{1-p/r} \left(\sum_{j=k}^{\infty} (m_j)^{(p/r-1)\alpha} \right)^{1/\alpha} \leq C,$$

where the last inequality follows from (1.1).

(ii) Clearly, (2.2) implies (2.3) whenever $p > 0$. If (2.3) holds we see, like in the proof of (i), that

$$(m_k)^{1-p} \left(\sum_{j=k}^{\infty} |a_j|^p \right)^{1/\alpha} \leq C(m_k)^{1-p} \left(\sum_{j=k}^{\infty} (m_j)^{(p-1)\alpha} \right)^{1/\alpha} < C,$$

because $p - 1 < 0$.

(iii) For $j \geq k$ and $1 < r$ we have $(m_j)^{1/r-1} \leq (m_k)^{1/r-1}$. Thus, assuming (2.2) with $p = 1$ we immediately obtain (2.1).

THEOREM 2.1. Let $0 < p \leq 1$ and $\phi \in L^{\infty}(\Gamma)$.

(a) If $p \leq r \leq 1$ and if

$$\sup_k (m_k)^{1-p/r} \left(\sum_{j=k}^{\infty} ((m_j)^{p/r-p} \|(\phi^j)^{\vee}\|_{K(1/p-1/r, r, p)}^p)^{2/(2-p)} \right)^{(2-p)/2} < \infty,$$

then $\phi \in \mathcal{M}(H^p)$.

(b) If $1 \leq r < \infty$ and

$$\sup_k (m_k)^{1-p} \left(\sum_{j=k}^{\infty} (\|(\phi^j)^\vee\|_{K(1/p-1/r, r, p)}^p)^{2/(2-p)} \right)^{(2-p)/2} < \infty,$$

then $\phi \in \mathcal{M}(H^p)$.

PROOF. Let a be a (p, ∞) atom with $\text{supp } a \subset G_n$ and

$$\|a\|_\infty \leq (\mu(G_n))^{-1/p}$$

for some $n \in \mathbb{Z}$. Fix $k \in \mathbb{Z}$, let $f = (\phi_k \hat{a})^\vee$ and $f^* = \sup_l |f * \Delta_l|$. Then

$$\begin{aligned} \|f\|_{H^p}^p &= \int_{G_n} (f^*(x))^p d\mu(x) + \int_{G \setminus G_n} (f^*(x))^p d\mu(x) \\ &= A + B, \quad \text{say.} \end{aligned}$$

We have

$$\begin{aligned} A &\leq \left(\int_{G_n} (f^*(x))^2 d\mu(x) \right)^{p/2} (\mu(G_n))^{1-p/2} \leq C \|f\|_2^p (m_n)^{p/2-1} \\ &\leq C \|\phi\|_\infty^p \|a\|_2^p (m_n)^{p/2-1} \leq C, \end{aligned}$$

because a is a (p, ∞) atom and $\phi \in L^\infty(\Gamma)$. To find a similar inequality for B we first observe that Kitada proved in [3] that

$$f^*(x) \leq \sum_{j=n}^{\infty} |(\phi^j)^\vee * a_j(x)|,$$

where $a_j = a * (\Delta_{j+1} - \Delta_j)$. Therefore,

$$B \leq \sum_{k=-\infty}^{n-1} \sum_{j=n}^{\infty} \int_{G_k \setminus G_{k+1}} |(\phi^j)^\vee * a_j(x)|^p d\mu(x).$$

In [3] Kitada also showed that for $x \in G_k \setminus G_{k+1}$ with $k \leq n-1$ we have

$$(\phi^j)^\vee * a_j(x) = (\phi^j)^\vee \chi_{G_k \setminus G_{k+1}} * a_j(x),$$

so that, after an application of Hölder's inequality, we obtain

$$(2.4) \quad B \leq \sum_{j=n}^{\infty} \sum_{k=-\infty}^{n-1} \left(\int_{G_k \setminus G_{k+1}} |(\phi^j)^\vee \chi_{G_k \setminus G_{k+1}} * a_j(x)|^r d\mu(x) \right)^{p/r} \cdot \mu(G_k \setminus G_{k+1})^{1-p/r}.$$

(a) Now we assume that $p \leq r \leq 1$. Since $\text{supp } \phi^j \subset \Gamma_{j+1}$ we see that $(\phi^j)^\vee \chi_{G_k \setminus G_{k+1}}$ is constant on the cosets of G_{j+1} in G whenever $k+1 \leq j+1$. Also, a_j is constant on the cosets of G_{j+1} in G and $\text{supp } a_j \subset G_n$ for $j \geq n$. Thus it follows from Lemma 2.1 that

(2.5)

$$\begin{aligned} B &\leq C \sum_{j=n}^{\infty} \sum_{k=-\infty}^{n-1} \left((m_j)^{1-r} \int_{G_k \setminus G_{k+1}} |(\phi^j)^\vee \chi_{G_k \setminus G_{k+1}}|^r * |a_j|^r(x) d\mu(x) \right)^{p/r} (m_k)^{p/r-1} \\ &\leq C \sum_{j=n}^{\infty} (m_j)^{p/r-p} \|a_j\|_r^p \sum_{k=-\infty}^{n-1} (m_k)^{p/r-1} \|(\phi^j)^\vee \chi_{G_k \setminus G_{k+1}}\|_r^p. \end{aligned}$$

Thus,

$$\begin{aligned} B &\leq C \sum_{j=n}^{\infty} (m_j)^{p/r-1} \|a_j\|_r^p \|(\phi^j)^\vee\|_{K(1/p-1/r, r, p)}^p \\ &\leq C \left(\sum_{j=n}^{\infty} \|a_j\|_r^2 \right)^{p/2} \left(\sum_{j=n}^{\infty} ((m_j)^{p/r-p} \|(\phi^j)^\vee\|_{K(1/p-1/r, r, p)}^p)^{2/(2-p)} \right)^{(2-p)/2}. \end{aligned}$$

Since $a_j = a * (\Delta_{j+1} - \Delta_j)$ implies that $\hat{a}_j = \hat{a} \chi_{\Gamma_{j+1} \setminus \Gamma_j}$, we see that

$$\begin{aligned} \sum_{j=n}^{\infty} \|a_j\|_2^2 &\leq \sum_{j=n}^{\infty} \|a_j\|_2^2 (\mu(G_n))^{2/r-1} = (m_n)^{1-2/r} \sum_{j=n}^{\infty} \|\hat{a}_j\|_2^2 \\ &\leq (m_n)^{1-2/r} \|\hat{a}\|_2^2 \leq (m_n)^{2/p-2/r}. \end{aligned}$$

Therefore, using the assumption of the theorem, we see that $B \leq C$ and we may conclude that $\phi \in \mathcal{M}(H^p)$.

(b) Next, assume that $1 < r < \infty$. Applying Young's inequality in inequality (2.4) we see that

$$B \leq C \sum_{j=n}^{\infty} \sum_{k=-\infty}^{n-1} (m_k)^{p/r-1} \|(\phi^j)^\vee \chi_{G_k \setminus G_{k+1}}\|_r^p \|a_j\|_1^p$$

$$\begin{aligned}
&\leq C \sum_{j=n}^{\infty} \|a_j\|_1^p \|(\phi^j)^\vee\|_{K(1/p-1/r, r, p)}^p \\
&\leq C \left(\sum_{j=n}^{\infty} \|a_j\|_1^2 \right)^{p/2} \left(\sum_{j=n}^{\infty} (\|(\phi^j)^\vee\|_{K(1/p-1/r, r, p)}^p)^{2/(2-p)} \right)^{(2-p)/2} \\
&\leq C(m_n)^{1-p} \left(\sum_{j=n}^{\infty} (\|(\phi^j)^\vee\|_{K(1/p-1/r, r, p)}^p)^{2/(2-p)} \right)^{(2-p)/2}.
\end{aligned}$$

Thus, by assumption, $B \leq C$ and we may again conclude that $\phi \in \mathcal{M}(H^p)$.

Our first observation is that Theorem 2.1 combined with Lemma 2.2 immediately implies Corollary 2.1 below. Corollaries 2.2 and 2.3 are simply restatements of Theorem 2.1 in case $p = 1 \leq r$ and in case $0 < p = r \leq 1$, respectively.

COROLLARY 2.1. *Let $0 < p < 1$ and $p < r < \infty$. If $\phi \in L^\infty(\Gamma)$ and if*

$$\sup_k (m_k)^{1/p-1} \|(\phi^k)^\vee\|_{K(1/p-1/r, r, p)} < \infty,$$

then $\phi \in \mathcal{M}(H^p)$.

COROLLARY 2.2. *If $\phi \in L^\infty(\Gamma)$ satisfies*

$$\sum_{j=-\infty}^{\infty} \|(\phi^j)^\vee\|_{K(1-1/r, r, 1)}^2 < \infty$$

for some $r \geq 1$, then $\phi \in \mathcal{M}(H^1)$.

COROLLARY 2.3. *Let $0 < p \leq 1$. If $\phi \in L^\infty(\Gamma)$ and if*

$$\sum_{j=-\infty}^{\infty} ((m_j)^{1/p-1} \|(\phi^j)^\vee\|_p)^{2p/(2-p)} < \infty,$$

then $\phi \in \mathcal{M}(H^p)$.

REMARK 1. Combining the techniques used in the proof of Theorem 2.1 with those used to prove Theorem (4.7) in [5] we can actually show that under the assumptions of Corollary 2.1 the function ϕ is a multiplier on the power-weighted Hardy spaces $H_\alpha^p(G)$ for α satisfying $-1 + p/r < \alpha \leq 0$. Thus, we can extend Corollary (4.8) in [5] from $0 < p < 1$ and $1 \leq r < \infty$ to $0 < p < 1$ and $p < r$.

REMARK 2. Corollary 2.1 with $0 < p < 1$ and $r = 1$ may be considered as the analogue on G of Theorem 3a in [1], in which Baernstein and Sawyer obtained a comparable result for multipliers on Hardy spaces defined on \mathbb{R}^n .

We now turn to a discussion of the sharpness of the preceding corollaries. We first consider Corollaries 2.1 and 2.2.

THEOREM 2.2. (a) *Let $0 < p < 1$ and $p < r < \infty$. There exists $\phi \in L^\infty(\Gamma)$ such that*

$$\sup_k (m_k)^{1/p-1} \|(\phi^k)^\vee\|_{K(1/p-1/r, r, q)} < \infty$$

for every $q > p$ and $\phi \notin \mathcal{M}(H^p)$.

(b) *Let $1 < r < \infty$. There exists $\phi \in L^\infty(\Gamma)$ such that*

$$(2.6) \quad \sum_{j=-\infty}^{\infty} \|(\phi^j)^\vee\|_{K(1-1/r, r, q)}^2 < \infty$$

for every $q > 1$ and $\phi \notin \mathcal{M}(H^1)$.

PROOF. For part (a) we may use the example described in the proof of Theorem (4.9) in [5]. A careful reading of this proof shows that the restriction $1 \leq r$ in that theorem can be relaxed to $p < r$. To prove (b), choose a sequence $(\varepsilon_l)_1^\infty$ with each $\varepsilon_l = \pm 1$ such that

$$\sum_{l=1}^{\infty} \frac{\varepsilon_l}{l} \left(1 - \frac{\mu(G_{-l+1})}{\mu(G_{-l})} \right)$$

converges. Define $\psi : \Gamma \rightarrow \mathbb{C}$ by

$$\psi(\gamma) = \sum_{l=1}^{\infty} \varepsilon_l l^{-1} \lambda(\Gamma_{-l})(F_{-l+1} - F_{-l})(\gamma),$$

where $F_k = (\chi_{G_k})^\wedge = (\lambda(\Gamma_k))^{-1} \chi_{\Gamma_k}$. If $\gamma \notin \Gamma_0$ then $\psi(\gamma) = 0$, whereas if $\gamma \in \Gamma_{-k+1} \setminus \Gamma_{-k}$ for some $k \geq 1$ then

$$|\psi(\gamma)| \leq |\varepsilon_k k^{-1} \lambda(\Gamma_{-k})(\lambda(\Gamma_{-k+1}))^{-1}| + \left| \sum_{l=k+1}^{\infty} \varepsilon_l l^{-1} \lambda(\Gamma_{-l}) \left((\lambda(\Gamma_{-l+1}))^{-1} - (\lambda(\Gamma_{-l}))^{-1} \right) \right|.$$

Thus, $\psi \in L^\infty(\Gamma) \cap L^1(\Gamma)$. Next, choose $\gamma_1 \in \Gamma_1 \setminus \Gamma_0$ and define $\phi : \Gamma \rightarrow \mathbb{C}$ by $\phi(\gamma) = \psi(\gamma - \gamma_1)$. Then $\phi \in L^\infty(\Gamma) \cap L^1(\Gamma)$ and $\text{supp } \phi \subset \Gamma_1 \setminus \Gamma_0$ so that $\phi^1 = \phi$ and $\phi^j = 0$ for $j \neq 1$. Furthermore,

$$\phi^\vee(x) = \sum_{l=1}^{\infty} \varepsilon_l l^{-1} \lambda(\Gamma_{-l}) (\chi_{G_{-l+1}} - \chi_{G_{-l}})(x) \gamma_1(x)$$

and for any $q > 1$ and any r with $1 \leq r < \infty$ we have

$$\begin{aligned} \|\phi^\vee\|_{K(1-1/r, r, q)}^q &= \sum_{l=1}^{\infty} ((m_{-l})^{1/r-1} l^{-1} \lambda(\Gamma_{-l}) \|\chi_{G_{-l+1}} - \chi_{G_{-l}}\|_r)^q \\ &\leq \sum_{l=1}^{\infty} l^{-q} (m_{-l})^{(1/r-1+1-1/r)q} < \infty. \end{aligned}$$

Thus (2.6) holds. Moreover,

$$\|\phi^\vee\|_1 = \sum_{l=1}^{\infty} l^{-1} \lambda(\Gamma_{-l}) (\mu(G_{-l+1}) - \mu(G_{-l})) = \infty,$$

that is $\phi^\vee \notin L^1(G)$. Next, if we define $g : G \rightarrow \mathbb{C}$ by $g(x) = \Delta_1(x) - \Delta_0(x)$ then g is a multiple of a $(1, \infty)$ atom so that $g \in H^1(G)$. Also, $\hat{g} = \chi_{r_1 \setminus r_0}$ and this implies that $(\phi \hat{g})^\vee = \phi^\vee$ with $\phi^\vee \notin H^1(G)$. This proves that $\phi \notin \mathcal{M}(H^1)$.

In the following theorem we consider the sharpness of Corollary 2.3.

THEOREM 2.3. *For every p with $0 < p \leq 1$ and every q with $2p/(2-p) < q \leq \infty$ there exists $\phi \in L^\infty(\Gamma)$ such that*

- (i) $\sum_{j=-\infty}^{\infty} ((m_j)^{1/p-1} \|(\phi^j)^\vee\|_p)^q < \infty$,
- (ii) $\phi \notin \mathcal{M}(H^p)$.

PROOF. For each $j \in \mathbb{N}$ decompose G_0 into the mutually disjoint cosets of G_j in G_0 , say,

$$G_0 = \bigcup_{i=1}^{m_j} b_{j,i} + G_j.$$

Define $g_j : G \rightarrow \mathbb{C}$ by

$$g_j(x) = \sum_{i=1}^{m_j} \left(\frac{m_{j+1}}{m_j} \chi_{b_{j,i} + G_{j+1}} - \chi_{b_{j,i} + G_j} \right)(x).$$

Clearly, $\text{supp } g_j \subset G_0$, $\int_G g_j(x) d\mu(x) = 0$ and $\|g_j\|_2 \leq P^{1/2}$, where $P = \sup_j (m_{j+1}/m_j)$. Moreover, since

$$(2.7) \quad (g_j)^\wedge(\gamma) = \sum_{i=1}^{m_j} \overline{\gamma(b_{j,i})} \frac{1}{m_j} (\chi_{\Gamma_{j+1}} - \chi_{\Gamma_j})(\gamma),$$

we see that $\text{supp } (g_j)^\wedge \subset \Gamma_{j+1} \setminus \Gamma_j$. Next, for each $n \in \mathbb{N}$ define $h_n : G \rightarrow \mathbb{C}$ by

$$h_n(x) = \sum_{j=1}^n g_j(x).$$

Then $\text{supp } h_n \subset G_0$, $\int_G h_n(x) d\mu(x) = 0$ and $\|h_n\|_2 \leq P^{1/2} n^{1/2}$. Thus h_n is a multiple of a $(p, 2)$ atom and $\|h_n\|_{H^p} \leq P^{1/2} n^{1/2}$.

We now turn to the definition of the function $\phi \in L^\infty(\Gamma)$ satisfying conditions (i) and (ii). For each $j \in \mathbb{N}$ choose an element $z_j \in G_{-j} \setminus G_{-j+1}$ and define $f_j : G \rightarrow \mathbb{C}$ by

$$f_j(x) = j^{-\alpha} (m_{j+1} \chi_{z_j + G_{j+1}} - m_j \chi_{z_j + G_j})(x),$$

where $\alpha = \frac{1}{2}((2-p)/2p + 1/q)$. Then $\|f_j\|_p \leq C j^{-\alpha} (m_j)^{1-1/p}$ and

$$(2.8) \quad (f_j)^\wedge(\gamma) = j^{-\alpha} \overline{\gamma(z_j)} (\chi_{\Gamma_{j+1}} - \chi_{\Gamma_j})(\gamma),$$

so that $\text{supp } (f_j)^\wedge \subset \Gamma_{j+1} \setminus \Gamma_j$ and $\|f_j\|_\infty \leq j^{-\alpha} \leq 1$. Define $\phi : \Gamma \rightarrow \mathbb{C}$ by

$$\phi(\gamma) = \sum_{j=1}^{\infty} (f_j)^\wedge(\gamma).$$

Clearly, $\phi \in L^\infty(\Gamma)$, $\phi^j = 0$ for $j \leq 0$ and $\phi^j = (f_j)^\wedge$ for $j \geq 1$; moreover, ϕ satisfies condition (i). Furthermore, for each $n \in \mathbb{N}$ and $x \in G$ we have

$$\begin{aligned} (\phi(h_n)^\wedge)^\vee(x) &= \left(\sum_{j=1}^{\infty} (f_j)^\wedge \sum_{j=1}^n (g_j)^\wedge \right)^\vee(x) \\ &= \sum_{j=1}^n (f_j * g_j)(x). \end{aligned}$$

Thus, $(\phi(h_n)^\wedge)^\vee \in L^1(G)$, and it follows immediately from (2.7) and (2.8) that for every $j \geq 1$,

$$(f_j * g_j)(x) = j^{-\alpha} \sum_{i=1}^{m_j} \left(\frac{m_{j+1}}{m_j} \chi_{z_j + b_{j,i} + G_{j+1}} - \chi_{z_j + b_{j,i} + G_j} \right)(x).$$

Finally, assume $\phi \in \mathcal{M}(H^p)$. Then there exists $C > 0$ so that

$$\begin{aligned} C \|h_n\|_{H^p}^p &\geq \|(\phi(h_n)^\wedge)^\vee\|_{H^p}^p \geq \|\phi(h_n)^\wedge\|_p^p \\ &\geq \sum_{j=1}^n j^{-\alpha p} \mu \left(z_j + \bigcup_{i=1}^{m_j} (b_{j,i} + G_j) \right) = \sum_{j=1}^n j^{-\alpha p} \\ &\geq C n^{1-\alpha p}, \end{aligned}$$

that is, $\|h_n\|_{H^p} \geq C n^{1/p-\alpha}$. Since $q > 2p/(2-p)$ implies $1/p - \alpha > 1/2$, we have a contradiction of the inequality $\|h_n\|_{H^p} \leq P^{1/2} n^{1/2}$. This shows that $\phi \notin \mathcal{M}(H^p)$, which completes the proof of Theorem 2.3.

REMARK 3. In Section 4 of [4] it was shown that if $\phi \in L^\infty(\Gamma)$ satisfies $\sum_{j=-\infty}^\infty \|(\phi^j)^\vee\|_1 < \infty$ then $\phi \in \mathcal{M}(H^1)$, and that there exists $\phi \in L^\infty(\Gamma)$ such that $\sup_j \|(\phi^j)^\vee\|_1 < \infty$ and $\phi \notin \mathcal{M}(H^1)$. Clearly, the case $p = 1$ of Corollary 2.3 and of Theorem 2.3 sharpen these results from [4].

In view of the fact that condition (i) in Theorem 2.3 is not sufficient to guarantee that $\phi \in \mathcal{M}(H^p)$ it is of some interest to determine what kind of additional condition would be sufficient to obtain $H^p(G)$ -multipliers. The following theorem gives one type of solution for this problem.

THEOREM 2.4. *Let $0 < p \leq 1$. Let $\phi \in L^\infty(\Gamma)$ satisfy*

$$\sum_{j=-\infty}^\infty ((m_j)^{1/p-1} \|(\phi^j)^\vee\|_p)^q < \infty$$

for some q with $2p/(2-p) \leq q \leq \infty$. Define β by $\beta = 2pq/((2-p)q - 2p)$ (if $q = \infty$ we take $\beta = 2p/(2-p)$), if $q = 2p/(2-p)$ we take $\beta = \infty$). Let $(\alpha_j)_{j=-\infty}^\infty \in l^\beta(\mathbb{Z})$ and define $\psi : \Gamma \rightarrow \mathbb{C}$ by $\psi(\gamma) = \sum_{j=-\infty}^\infty \alpha_j \phi^j(\gamma)$. Then $\psi \in \mathcal{M}(H^p)$.

PROOF. We have

$$\begin{aligned} &\sum_{j=-\infty}^\infty ((m_j)^{1/p-1} \|(\psi^j)^\vee\|_p)^{2p/(2-p)} \\ &\leq \left(\sum_{j=-\infty}^\infty ((m_j)^{1/p-1} \|(\phi^j)^\vee\|_p)^q \right)^{2p/(2-p)q} \left(\sum_{j=-\infty}^\infty |\alpha_j|^{2pq/(q(2-p)-2p)} \right)^{1-2p/(2-p)q} \\ &< \infty. \end{aligned}$$

Thus it follows from Corollary 2.4 that $\psi \in \mathcal{M}(H^p)$.

We explicitly state the most interesting case of Theorem 2.4, namely the case when $q = \infty$.

COROLLARY 2.4. *Let $0 < p \leq 1$. Let $\phi \in L^\infty(\Gamma)$ satisfy*

$$\sup_j (m_j)^{1/p-1} \|(\phi^j)^\vee\|_p < \infty$$

and let $(\alpha_j)_{-\infty}^\infty \in l^{2p/(2-p)}(\mathbb{Z})$. If $\psi = \sum_{-\infty}^\infty \alpha_j \phi^j$ then $\psi \in \mathcal{M}(H^p)$.

REMARK 4. Corollary 2.4 is an extension to H^p -spaces, $0 < p \leq 1$, (and on locally compact Vilenkin groups instead of on \mathbb{R}^n) of Theorem 2 in [2], in which a similar result was obtained for multipliers on $L^p(\mathbb{R}^n)$ -spaces, $1 < p < \infty$.

In the next theorem we show, at least for the case $p = 1$, the sharpness of Corollary 2.4.

THEOREM 2.5. *Let $(\alpha_j)_{-\infty}^\infty \in l^\infty(\mathbb{Z}) \setminus l^2(\mathbb{Z})$. Then there exists $\phi \in L^\infty(\Gamma)$ such that $\sup_j \|(\phi^j)^\vee\|_1 < \infty$ and $\psi = \sum_{-\infty}^\infty \alpha_j \phi^j \notin \mathcal{M}(H^1)$.*

PROOF. We consider the case when $\sum_1^\infty |\alpha_j|^2 = \infty$. Then there exists a sequence $(\lambda_j)_1^\infty$ in $l^2(\mathbb{N})$ such that $\sum_1^\infty |\alpha_j \lambda_j| = \infty$. Assume $|\alpha_1 \lambda_1| > 0$. We define a sequence $(N_k)_0^\infty$ inductively. Let $N_0 = 1$ and, assuming $N_k \in \mathbb{N}$ has been defined, define $N_{k+1} \in \mathbb{N}$ so that $N_{k+1} > N_k$ and

$$\sum_{j=N_k+1}^{N_{k+1}} |\alpha_j \lambda_j| > 2^{k+1} |\alpha_1 \lambda_1|.$$

Next, for each $j \in \mathbb{N}$ choose a character $\gamma_j \in \Gamma_{j+1} \setminus \Gamma_j$ and define $\phi : \Gamma \rightarrow \mathbb{C}$ by

$$\phi(\gamma) = \sum_{j=1}^\infty (A_{-j})^\wedge(\gamma - \gamma_j),$$

where, for $n \in \mathbb{Z}$, we set

$$A_n(x) = (\mu(G_n \setminus G_{n+1}))^{-1} \chi_{G_n \setminus G_{n+1}}(x).$$

Then $\phi \in L^\infty(\Gamma)$ and $(\phi^j)^\vee(x) = 0$ if $j \leq 1$ and $(\phi^j)^\vee(x) = \gamma_j(x) A_{-j}(x)$ if $j \geq 1$. Clearly, $\sup_j \|(\phi^j)^\vee\|_1 = 1$.

Next, for $k \in \mathbb{N}$ define $g_k : \Gamma \rightarrow \mathbb{C}$ by

$$g_k(\gamma) = \sum_{j=N_k+1}^{N_{k+1}} \lambda_j \chi_{\Gamma_{-N_k}}(\gamma - \gamma_j).$$

If $h_k = (g_k)^\vee$ then

$$h_k(x) = \sum_{j=N_k+1}^{N_{k+1}} \lambda_j \gamma_j(x) \Delta_{-N_k}(x)$$

and we have $\int_G h_k(x) d\mu(x) = 0$ and

$$\|h_k\|_2^2 = \sum_{j=N_k+1}^{N_{k+1}} |\lambda_j|^2 m_{-N_k}.$$

Also, since $\text{supp } h_k \subset G_{-N_k}$, we see that

$$\|(| \cdot | h_k)\|_2^2 \leq (m_{-N_k})^{-2} \|h_k\|_2^2.$$

Therefore,

$$\|h_k\|_2^{1/2} \|(| \cdot | h_k)\|_2^{1/2} \leq \left(\sum_{j=1}^{\infty} |\lambda_j|^2 \right)^{1/2}.$$

Consequently h_k is a $(1, 2, 1)$ molecule centered at $0 \in G$ (see [4] for a definition of $(1, 2, 1)$ molecules on G); this implies that $h_k \in H^1(G)$ and that there exists a constant $C_1 > 0$, C_1 independent of h_k , so that $\|h_k\|_{H^1} \leq C_1 (\sum_1^\infty |\lambda_j|^2)^{1/2}$.

Now define $\psi : \Gamma \rightarrow \mathbb{C}$ by $\psi = \sum_1^\infty \alpha_j \phi^j$ and assume that $\psi \in \mathcal{M}(H^1)$. Then there exists a constant $C_2 > 0$ so that for every $h \in H^1(G)$ we have $\|(\psi \hat{h})^\vee\|_{H^1} \leq C \|h\|_{H^1}$. Choose $k_0 \in \mathbb{N}$ so that

$$2^{k_0+1} |\alpha_1 \lambda_1| \geq 2C_1 C_2 \left(\sum_1^\infty |\lambda_j|^2 \right)^{1/2}.$$

Now

$$(\psi \hat{h}_{k_0})^\vee(x) = \sum_{j=N_{k_0}+1}^{N_{k_0+1}} \alpha_j \lambda_j \gamma_j(x) A_{-j}(x)$$

and we see that

$$\|(\psi \hat{h}_{k_0})^\vee\|_{H^1} \geq \|(\psi \hat{h}_{k_0})^\vee\|_1 = \sum_{j=N_{k_0}+1}^{N_{k_0+1}} |\alpha_j \lambda_j| \geq 2C_2 \|h_{k_0}\|_{H^1},$$

a contradiction. Thus we have shown that $\psi \notin \mathcal{M}(H^1)$.

Finally, if $\sum_{j=-\infty}^{-1} |\alpha_j|^2 = \infty$, then except for some minor changes, an argument like the preceding one leads again to functions ϕ and ψ with the required properties. This completes the proof of Theorem 2.5.

As our final result we present a theorem whose proof is a minor variation of the proof of Theorem 2.1. We then briefly indicate how this theorem is related to a result of Seeger in [6] about multipliers for $L^p(\mathbb{R}^n)$ -spaces.

THEOREM 2.6. *Let $0 < p \leq 1$. Assume $\phi \in L^\infty(\Gamma)$ satisfies*

$$\sup_k \sum_{j=k}^{\infty} ((m_j)^{1/p-1} \|(\phi^j)^\vee \chi_{G \setminus G_k}\|_p)^{2p/(2-p)} < \infty.$$

Then $\phi \in \mathcal{M}(H^p)$.

PROOF. We use the same notation as in the proof of Theorem 2.1(a) and we consider the case $r = p$. Then $A \leq C$ and we have, according to (2.5),

$$\begin{aligned} B &\leq C \sum_{j=n}^{\infty} (m_j)^{1-p} \|a_j\|_p^p \sum_{k=-\infty}^{n-1} \|(\phi^j)^\vee \chi_{G_k \setminus G_{k+1}}\|_p^p \\ &= C \sum_{j=n}^{\infty} (m_j)^{1-p} \|a_j\|_p^p \|(\phi^j)^\vee \chi_{G \setminus G_n}\|_p^p \\ &\leq C \left(\sum_{j=n}^{\infty} \|a_j\|_p^2 \right)^{p/2} \left(\sum_{j=n}^{\infty} ((m_j)^{1-p} \|(\phi^j)^\vee \chi_{G \setminus G_n}\|_p^p)^{2/(2-p)} \right)^{(2-p)/2} \leq C. \end{aligned}$$

Thus $\phi \in \mathcal{M}(H^p)$.

COROLLARY 2.5. *Let $0 < p \leq 1$ and $\phi \in L^\infty(\Gamma)$. If there exists $\varepsilon > 0$ such that for every $n \in \mathbb{Z}$,*

$$\sup_j (m_j)^{\varepsilon+1/p-1} \|(\phi^j)^\vee \chi_{G \setminus G_n}\|_p \leq C(m_n)^\varepsilon,$$

then $\phi \in \mathcal{M}(H^p)$.

PROOF. For each $n \in \mathbb{Z}$ we have

$$\sum_{j=n}^{\infty} ((m_j)^{1/p-1} \|(\phi^j)^\vee \chi_{G \setminus G_n}\|_p)^{2p/(2-p)} \leq C(m_n)^{\varepsilon 2p/(2-p)} \sum_{j=n}^{\infty} (m_j)^{-\varepsilon 2p/(2-p)} \leq C,$$

by inequality (1.1), because $\varepsilon 2p/(2-p) > 0$. Thus we may conclude that $\phi \in \mathcal{M}(H^p)$.

REMARK 5. In [6, Theorem 1] Seeger used a restriction on

$$(2.9) \quad \sup_{t>0} |(\phi m(t \cdot))^\vee \chi_{|x| \geq \omega}|_1$$

to prove that certain $m \in L^\infty(\mathbb{R}^n)$ are multipliers for $L^p(\mathbb{R}^n)$, $1 < p < \infty$, see [6, Section 1] for details. On G the analogue of (2.9) is

$$\sup_n \|(\phi^j)^\vee \chi_{G \setminus G_n}\|_1.$$

Thus, Corollary 2.5 may be considered as a version on locally compact Vilenkin groups G of an extension to Hardy spaces H^p , $0 < p \leq 1$, of Seeger's multiplier theorem for $L^p(\mathbb{R}^n)$ -spaces, $1 < p < \infty$.

CONCLUDING REMARK. At various places throughout this paper we have compared our results to certain multiplier theorems for Lebesgue or Hardy spaces defined on \mathbb{R}^n . The results presented here raise obvious questions and conjectures for possible additional multiplier theorems for the $H^p(\mathbb{R}^n)$ -spaces. We intend to report on some of these questions elsewhere.

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