J. Austral. Math. Soc. (Series A) **55** (1993), 369–385

COINCIDENCE AND COMMON FIXED POINTS OF HYBRID CONTRACTIONS

M. S. KHAN, Y. J. CHO, W. T. PARK and T. MUMTAZ

(Received 10 January 1991; revised 15 April 1991)

Communicated by J. H. Rubinstein

Abstract

In this paper, we show the existence of solutions of functional equations $f_i x \in Sx \cap Tx$ and $x = f_i x \in Sx \cap Tx$ under certain nonlinear hybrid contraction and asymptotic regularity conditions, generalize and improve a recent result due to Kaneko concerning common fixed points of multivalued mappings weakly commuting with a single-valued mapping and satisfying a generalized contraction type. Some related results are also obtained.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 54 H 25.
Keywords and phrases: Coincidence and common fixed points, Hausdorff metric, orbital complete, proximinal subsets and weakly commuting mappings.

1. Introduction

Nadler [12] proved a fixed point theorem for multi-valued contraction mappings, which is called Nadler's contraction principle. Subsequently, a number of generalization of Nadler's contraction principle were obtained by Ciric [23], Khan [8], Kubiak [9], Kaneke [6, 7], Sessa [14, 19], Singh [10] and many others [22, 24].

Recently, non-linear hybrid contractions, that is, contraction types involving single-valued and multi-valued mappings have been studied by Mukherjee [11], Naimpally *et al.* [13], Rhoades *et al.* [16] and Sessa *et al.* [20, 19].

In this paper, we show the existence of solutions of functional equations

$f_i x \in Sx \cap Tx$ and $x = f_i \in Sx \cap Tx$ under certain non-linear hybrid contraction and asymptotic regularity conditions where $\{f_i\}$ is a family of single-valued mappings on a metric space, S and T are multi-valued mappings on a metric space. Our results are generalizations and improvements of some results due to Kaneko, Kubiak, Mukherjee, Naimpally *et al.*, Rhoades *et al.* and many others. Also, we obtain other related results by using the proximinality of sets.

2. Preliminaries

Let (X, d) be a metric space. A subset K of X is said to be *proximinal* if for each $x \in X$, there exists a point $y \in K$ such that $d(x, y) = D(x, K)$, where $D(x, A)$ denotes the ordinary distance between $x \in X$ and a non-empty subset A of X. We shall use the following notation and definitions:

$$\begin{aligned} CL(X) &= \{A : A \text{ is a non-empty closed subset of } X\}, \\ CB(X) &= \{A : A \text{ is a non-empty closed and bounded subset of } X\}, \\ C(X) &= \{A : A \text{ is a non-empty compact subset of } X\} \text{ and} \\ P_x(X) &= \{A : A \text{ is a non-empty proximinal subset of } X\}. \end{aligned}$$

For $A, B \in CL(X)$ and $\varepsilon > 0$.

$$\begin{aligned} N(\varepsilon, A) &= \{x \in X : d(x, a) < \varepsilon \text{ for some } a \in A\}, \\ E_{A,B} &= \{\varepsilon > 0 : A \subseteq N(\varepsilon, B) \text{ and } B \subseteq N(\varepsilon, A)\}, \text{ and} \\ H(A, B) &= \begin{cases} \inf E_{A,B}, & \text{if } E_{A,B} \neq \emptyset \\ +\infty, & \text{if } E_{A,B} = \emptyset. \end{cases} \end{aligned}$$

H is called the generalized Hausdorff distance function for $CL(X)$ induced by the metric d, and H defined on $CB(X)$ is said to be the Hausdorff metric induced by d.

It is well-known that $P_x(X) \subset CL(X)$ and $C(X) \subset P_X(X)$ ([21]).

Let $\{f_i\}$ be a family of a singled-valued mappings from X to itself and S, T be multi-valued mappings from X to the non-empty subsets of X.

DEFINITION 2.1. If, for $x_0 \in X$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned} y_{2n} &= f_{2n} x_{2n-1} \in S x_{2n} \quad \text{for every } n \in \mathbb{N}, \\ y_{2n+1} &= f_{2n+1} x_{2n} \in T x_{2n+1} \quad \text{for every } n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \end{aligned}$$

then $0_{f_i}(x_0) = \{y_n : n = 0, 1, 2, \ldots\}$ is said to be the *orbit for* $(S, T; f_i)$ *at* x_0. Further, $0_{f_i}(x_0)$ is called a *regular orbit for* $(S, T; f)$ if

$$d(y_n, y_{n+1}) \leq \begin{cases} H(Sx_{n-1}, Tx_n), & \text{if } n \text{ is odd,} \\ H(Tx_{n-1}), Sx_n), & \text{if } n \text{ is even.} \end{cases}$$

DEFINITION 2.2. If, for $x_0 \in X$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that every Cauchy sequence of the form $o_{f_i}(x_0)$ converges in X, then X is called $(S, T; f_i)$*-orbitally complete with respect to* x_0 *or simply* $(S, T; f_i, x_0)$*-orbitally complete*.

If, for each $i \in \mathbb{N}$, f_i is an identity mapping on X, then $0_{f_i}(x_0)$ is denoted by $0(x_0)$ and $(S, T; f_i, x_0)$-orbitally completeness by $(S, T; x_0)$-orbital completeness.

DEFINITION 2.3. A pair (S, T) is said to be *asymptotically regular at* $x_0 \in X$ if for any sequence $\{x_n\}$ in X and each sequence$\{y_n\}$ in X such that $y_n \in Sx_{n-1} \cup Tx_{n-1}$, $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

We remark that Definitions 2.1–2.3 with $S = T$ and $f_i = f$ for each $i \in \mathbb{N}$ reduce to Definitions 4, 6 and 7 of [16], respectively, and orbital completeness need not imply the completeness of the space. Evidently, a complete space is orbitally complete.

DEFINITION 2.4. For each $i \in \mathbb{N}$, the mappings f_i and S are said to be *commuting at a point* $x \in X$ ([2]) if $f_i Sx \subseteq Sf_i x$. The mappings f_i and S are said to be *commuting on* X if $f_i Sx \subseteq Sf_i x$ for every point $x \in X$.

In [17], Sessa introduced the concept of weak commutativity for single-valued mappings on a metric space and Sessa *et al.* [19] extended this concept to the setting of a single-valued mapping and a multi-valued mapping on a metric space.

DEFINITION 2.5. For each $i \in \mathbb{N}$, the mappings f_i and S are said to be *weakly commuting at* $x \in X$ if $H(f_i Sx, Sf_i x) \leq D(f_i x, Sx)$. The mapping f and S are said to be *weakly commuting on* X if they are weakly commuting at every $x \in X$.

Note that commutativity implies weak commutativity, but the converse need not be true even in the case of singled-valued mappings as shown in Section 4.

EXAMPLE. Let $x = \{1, 2, 3, 4\}$, define a metric d on x and mappings f, S on X as follows:

$$d(1,2) = d(3,4) = 2, \qquad d(1,3) = d(2,4) = 1$$
$$d(1,4) = d(2,3) = 3/2,$$

$$S(1) = S(3) = \{4\}, \qquad S(2) = S = S(4) = \{3\},$$
$$f(1) = f(2) = f(4) = 2, \qquad f(4) = 1, \text{ respectively.}$$

Then we have $Sf(1) = \{3\}$ and $fS(1) = \{1\}$ and so f and S do not commute at $x = 1$, but f and S are weakly commuting at $x = 1$ since

$$H(Sf(1), fS(1)) = D(f(1), S(1)) = 2.$$

Let $\mathscr{F}$ be the family of all mappings ϕ from the set R^+ of non-negative real numbers to itself such that $\emptyset$ is upper-semicontinuous, non-decreasing and $\phi(t) < t$ for any $t > 0$.

The following theorem is an interesting result for the existence of coincidence points of non-linear hybrid contractions:

THEOREM 2.1. [16] *Let T be a multi-valued mapping from a metric space X into $CL(X)$. If there exists a mapping f from X into itself such that $T(X) \subseteq f(X)$, for each $x, y \in X$ and $\phi \in \mathscr{F}$,*

(2.1) $H(Tx, Ty) \leq$
$$\phi(\max(D(fx, Tx), D(fy, Ty), D(fx, Ty), D(fy, Tx)d(fx, fy))),$$

(2.2) $\phi(t) < qt$ *for each* $t > 0$ *and for some* $0 < q < 1$,

(2.3) *there exists a point $x_0 \in X$ such that T is asymptotically regular at x_0 and $f(X)$is $(T; f, x_0)$-arbitrally complete, then f and T have a coincidence point in X.*

If f is not the identity mapping, then the commuting mappings f and T satisfying the hypotheses of Theorem 2.1 need not have a common fixed point in X.

Now we can consider: What additional conditions will guarantee the existence of a common fixed point of f and T?

In [16], Rhoades *et al.* gave the solution to this problem. In this paper, we also investigate different sets of conditions under which the fixed point equation

$$x = f_i x \in Sx \cap Tx \quad \text{for } x \in X \text{ and each } i \in \mathbb{N}$$

possesses a solution.

3. Results

Now we are ready to give our main theorems:

THEOREM 3.1. *Let S and T be multi-valued mappings from a metric space X into $P_X(x)$ and let $\{f_i\}$ be the family of all continuous mappings from X into itself such that*

(3.1) $$S(X) \cup T(X) \subset f_i(X) \text{ for each } i \in \mathbb{N},$$

(3.2) $$H(Sx, Tx) \leq \phi(\max(D(f_ix, Sx), D(f_jy, Ty), D(f_ix, Ty), D(f_jy, Sx), d(f_ix, f_jy))) \quad \text{for each } x, y \in X, i, j \in \mathbb{N}, i \neq j \text{ and } \phi \in \mathscr{F},$$

(3.3) $$\phi(t) \leq qt \text{ for each } t > 0 \text{ and for some fixed } q \in (0, 1),$$

(3.4) *there exists a point $x_0 \in X$ such that the pair (S, T) is asymptotically regular at x_0, and*

(3.5) *for each $i \in \mathbb{N}$, $f_i(X)$ is $(S, T; f_i, x_0)$-orbitally complete.*

Then (1) *f_i, S and T have a coincidence point in X.*

If z is a coincidence point of f_i, S and T and f_iz is a fixed point of f_i, then we have:

(2) *f_iz is also a fixed point of S provided f_i is weakly commuting with S at z for every even $i \in \mathbb{N}$ for any odd $j \in \mathbb{N}$;*

(3) *f_iz is also a fixed point of T provided f_i is weakly commuting with T at z;*

(4) *f_iz is a common fixed point of S and T provided f_i is weakly commuting with each of S and T at z.*

PROOF. (1) Let x_0 be a point in X satisfying (3.4). Since $S(X) \cup T(X) \subset f_i(X)$, for each $i \in \mathbb{N}$, we can find sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned} y_{2n} &= f_{2n}x_{2n-1} \in Sx_{2n} \quad \text{for every } n \in \mathbb{N}, \\ y_{2n+1} &= f_{2n+1}x_{2n} \in Tx_{2n+1} \quad \text{for every } n \in \mathbb{N}_0, \\ d(y_{2n}, y_{2n+1}) &\leq q^{-1/2}H(Sx_{2n}, Tx_{2n+1}) \quad \text{and} \\ d(y_{2n+1}, y_{2n+2}) &\leq q^{-1/2}H(Tx_{2n+1}, Sx_{2n+2}). \end{aligned}$$

By (3.4), we have $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

Now, we want to show that $\{y_{2n}\}$ is a Cauchy sequence in $f_i(X)$. Suppose that the sequence $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists a positive number ε such that, for each positive integer κ, there exist integers $n(\kappa)$ and $m(\kappa)$ such that

$$\kappa \leq n(\kappa) < m(\kappa) \tag{3.a}$$

and

$$d(y_{2n(\kappa)}, y_{2m(\kappa)}) \geq \varepsilon. \tag{3.b}$$

Let $d_{i,j} = d(y_i, y_j)$ and $d_i = d(y_i, y_{i+1})$ for each $i, j \in \mathbb{N}$. Then, for each integer κ, we have

$$\varepsilon \leq d_{2n(\kappa),2m(\kappa)} \leq d_{2n(\kappa),2m(\kappa)-2} + d_{2m(\kappa)-2} + d_{2m(\kappa)-1}. \tag{3.c}$$

For each integer κ, let $m(\kappa)$ denote the smallest integer satisfying (3.a) and (3.b) for some $n(\kappa)$. Then we have $d_{2n(\kappa),2m(\kappa)-2} < \varepsilon$ and it follows from (3.c) that

$$\lim_{\kappa\to\infty} d_{2n(\kappa),2m(\kappa)} = \varepsilon. \tag{3.d}$$

Using the triangle inequality, we get

$$|d_{2n(\kappa),2m(\kappa)-1} - d_{2n(\kappa),2m(\kappa)}| \leq d_{2m(\kappa)-1} \quad \text{and}$$
$$|d_{2n(\kappa)+1,2m(\kappa)-1} - d_{2n(\kappa),2m(\kappa)}| \leq d_{2n(\kappa)} + d_{2m(\kappa)-1},$$

which yield

$$\lim_{\kappa\to\infty} d_{2n(\kappa),2m(\kappa)-1} = \lim_{\kappa\to\infty} d_{2n(\kappa)+1,2m(\kappa)-1} = \varepsilon$$

in view of (3.4) and (3.d) and so, by (3.2), we have

$$\begin{aligned}
d_{2n(\kappa),2m(\kappa)} &\leq d_{2n(\kappa)} + d_{2n(\kappa)+1,2m(\kappa)} \\
&\leq d_{2n(\kappa)} + q^{-1/2} H(Sx_{2m(\kappa)}, Tx_{2n(\kappa)+1}) \\
&\leq d_{2n(\kappa)} + q^{-1/2}\phi\Big(\max\big(D(f_{2m(\kappa)+1}x_{2m(\kappa)}, Sx_{2m(\kappa)}), \\
&\qquad D(f_{2n(\kappa)+2}x_{2n(\kappa)+1}, Tx_{2n(\kappa)+1}), D(f_{2n(\kappa)+2}x_{2n(\kappa)+1}, Sx_{2m(\kappa)}), \\
&\qquad D(f_{2m(\kappa)+1}x_{2m(\kappa)}, Tx_{2n(\kappa)+1}), d(f_{2m(\kappa)+1}x_{2m(\kappa)}, f_{2n(\kappa)+2}x_{2n(\kappa)+1})\big)\Big), \\
&\leq d_{2n(\kappa)} + q^{-1/2}\phi\Big(\max\big(d_{2m(\kappa)}, d_{2n(\kappa)+1}, d_{2n(\kappa)+2,2m(\kappa)}, \\
&\qquad d_{2n(\kappa)+1,2m(\kappa)+1}, d_{2n(\kappa)+2,2m(\kappa)+1}\big)\Big).
\end{aligned}$$

Using the upper semicontinuity of ϕ and letting $\kappa \to \infty$, this inequality yields

$$\varepsilon \leq q^{-1/2}\phi(\varepsilon) \leq q^{-1/2} q\varepsilon < \varepsilon$$

since $\varepsilon > 0$ and $q^{1-1/2} < 1$, which contradicts the choice of ε and so the sequence $\{y_{2n}\}$ is a Cauchy sequence. Similarly, we can prove that $\{y_{2n+1}\}$ is also a Cauchy sequence in $f_i(X)$.

Since $f_i(X)$ is $(S, T; f_i, x_0)$-orbitally complete, the Cauchy sequence $\{y_{2n}\}$ has a limit u in $f_i(X)$ for each $i \in \mathbb{N}$. By condition (3.4), we have

$$0 = \lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = \lim_{n\to\infty} d(u, y_{2n+1})$$

and so $\{y_{2n+1}\}$ also converges to u.

Hence there is a point $z \in X$ such that $f_i z = u$. By condition (3.2), for any even $i \in \mathbb{N}$,

$$\begin{aligned} D(f_i z, Sz) &\le d(f_i z, y_{2n+1}) + D(y_{2n+1}, Sz) \\ &\le d(f_i z, y_{2n+1}) + H(Sz, Tx_{2n+1}) \\ &\le d(f_i z, y_{2n+1}) + \phi\Big(\max\big(D(f_i z, Sz), D(f_{2n+2}x_{2n+1}, Tx_{2n+1}), \\ &\qquad D(f_i z, Tx_{2n+1}), D(f_{2n+2}x_{2n+1}, Sz), d(f_i z, f_{2n+2}x_{2n+1})\big)\Big). \\ &\le d(f_i z, y_{2n+1}) + \phi\Big(\max\big(D(f_i z, Sz), D(y_{2n+2}, Tx_{2n+1}), \\ &\qquad D(f_i z, Tx_{2n+1}), D(y_{2n+2}, Sz), d(f_i z, y_{2n+2})\big)\Big). \\ &\le d(f_i z, y_{2n+1}) + \phi\Big(\max\big(D(f_i z, Sz), d(y_{2n+2}, y_{2n+1}), \\ &\qquad d(f_i z, y_{2n+1}), d(y_{2n+2}, f_i z) + D(f_i z, Sz), d(f_i z, y_{2n+2})\big)\Big). \end{aligned}$$

Letting $n \to \infty$, this inequality yields

$$D(f_i z, Sz) \le \phi\Big(\max\big(D(f_i z, Sz)0, 0, D(f_i z, Sz), 0\big)\Big).$$

If $f_i z \notin Sz$ for any even $i \in \mathbb{N}$, then $D(f_i z, Sz) > 0$ and the above inequality implies

$$D(f_i z, Sz) \le \phi(D(f_i z, Sz)) < D(f_i z, Sz),$$

which is a contradiction.

Hence $f_i z \in Sz$ for every even $i \in \mathbb{N}$ since every proximal set is closed. Similarly, we have $f_i z \in Tz$ for any odd $i \in \mathbb{N}$. Therefore, z is a coincidence point of f_i, S and T for each $i \in \mathbb{N}$.

(2) If for any even $i \in \mathbb{N}$, $u = f_i z$ is a fixed point of f_i, then $u = f_i u = f_i f_i z \in f_i Sz$. If f_i is weakly commuting with S at z, then $f_i Sz = Sf_i z$ since

$f_i z \in Sz$. Therefore, we have $u \in f_i Sz = Sf_i z = Su$; that is, u is a fixed point of S.

We can also prove (3) by the same techniques, and by (2) and (3), we have (4).

Since (2.3) implies (3.2), Theorem 3.1 with $S = T$ and $f_i = I_X$ (the identity mapping on X) improves slightly Theorem 2.1. Theorem 3.1 remains true when we replace $P_x(X)$ by $CL(X)$.

Replacing the condition (3.1) of Theorem 3.1 by orbital regularity, we have the following:

THEOREM 3.2. *Let S and T be multi-valued mappings from a metric space X into $P_x(X)$ and let $\{f_i\}$ be the family of all continuous mappings from X into itself such that the conditions* (3.2) *holds,*

(3.6) $\phi(t) < t$ *for each* $t > 0$ *and some* $\phi \in \mathscr{F}$,

(3.7) *for some $x_0 \in X$, and each $i \in \mathbb{N}$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that the orbit $0_{f_i}(x_0)$ is regular, the pair (S, T) is asymptotically regular at x_0 and $f(X)$ is $(S, T; f_i, x_0)$-orbitally complete. Then we have:*

(1) *for each $i \in \mathbb{N}$, f_i, S and T have a coincidence point in X,*

(2) *if the limit of $0_{f_i}(x_0)$ is a fixed point of f_i, then the conclusions* (2) *and* (3) *of Theorem* 3.1 *are also true.*

We remark that Theorem 3.2 with $S = T$ and $f = f_i$ for each $i \in \mathbb{N}$ is Theorem 2 in [16].

It is well known that if S is a multi-valued mapping from X into $C(X)$, then for every $x, y \in X$ and $u \in Sx$, there exists a point $v \in Sy$ such that

$$d(u, v) \leq H(Sx, Sy).$$

Hence, if S and T are multi-valued mappings from X into $CL(X)$, then orbital regularity in Theorem 3.2 can be dropped.

THEOREM 3.3. *Let S and T be multi-valued mappings from a metric space X into $C(X)$ and let $\{f_i\}$ be the family of continuous mappings fro X into itself such that the conditions* (3.1), (3.2), (3.4), (3.5) *and* (3.6) *hold. Then all the conclusions of Theorem* 3.2 *are also true.*

Consider the following condition:

(3.8) $H(Sx, Ty) \le$
$$\phi\big(\max(D(f_i x, Sx), D(f_j y, Ty), D(f_i x, Ty), D(f_j y, Sx), d(f_i x, f_j(y))\big)$$

for each $x, y \in X, i, j \in \mathbb{N}, i \neq j$, and $\phi \in \mathscr{F}$.

In Theorem 3.1, if we replace (3.2) by (3.8), the asymptotic regularity of (S, T), (3.4), is not needed.

In fact, we have the following:

THEOREM 3.4. *Let S and T be multi-valued mappings from a metric space X into $P_x(X)$ and let $\{f_i\}$ be the family of continuous mappings from X into itself such that the conditions* (3.1), (3.5) *and* (3.8) *hold. Then all the conclusions of Theorem* 3.1 *are also true.*

In Theorems 3.1 and 3.3, taking $f = f_i$ on X for each $i \in \mathbb{N}$ and defining $\phi(t) = qt, 0 < q < 1$, we have the following:

COROLLARY 3.5. *Let S and T be multi-valued mappings from a metric space into $P_x(X)$ and let f be a continuous mapping from X into itself such that the condition* (3.4) *holds, as well as*

(3.9) $$S(X) \cup T(X) \subseteq f(X);$$

(3.10) $H(Sx, Ty) \le$
$$q \max\big(D(fx, Sx), D(fy, Ty), D(fx, Ty), D(fy, Sx), d(fx, fy)\big)$$
for each $x, y \in X$ and for some $0 < q < 1$,

(3.11) *$f(X)$ is (S, T, f, x_0)-orbitally complete, and*

(3.12) *f is weakly commuting with each of S and T.*

Then f, S and T have a common fixed point in X.

COROLLARY 3.6. *Let S and T be multi-valued mappings from a metric space X into $P_x(X)$ and let f be a continuous mapping from X into itself such that the conditions* (3.9) *and* (3.12) *hold, and*

(3.13) $H(Sx, Ty) \le$
$$q \max\big(D(fx, Sx), D(fy, Ty), \tfrac{1}{2}(D(fy, Sx) + D(fx, Ty)), d(fx, fy)\big)$$
for each $x, y \in X$ and for some $0 < q < 1$.

Then f, S and T have a common fixed point in X.

The following theorem is an extension of a recent result due to Kubiak [9]:

THEOREM 3.7. *Let S, T be multi-valued mappings from a metric space X into $P_x(f(X) \cap g(X))$ and f, g be continuous mappings from X into itself such that the conditions* (3.12) *and* (3.13) *hold. Then f, g, S and T have a common fixed point in X.*

REMARK. Our results extend some theorems of Hadzic [1] and Sessa *et al.* [20] to the version of hybrid contraction mappings.

4. Some related results

Let X be a reflexive Banach space and $WC(X)$ denote the family of non-empty weakly compact subsets of X. Note that every non-empty weakly compact subset of a reflexive Banach space is proximinal [21] and so closed.

We need the following lemma for our main theorems:

LEMMA 4.1. ([10]) *Let $f \in \mathscr{F}$ and $t_0 > 0$. If $t_{n+1} \le f(t_n)$ for all $n \in \mathbb{N}$, then the sequence $\{t_n\}$ converges to* 0.

The following theorem is an extension of a result due to Kaneko [7]:

THEOREM 4.2. *Let X be a reflexive Banach space and let $\{S_n\}$, $\{T_n\}$ be sequences of mappings from X into $WC(X)$ such that*

$$(4.1)\quad H(S_m x, T_n y) \le \phi\Big(\max\big(D(x, S_m x), D(y, T_n y), \tfrac{1}{2}(D(x, T_n y) + D(y, S_m x)), d(x, y)\big)\Big)$$

for each $x, y \in X$ and for some $\phi \in \mathscr{F}$.

Then there exists a point $z \in X$ such that $z \in S_m z \cap T_n z$ for each $m, n \in \mathbb{N}$.

PROOF. First, we assume that $h = 0$. Let $x_0 = X$ be an arbitrary point. Then there exists a point $x_1 \in S_1 x_0$ such that $d(x_0, x_1) = D(x_0, Sx_0)$. This is possible because $S_1 x_0$ is a non-empty weakly compact subset of a reflexive Banach space and so is proximinial.

Now, for all $n \in \mathbb{N}$, we have $D(x_1, T_n x_1) \le H(S_1 x_0, T_n x_1) = 0$ and hence we have $x_1 \in T_n x_1$. Similarly, for all $n \in \mathbb{N}$,

$$D(x_1, S_1 x_1) \le H(T_n x_1, S_1 x_1) = 0 \quad \text{yields } x_1 \in S_n x_1.$$

Thus x_1 is a common fixed point of the two equations $\{S_n\}$ and $\{T_n\}$.

Next, we assume that $h \neq 0$. Let $x_0 \in X$ be arbitrary but fixed.

Construct a sequence $\{x_n\}$ in X such that

$$x_{2n-1} \in S_n x_{2n-2}, \qquad x_{2n} \in T_n x_{2n-1},$$
$$d(x_{2n-2}, x_{2n-1}) = D(x_{2n-2}, S_n x_{2n-2}), \qquad \text{and}$$
$$d(x_{2n-1}, x_{2n}) = D(x_{2n-1}, T_n x_{2n}).$$

The existence of such a sequence in X is guaranteed by the proximinality of $S_m(x)$ and $T_n(x)$ for each $m, n \in \mathbb{N}$.

Now, suppose that $x_n = x_{n+1}$ for some $n \in \mathbb{N}$. If $n \in \mathbb{N}$ is even, then we have $x_{2n} \in S_{n+1}x_{2n}$.

Further, for each $m \in \mathbb{N}$, we have

$$\begin{aligned} D(x_{2n}, T_m x_{2n}) &\leq H(S_{n+1}x_{2n}, T_m x_{2n}) \\ &\leq \phi\Big(\max\big(D(x_{2n}, S_{n+1}x_{2n}), D(x_{2n}, T_m x_{2n}), \\ &\qquad \tfrac{1}{2}(D(x_{2n}, T_m x_{2n}) + D(x_{2n}, S_{n+1}x_{2n})), d(x_{2n} x_{2n})\big)\Big) \\ &= \phi\big(D(x_{2n}, T_m x_{2n})\big). \end{aligned}$$

If $D(x_{2n}, T_m x_{2n}) > 0$, then it follows from the above inequality that

$$D(x_{2n}, T_m x_{2n}) < D(x_{2n}, T_m x_{2n}),$$

which is a contradiction.

Thus, we have $x_{2n} \in T_m x_{2n}$ for each $m \in \mathbb{N}$.

Similarly, $D(x_{2n}, S_m x_{2n}) \leq H(S_m x_{2n}, T x_{2n}) \leq \phi(D(x_{2n}, S_m x_{2n}))$ gives $x_{2n} \in S_m x_{2n}$ for each $m \in \mathbb{N}$.

For an odd number $n \in \mathbb{N}$, we have also the same results.

Therefore, in each case, we have a common fixed point for the sequences $\{S_m\}$ and $\{T_n\}$.

Suppose now that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. We shall show that $\{x_n\}$ is a Cauchy sequence in X. Let us first observe that

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= D(x_{2n}, S_{n+1}x_{2n}) \\ &\leq H(T_n x_{2n-1}, S_{n+1}x_{2n}) \\ &\leq \phi\Big(\max\big(D(x_{2n-1}, S_{n+1}x_{2n}), D(x_{2n-1}, T_n x_{2n-1}), \\ &\qquad \tfrac{1}{2}(D(x_{2n}, T_n x_{2n-1}) + D(x_{2n-1}, S_{n+1}x_{2n})), d(x_{2n}, x_{2n-1})\big)\Big) \end{aligned}$$

$$= \phi\Big(\max\big(d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}), \tfrac{1}{2}d(x_{2n-1}, x_{2n+1})\big)\Big)$$
$$\leq \phi\Big(\max\big(d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}), \tfrac{1}{2}(d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1}))\big)\Big).$$

If $d(x_{2n-1}, x_{2n}) < d(x_{2n}, x_{2n+1})$, then the above inequality yields

$$d(x_{2n}, x_{2n+1}) \leq \phi(d(x_{2n}, x_{2n+1})) < d(x_{2n}, x_{2n+1}),$$

which is a contradiction. Therefore, we get

$$d(x_{2n}, x_{2n+1}) \leq \phi(d(x_{2n-1}, x_{2n})).$$

Similarly, we can show that

$$d(x_{2n+1}, x_{2n+2}) \leq \phi(d(x_{2n}, x_{2n+1})).$$

It follows from the above relations that

$$d(x_n, x_{n+1}) \leq \phi(d(x_n, x_{n-1})).$$

By Lemma 4.1, we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, which yields that $\{x_n\}$ is a Cauchy sequence in X. Since X is a Banach space, this Cauchy sequence converges to a point in X. Let $\lim_{n\to\infty} x_n = z$. Then we have

$$\begin{aligned} D(x_{2n-1}, T_m z) &\leq H(S_n x_{2n-2}, T_m z) \\ &\leq \phi\Big(\max\big(D(x_{2n-2}, S_n x_{2n-2}), D(z, T_m z), \\ &\qquad \tfrac{1}{2}(D(x_{2n-2}, T_m z) + D(z, S_n x_{2n-2})), d(x_{2n-2}, z)\big)\Big) \\ &\leq \phi\Big(\max\big(d(x_{2n-2}, x_{2n-1}), D(z, T_m z), \\ &\qquad \tfrac{1}{2}(D(x_{2n-2}, T_m z) + d(z, x_{2n-1})), d(x_{2n-2}, z)\big)\Big). \end{aligned}$$

Letting $n \to \infty$, the above inequality yields

$$D(z, T_m z) = 0.$$

Since $T_m z$ is closed, $z \in T_m z$ for all $m \in \mathbb{N}$.

Similarly, we have also $z \in S_m z$ for all $m \in \mathbb{N}$.

Therefore, z is a common fixed point of the sequence $\{S_n\}$ and $\{T_n\}$. This completes the proof.

Let f, g be single-valued mappings from a metric space X into itself. Recall that the mappings f and g are said to be weakly commuting [17] if $d(fgx, gfx) \leq d(fx, gx)$ for any $x \in X$.

Clearly two commuting mappings [4] (that is, $fgx = gfx$ for any $x \in X$) are weakly commuting but the converse is not true.

EXAMPLE. Let $X = [0, 1]$ with the Euclidean metric d and let $f, g\ X \to X$ be defined by $f(x) = 2x/(x+1)$ and $g(x) = x/(2x+1)$ for all $x \in X$. Then the mappings f and g are weakly commuting but not commuting.

Some fixed point theorems for commuting and weakly commuting mappings may be found in [3–5] and [14–18, 20].

COROLLARY 4.3. *Let f, g, S and T be single-valued mappings from a complete metric space X into itself such that*

$$(4.2)\quad d(fx, gy) \leq \varphi\Big(\max\big(d(fx, Sx), d(gy, Ty), \tfrac{1}{2}(d(fx, Ty) + d(gy, Sx)), d(Sx, Ty)\big)\Big)$$

for each $x, y \in X$ and for some $\varphi \varepsilon \mathscr{F}$,

$$(4.3)\qquad (f, S) \text{ and } (g, T) \text{ are weakly commuting pairs.}$$

Then f, g, S and T have a unique common fixed point in X.

PROOF. Define two mappings A, B by $Ax = \{fx\}$ and $Bx = \{gx\}$ for any $x \in X$. Then (A, S) and (B, T) are weakly commuting pairs. Now in view of Theorem 4.2, there exists a point $z \in X$ such that $Sz \in Az$ and $Tz \in Bz$, which yields $Sz = fz$ And $Tz = gz$. Substituting in (4.2), we obtain

$$\begin{aligned} d(fz,gz) &\leq \varphi\Big(\max\big\{d(fz,Sz), d(gz,Tz), \tfrac{1}{2}[d(fx,Tz)+d(gz,Sz)], d(Sz,Tz)\big\}\Big) \\ &= \varphi\Big(\max\big\{0, 0, d(fx, gz), d(fz, gz)\big\}\Big) = \phi(d(fz, gz)). \end{aligned}$$

If $fz \neq gz$, then we obtain the contradiction $d(fz, gz) < d(fz, gz)$. Therefore z is a common coincidence point of f, g, S, and T.

We shall now show that fz is a common fixed point of f, g, S and T.

Since f and S weakly commute, $d(Sfz, fSz) \leq d(Sx, fz) = 0$ and $Sfz = fSz$. But $Sz = fz$. Therefore $Sfz = fSz = f^2z$ and fz is a common coincidence point of f and S.

Since g and T weakly commute, $d(gtZ, Tgz) \leq d(Tz, gz) = 0$ and $gTz = Tgz$. Since $gz = Tz$ we obtain $Tgz = gTz = g^2z$ and gz is a common coincidence point of f, g, S and T. Since $fz = gz$, fz is a common coincidence point of f, g, S and T.

From (4.2), we obtain

$$\begin{aligned} d(fz, gfz) &\leq \varphi\Big(\max\big\{d(fz, Sz), d(gfz, Tfz), \\ &\qquad \tfrac{1}{2}[d(fz, Tfz) + d(gfz, Sz)], d(Sz, Tfz)\big\}\Big) \\ &= \varphi\Big(\max\big\{0, 0, d(fz, gfz), d(fz, gfz)\big\}\Big) \\ &= \varphi(d(fz, gfz)), \end{aligned}$$

which forces $fz = gfz$ and fz is a fixed point of g. From $Tgz = g^2z$ and $fz = gz$, it follows that fz is also a fixed point of T.

If we can show that fz is also a fixed point of f, then, from $Sfz = f^2z = fz$, it follow that fz is a also a fixed point of S. So consider

$$\begin{aligned} d(ffz, fz) &= d(ffz, gz) \\ &\leq \varphi\Big(\max\big\{d(ffz, Sfz), d(gz, Tz), \\ &\qquad \tfrac{1}{2}[d(ffz, Tz) + d(gz, Sfz)], d(Sfz, Tz)\big\}\Big) \\ &= \varphi\Big(\max\big\{0, 0, d(ffz, fz), d(ffz, fz)\big\}\Big) \\ &= \varphi(d(ffz, fz)), \end{aligned}$$

which implies that $ffz = fz$.

We shall now show uniqueness. Suppose that u and v are common fixed points of f, g, S and T. Then, from (4.2),

$$\begin{aligned} d(u, v) &= d(fu, gv) \\ &\leq \phi\Big(\max\big\{d(fu, Su), d(gv, Tv), \\ &\qquad \tfrac{1}{2}[d(fu, Tv) + d(gv, Su)], d(Su, Tv)\big\}\Big) \\ &= \phi\Big(\max\big\{0, 0, d(u, v), d(u, v)\big\}\Big) \\ &= \phi(d(u, v)), \end{aligned}$$

which implies that $u = v$.

COROLLARY 4.4. [20] *Let S and T be two continuous mappings from a complete metric space X into itself. Then S and T have a common fixed point in X if and only if there are two self-mappings A and B on X such that*

$$A(X) \cup B(X) \subset S(X) \cap T(X) \tag{4.4}$$

$$d(Ax, By) \leq \varphi(\max(d(Sx, Ax), d(Ty, By), \tfrac{1}{2}(d(Sx, By) + d(Ty, Ax)), d(Sx, Ty))) \tag{4.5}$$

for each $x, y \in X$ and for some $\phi \in \mathscr{F}$,

(4.6) *(A, S) and (B, T) are weakly commuting pairs.*

Further, z is the unique common fixed point of A, B, S and T.

THEOREM 4.5. *Let X be a compact metric space and let S, T be multi-valued mappings from X into $P_x(X)$ such that either S or T is continuous,*

$$H(Sx, Ty) \leq \phi\Big(\max\big(d(x, Sx), d(y, Ty), \tfrac{1}{2}(d(x, Ty) + d(y, Sx)), d(x, y)\big)\Big) \tag{4.7}$$

for each $x, y \in X$, $x \neq y$,

and for some $\phi \in \mathscr{F}$. Then either S or T has a fixed point in X.

PROOF. Suppose that S is continuous. Define $\varphi\, X \to \mathbb{R}$ by $\varphi(x) = D(x, Sx)$ for all $x \in X$. Then φ is continuous on X. Since X is compact, there exists a point $x_0 \in X$ such that $\varphi(x_0) = \min\{\varphi(x) : x \in X\}$. Now, choose $x_1 \in Sx_0$ such that $d(x_0, x_1) = D(x_0, Sx_0)$. This is possible because Sx_0 is proximimal. Similarly, we can choose $x_2 \in Tx_1$ and $x_3 \in Sx_2$ such that

$$d(x_1, x_2) = D(x_1, Tx_1) \quad \text{and} \quad d(x_2, x_3) = D(x_2, Sx_2).$$

Let us suppose that $d(x_0, Sx_0) > 0$ and $d(x_1, T_1x) > 0$. Then we have

$$\begin{aligned} d(x_1, x_2) &\leq H(Sx_0, Tx_1) \\ &< \phi\Big(\max\big(D(x_0, Sx_0), D(x_1, Tx_1), \\ &\qquad \tfrac{1}{2}(D(x_0, Tx_1) + D(x_1, Sx_0)), d(x_0, x_1)\big)\Big) \\ &\leq \phi\Big(\max\big(d(x_1, x_2), \tfrac{1}{2}d(x_0, x_2), d(x_0, x_1)\big)\Big), \end{aligned}$$

which yield $d(x_1, x_2) < d(x_0, x_1)$.

Similarly, we have also $d(x_2, x_3) < d(x_1, x_2)$. Therefore, we have $d(x_2, x_3) < d(x_0, x_1) = \varphi(x_0)$, which is a contradiction to the minimality of $\varphi(x_0)$. Hence either $D(x_0, Sx_0) = 0$ or $D(x_1, Tx_1) = 0$. Since the proximal sets Sx_0 and Tx_1 are closed, it follows that either $x_0 \in Sx_0$ or $x_1 \in Tx_1$. This completes the proof.

Acknowledgement

The authors thank the learned referee for several useful comments.

References

[1] O. Hadzic, 'Common fixed point theorems for family of mappings in complete metric spaces', *Math. Japon.* **29** (1984), 127–134.
[2] S. Itoh and W. Takahashi, 'Single-valued mappings, multi-valued mappings and fixed point theorems', *J. Math. Anal. Appl.* **59** (1977), 261–263.
[3] G. Jungck, 'Common fixed points for commuting and compatible maps on compacts', to appear.
[4] ———, 'Commuting mapping and fixed points', *Amer. Math. Monthly* **83** (1976), 261–263.
[5] ———, 'Periodic and fixed points, and commuting mappings', *Proc. Amer. Math. Soc.* **76** (1979), 333–338.
[6] H. Kaneko, 'Single-valued and multi-valued f-contractions', *Boll. Un. Mat. Ital. A* **4** (1985), 29–33.
[7] ———, 'A comparison of contractive conditions for multi-valued mappings', *Kobe J. Math.* **3** (1986), 37–45.
[8] M. S. Khan, 'Common fixed point theorems for multi-valued mappings', *Pacific J. Math.* **95** (1981), 337–347.
[9] T. Kubiak, 'Fixed point theorems for contractive type multi-valued mapping', *Math. Japon.* **30** (1985), 89–101.
[10] B. A. Meade and S. P. Singh, 'On common fixed point theorems', *Bull. Austral. Math. Soc.* **16** (1977), 49–53.
[11] R. N. Mukherjee, 'On fixed points of single-valued and set-valued mappings', *J. Indian Acad. Math.* **4** (1982), 101–103.
[12] S. B. Nadler, 'Multivalued contraction mappings', *Pacific J. Math.* **30** (1969), 475–488.
[13] S. A. Naimpally, S. L. Singh and J. H. M. Whitfield, 'Coincidence theorems for hybrid contractions', *Math. Nachr.* **127** (1986), 177–180.
[14] B. E. Rhoades and S. Sessa, 'Common fixed point theorems for three mappings under a weak commutativity condition', *Indian J. Pure Appl. Math.* **17** (1986), 47–57.
[15] B. E. Rhoades, S. Sessa, M. S. Khan and M. Swaleh, 'On fixed points of asymptotically regular mappings', *J. Austral. Math. Soc.* (*Series A*) **43** (1987), 328–346.

[16] B. E. Rhoades, S. L. Singh and C. Kulshrestha, 'Coincidence theorems for some multi-valued mappings', *Internat. J. Math. Sci.* **7** (1984), 429–434.

[17] S. Sessa, 'On a weak commutativity condition in fixed point considerations', *Publ. Inst. Math.* **32** (1982), 149–153.

[18] S. Sessa and B. Fisher, 'On common fixed points of weakly commuting mappings and set-valued mappings', *Internat. J. Math. Soc.* **9** (1986), 323–329.

[19] S. Sessa, M. S. Khan and M. Imada, 'A common fixed point theorem with a weak commutativity condition', *Glas. Mat. Ser.* III **21** (1986), 225–235.

[20] S. Sessa, R. N. Mukherjee and T. Som, 'A common fixed point theorem for weakly commuting mappings', *Math. Japon.* **31** (1986), 235–245.

[21] I. Singer, 'The theory of best approximation and functional analysis', in: *CBMS-NSF Regional Conf. Ser. in Appl. Math.* 13 (SIAM, Philadelphia, 1974).

[22] R. E. Smithon, 'Fixed points for contractive multifunctions', *Proc. Amer. Math. Soc.* **27** (1971), 192–194.

[23] Lj. B. Čirič, 'Fixed points for generalized multivalued contractions', *Math. Vesnik* **9** (1972), 265–272.

[24] R. Wegrzyk, 'Fixed point theorems for multi-valued functions and their applications to functional equations', *Dissertatione Math.* (Rozprawy Mat.) **201** (1982), 1–28.

Sultan Qaboos University
Department of Mathematics
and Computing
College of Science
P.O. Box 32486
Al-Khod, Muscat
Sultanate of Oman

Gyeongsang National University
Jinju 660-701
Korea

Gyeongsang National University
Jinju 660-701
Korea

Aligarh Muslim University
Aligarh 202002
India