

## TWISTOR DIAGRAMS AND THE ALGEBRAIC DE RHAM THEOREM

RICHARD JOZSA and JOHN RICE

(Received 20 June 1990; revised 15 March 1991)

Communicated by J. H. Rubinstein

### Abstract

A method for computing the number of contours for a twistor diagram, using Grothendieck's algebraic de Rham theorem, is described and some examples are given.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 32 L 25, 32 L 10; secondary 55 N 30.

### 1. Introduction

The formalism of twistor diagrams ([11, 12], and the review article [6]) embodies the twistor description of scattering amplitudes. A twistor diagram  $D$  describes (in diagrammatic notation) a rational function

$$\phi(Z_1, \dots, Z_n, W_1, \dots, W_m)$$

of  $n$  twistors  $Z_i \in \mathbb{C}^4$  and  $m$  dual twistors  $W_j \in \mathbb{C}^{4*}$  which is homogeneous of degree  $-4$  in each of these variables. If  $M = (CP^3)^n \times (CP^{3*})^m$  then the rational differential form

$$w = \phi d^4 Z_1 \wedge \dots \wedge d^4 Z_n \wedge d^4 W_1 \wedge \dots \wedge d^4 W_m$$

determines a meromorphic differential form on  $M$  of degree  $k = \dim M$ , as described in Section 3. We let  $D$  also denote the polar set of  $\phi$ , that is, the zero set of its denominator, so that  $w$  is holomorphic in  $M - D$ .

A scattering amplitude is then obtained by integrating  $w$  over a closed  $k$ -dimensional contour in  $M - D$ . Clearly one basic property of a twistor diagram is the *number* of independent contours in  $M - D$  over which  $w$  may be integrated, that is, knowledge of the homology space  $H_k(M - D, \mathbb{C})$ . In some cases (for example, massless Moller scattering [6, 7]) the existence of more than one contour is desirable to provide a unified description of the various “channels” of the given process.

Throughout the development of twistor diagram theory, the problem of computing the homology  $H_k(M - D, \mathbb{C})$  for a given diagram has proved to be a difficult one, and all existing methods have been applied successfully only to relatively small diagrams. These methods, developed originally by Sparling [12], use topological constructions (the relative homology exact sequence, the Mayer-Vietoris exact sequence and the Leray exact sequence) to break down the space  $M - D$ . Eastwood [1] has also described how the Leray spectral sequence can be used in this problem.

In this paper we introduce a further technique based on Grothendieck’s algebraic de Rham theorem [5] which provides a purely algebraic approach, in contrast to the above topological methods. To determine the homology  $H_k(M - D, \mathbb{C})$  we may, by Poincaré duality, determine instead the cohomology  $H^k(M - D, \mathbb{C})$ . Let  $A^p$  denote the space of smooth  $p$ -forms on  $M - D$ . By de Rham’s theorem  $H^k(M - D, \mathbb{C})$  is isomorphic to the cohomology of the complex  $A^*$ . Let  $W_D^p$  denote the space of meromorphic  $p$ -forms on  $M$  which are holomorphic on  $M - D$ . Grothendieck’s algebraic de Rham theorem states that  $H^k(M - D, \mathbb{C})$  is also isomorphic to the cohomology of the complex  $W_D^*$ . In Section 2 we shall sketch the main lines of the proof of this theorem. In Section 3 we set up a representation of meromorphic forms on projective space in terms of homogeneous polynomials, which gives a simple discretely parameterised countable basis for the spaces  $W_D^p$ . This is used in Sections 4 and 5 to compute explicitly the quotient of closed meromorphic  $k$ -forms by exact ones, in the top dimension  $k$ , for various twistor diagrams, illustrating the new approach advocated here. By contrast, the spaces  $A^p$ , having uncountable dimension, cannot be handled in this simple algebraic way.

A more sophisticated approach to the interpretation of twistor diagrams incorporates the fact that, in twistor theory, massless fields are described by analytic sheaf cohomology classes on twistor space [2, 10], so that the scattering amplitudes ought to be described by functionals on products of the cohomology groups representing the fields. In our discussion above, we have represented the fields instead by meromorphic functions, which are representative cocycles for these cohomology classes, and we have not taken into

consideration the associated coboundary freedom involved in the choice of representative. This freedom will in general reduce the number of acceptable contours. Some developments towards a cohomological theory of twistor diagrams have been made by Ginsberg [3, 4] and more recently by Huggett and Singer [8]. In particular, Huggett and Singer have shown that if massless fields are treated as cohomology classes, then the problem of the number of contours for a diagram can be reduced to computing the homology of a space of the form  $\Lambda - D$  where  $\Lambda$  is the product of complex projective lines (rather than the full  $\mathbb{CP}^3$  twistor spaces in  $M - D$ ) and it is expected that the method described in this paper will have even more effective applicability in this simplified case.

## 2. The algebraic de Rham theorem

Let  $M$  be a projective (or, more generally, compact Kähler) manifold and  $D \subset M$  a positive divisor. Let  $\Omega_{\text{alg}}^\bullet$  be the complex of sheaves of meromorphic differential forms on  $M$  which are defined on  $M - D$ , that is, whose poles lie in  $D$ . Over  $M - D$ , it is a subcomplex of the de Rham complex  $A^\bullet$  of  $M - D$ , which consists of the sheaves of all smooth differential forms on  $M - D$ . We may regard  $\Omega_{\text{alg}}^\bullet$  as the “algebraic part” of the de Rham complex of  $M - D$ . Let  $\Gamma$  be the global sections functor.

The Algebraic de Rham Theorem states the the inclusion of complexes  $\Gamma\Omega_{\text{alg}}^\bullet(M) \rightarrow \Gamma A^\bullet(M - D)$  induces an isomorphism between their cohomology groups. Otherwise said, the de Rham cohomology of  $M - D$  is isomorphic to the cohomology of its algebraic part, so by de Rham’s Theorem,

$$H^p(M - D, \mathbb{C}) = H^p(\Gamma\Omega_{\text{alg}}^\bullet(M - D)).$$

The Algebraic de Rham Theorem is proved from a standard “local to global” argument of cohomology theory. First one shows that for any suitably small open set  $U \subset M$  the theorem is true, that is, the inclusion

$$\Omega_{\text{alg}}^\bullet(U) \hookrightarrow A^\bullet(U - D)$$

induces an isomorphism in the cohomology. Then one has to show that this local result globalises to the case  $U = M$ . Let us deal with the local result first. Any point  $x \in M - D$  has a polydisc neighborhood  $U$ . By the Poincaré Lemma, the de Rham cohomology of  $U$  is zero in every degree, and there is a holomorphic Poincaré Lemma to show the same for  $\Omega_{\text{alg}}^\bullet(U)$ . Hence the local result holds true for such open sets. Next we consider the case where  $D$  is a normal crossing divisor. By definition, each  $x \in D$  has a neighborhood  $U$  such that  $U - D = (\mathbb{C} - 0)^j \times \mathbb{C}^{n-j}$  where  $n$  is the dimension

of  $M$ . The forms  $dz^\alpha/z^\alpha$  for  $\alpha = 1, \dots, j$  generate the cohomology of both  $\Omega_{\text{alg}}^\bullet(U)$  and of  $A^\bullet(U - D)$  in this case, and by direct computation ([5, p. 451]) one shows that meromorphic forms which are smoothly exact must be meromorphically exact. Hence the local result holds in this case also. The general case is reduced to the normal crossing case by Hironaka's Theorem, which asserts that there is a projective manifold  $\widetilde{M}$  with normal crossing divisor  $\widetilde{D}$  and a holomorphic map  $\widetilde{M} \xrightarrow{\pi} M$  such that  $\pi(\widetilde{D}) = D$  and  $\widetilde{M} - \widetilde{D} \xrightarrow{\pi} M - D$  is biholomorphic. In this case, one also has that  $\Omega_{\text{alg}}^\bullet(\widetilde{M})$  and  $\Omega_{\text{alg}}^\bullet(M)$  have the same cohomology, as do  $A^\bullet(\widetilde{M} - \widetilde{D})$  and  $A^\bullet(M - D)$ . Thus the result for general divisors is proved by the normal crossing case.

The global result is derived from the local result as an application of the Acyclicity Theorem. The morphisms  $\Omega_{\text{alg}}^\bullet(U) \mapsto A^\bullet(U - D)$  for each sufficiently small open set  $U$  constitute a homomorphism of complexes of sheaves  $i: \Omega_{\text{alg}}^\bullet \mapsto j_* A^\bullet$ , where  $j: M - D \mapsto M$  is the inclusion and

$$j_* A^p(U) = A^p(U - D).$$

The local result says that this morphism includes an isomorphism between the cohomology sheaves of these complexes, that is,  $i$  is a *quasi-isomorphism* in the category of sheaves on  $M$ . Taking global sections we get

$$\Gamma \Omega_{\text{alg}}^\bullet \xrightarrow{\Gamma(i)} \Gamma A^\bullet.$$

According to the Acyclicity Theorem,  $\Gamma(i)$  will be a quasi-isomorphism (so that  $\Gamma \Omega_{\text{alg}}^\bullet$  and  $\Gamma A^\bullet$  will have the same cohomology) if each of the sheaves in  $\Omega_{\text{alg}}^\bullet$  and  $j_* A^\bullet$  is  $\Gamma$ -acyclic, that is, have zero sheaf cohomology in each degree except degree zero. The sheaves  $j_* A^p$  admit partitions of unity which implies that they are  $\Gamma$ -acyclic. The sheaves  $\Omega_{\text{alg}}^p$  can be understood as

$$\Omega_{\text{alg}}^p = \varinjlim_k \Omega^p(k[D]),$$

where  $[D]$  is the line bundle associated with the divisor  $D$ . Cohomology commutes with direct limits, so we need to show that for large enough  $k$  the sheaf  $\Omega^p(k[D])$  is  $\Gamma$ -acyclic. This is the content of the Kodaira Theorem in the case that  $M$  is compact Kähler and  $D$  is a positive divisor, thus completing the proof of the Algebraic de Rham Theorem.

### 3. Meromorphic forms on $\mathbb{CP}^n$

The de Rham complex of meromorphic forms on  $\mathbb{CP}^n$  can be identified with a sub-complex of the de Rham complex of meromorphic forms

on  $\mathbb{C}^{n+1} - 0$ , namely its image under pull back by the natural projection  $\mathbb{C}^{n+1} - 0 \xrightarrow{\pi} \mathbb{C}P^n$ . This identification will be useful for calculating explicitly with these forms. Because  $\pi$  is a submersion  $\pi^*$  is injective. If  $\omega$  is meromorphic form on  $\mathbb{C}P^n$  then  $\pi^*(\omega)$  is one on  $\mathbb{C}^{n+1} - 0$  which annihilates the tangent space to the fibres of  $\pi$  and is invariant under dilation, the action of  $\mathbb{C}^*$ . It is easy to show that these two conditions characterise forms of the type  $\pi^*\omega$ ; this is a well known argument which, for any principal bundle  $P \xrightarrow{\pi} M$ , characterises the image of the de Rham complex  $\Omega^\bullet(M)$  under  $\pi^*$  as the subcomplex in  $\Omega^\bullet(P)$  of the so-called basic forms.

For dilation invariance, we observe that if  $\lambda$  denotes the map of multiplication by  $\lambda \in \mathbb{C}^*$  then its derivative map  $\lambda_*$  is also multiplication by  $\lambda$  in each tangent space. It follows that

$$\lambda^* \left( \sum_{i_1 < \dots < i_k} f_{i_1 \dots i_k}(z) dz_{i_1} \wedge \dots \wedge dz_{i_k} \right) = \sum_{i_1 < \dots < i_k} \lambda^k f(\lambda z) dz_{i_1} \wedge \dots \wedge dz_{i_k},$$

so that such a differential  $k$ -form is invariant if and only if each coefficient function  $r_{i_1 \dots i_k}$  is homogeneous of degree  $-k$ .

The tangent space to the fibres of  $\mathbb{C}_{n+1} - 0 \xrightarrow{\pi} \mathbb{C}P^n$  is spanned by the Euler vector field  $\Upsilon = \sum_{i=0}^n z_i \frac{\partial}{\partial z_i}$ . For any form  $\eta$  on  $\mathbb{C}^{n+1} - 0$ , its contraction with  $\Upsilon$ , denoted  $\Upsilon \lrcorner \eta$ , must annihilate  $\Upsilon$  and hence the tangent space to the fibres of  $\pi$ . By expressing forms as the wedge products of 1-forms dual to a basis containing  $\Upsilon$  it is easy to see that the converse is also true, that is,

$$\Upsilon \lrcorner \omega = 0 \quad \text{if and only if} \quad \omega = \Upsilon \lrcorner \eta \quad \text{for some } \eta.$$

In fact, what we have argued here is the exactness of the Koszul complex for the regular sequence  $\{z_0, \dots, z_n\}$ , since it corresponds to the de Rham complex but with differential  $\Upsilon \lrcorner$  instead of  $d$ . Furthermore, the argument shows that if  $\omega$  is dilation invariant then  $\eta$  may be chosen to be dilation invariant.

We conclude that, under  $\pi^*$ , there is a bijection between meromorphic  $k$ -forms on  $\mathbb{C}P^n$  and  $k$ -forms on  $\mathbb{C}^{n+1} - 0$  of the type

$$\sum f_{i_1 \dots i_{k+1}} (\Upsilon \lrcorner dz_{i_1} \wedge \dots \wedge dz_{i_{k+1}})$$

where each  $f$  is homogeneous of degree  $-(k+1)$ . In particular this makes explicit the well known correspondence between functions on  $\mathbb{C}^{n+1} - 0$  homogeneous of degree  $-(n+1)$ , and meromorphic  $n$ -forms on  $\mathbb{C}P^n$ , namely, to such a function  $f$  corresponds the push forward under  $\pi$  of

$$f \Upsilon \lrcorner dz_0 \wedge \dots \wedge dz_n = f \left( \sum_{j=0}^n z_j dz_0 \wedge \dots \wedge d\tilde{z}_j \wedge \dots \wedge dz_n \right).$$

Observe that for a dilation invariant form  $\omega$ , we have  $L_Y \omega = 0$  where  $L_Y$  denotes Lie derivative along  $Y$ . By Cartan's relation

$$L_Y = dY \lrcorner + Y \lrcorner d,$$

we have

$$d(Y \lrcorner \omega) = -Y \lrcorner d\omega.$$

In particular

$$d \left( Y \lrcorner \sum_{i=0}^n g_i dz_0 \wedge \cdots \wedge d\hat{z}_1 \cdots \wedge dz_n \right) = \left( - \sum \frac{\partial g_i}{\partial z_i} \right) Y \lrcorner dz_0 \wedge \cdots \wedge dz_n$$

so that a meromorphic  $n$ -form  $f Y \lrcorner dz_0 \wedge \cdots \wedge dz_n$  is exact if and only if  $f$  is a divergence, that is, there exist functions  $g_0, \dots, g_n$  homogeneous of degree  $-n$  such that

$$f = \sum_{i=0}^n \frac{\partial g_i}{\partial z_i}.$$

#### 4. Application to twistor diagrams

Let  $M$  be a product of projective spaces  $\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_r}$ . A function  $f$  on  $(\mathbb{C}^{n_1+1} - 0) \times \cdots \times (\mathbb{C}^{n_r+1} - 0)$  is called homogeneous of degree  $(i_1, i_2, \dots, i_r)$  if it is homogeneous of degree  $i_j$  as a function on  $\mathbb{C}^{n_j+1} - 0$  when all the other variables are fixed. The discussion of meromorphic forms given above generalises to the product of projective spaces by treating the factors separately. In particular, with  $k = \dim M - n_1 - \cdots - n_r$ , a meromorphic  $k$ -form corresponds to a meromorphic function homogeneous of 'top' degree  $(-n_1 - 1, -1, -n_2 - 1, \dots, -n_r - 1)$ . All such forms are closed of course. They are exact if and only if they are divergences.

Let  $D \subset M$  be given by the zero set of some homogeneous polynomial  $f$ . We appeal to the Algebraic de Rham Theorem to compute  $H^k(M - D, \mathbb{C})$  as the space of closed meromorphic  $k$ -forms whose poles are in  $D$ , modulo the exact ones. From our discussion above this is the same as the quotient of the space  $V_D$  of all meromorphic functions on  $(\mathbb{C}^{n_1+1} - 0) \times \cdots \times (\mathbb{C}^{n_r+1} - 0)$  homogeneous of top degree whose poles lie in  $D$ , by the space  $B_D$  of such functions which are divergences. From Chow's theorem [5] a homogeneous function meromorphic on  $M$  with poles in  $D$  must have the form  $h/f^m$  where  $h$  is also a homogeneous polynomial. We therefore conclude

**(4.1) THEOREM.** *Let  $M = \mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_r}$  and  $X = (\mathbb{C}^{n_1+1} - 0) \times \cdots \times (\mathbb{C}^{n_r+1} - 0)$ . Let  $f$  be a homogeneous polynomial on  $X$  whose zero set*

determines the variety  $D \subset M$ . Then

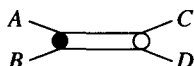
$$H^k(M - D, \mathbb{C}) = V_D/B_D$$

where  $k = n_1 \times \dots \times n_r$  and

$V_D$  = homogeneous functions on  $X$  of top degree and of the form  $h/f^m$ , where  $m \geq 0$  and  $h$  is a homogeneous polynomial,

$B_D$  = functions in  $V_D$  which are divergences  $\sum_{i=1}^{k+r} \frac{\partial g_i}{\partial z_i}$ , where  $g_i$ , of degree  $(-n_1 - 1, \dots, -n_i, \dots, -n_r - 1)$ , is of the form  $h_i/f^{m_i}$  for homogeneous polynomials  $h_i$ .

As an explicit example consider the scalar product diagram



which is the diagrammatic notation for the meromorphic form corresponding to

$$\phi(Z, W) = \frac{d^4 Z \wedge d^4 W}{(Z.A)(Z.B)(Z.W)^2(W.C)(W.D)}$$

on  $CP^3 \times CP^{3*}$ . In this case  $f(Z, W) = (A.Z)(B.Z)(Z.W)^2(C.W)(D.W)$  on  $(\mathbb{C}^4 - 0) \times (\mathbb{C}^{4*} - 0)$  and by cancellation against factors in  $f^m$ ,  $V_D$  is spanned by the functions

$$\frac{m(Z, W)}{(A.Z)^a(B.Z)^b(Z.W)^e(C.W)^c(D.W)^d}$$

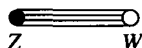
where  $m(Z, W)$  ranges over all monomials of degree  $(a + b + e - 4, c + d + e - 4)$ . We need to reduce these functions modulo divergences to determine a basis for  $H^4(M - D, \mathbb{C})$ .

Before treating the scalar product in detail, we give a simpler example, which follows readily from the useful result

**(4.2) PROPOSITION.** Consider a rational function of the form  $f(x^\alpha) \cdot (\partial/\partial x)_I g(x^\alpha)$  where  $f, g$  are rational and  $I = \alpha_1 \cdots \alpha_k$  is a multi-index denoting a  $k$ th order derivative. Then  $f(\partial/\partial x)_I g$  and  $(-1)^k g(\partial/\partial x)_I f$  differ by a rational divergence.

**PROOF.** We use the product rule successively to change  $\partial_{\alpha_i}$  on  $g$  into  $(-\partial_{\alpha_i})$  on  $f$ , the correction term, each time, being a rational divergence.

**(4.3) EXAMPLE.** Consider the twistor diagram



corresponding to

$$\phi(Z, W) = \frac{d^4 W \wedge d^4 Z}{(W \cdot Z)^4},$$

which is well known to have only one contour. To see this using the above approach, we look at rational functions on  $CP_3 \times CP_3^* - \{Z \cdot W = 0\}$ . A basis for these is given by

$$b_{JK}(Z, W) = \frac{Z^J W_K}{(W \cdot Z)^n}, \quad |J| = |K| = n - 4$$

(here  $J = \alpha_1 \cdots \alpha_{n-4}$ ,  $Z^J = Z^{\alpha_{n-4}} \cdots Z^{\alpha_1}$  etcetera). Hence

$$b_{JK} = \text{const } W_K \left( \frac{\partial}{\partial W} \right)_J \left( \frac{1}{(W \cdot Z)^4} \right),$$

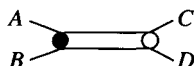
so up to a rational divergence (using Proposition 4.2),

$$\begin{aligned} b_{JK} &\equiv \frac{\text{const}}{(W \cdot Z)^4} \cdot \left( \frac{\partial}{\partial W} \right)_J (W_K) \\ &= \text{const} \cdot \frac{1}{(W \cdot Z)^4} \cdot \delta_K^J, \quad (\text{since } |J| = |K|), \end{aligned}$$

that is, any rational function in  $V_D$  is in the span of  $1/(W \cdot Z)^4$  modulo a rational divergence. Thus the quotient space is one dimensional, giving the existence of only one contour.

## 5. The scalar product diagram and related examples

Consider the twistor diagram



where  $A_\alpha, B_\alpha, C^\alpha, D^\alpha$  are assumed to be in general position.

We have already observed that  $V_D$  in this case is spanned by functions

$$\frac{Z^J W_K}{(A \cdot Z)^a (B \cdot Z)^b (Z \cdot W)^e (C \cdot W)^c (D \cdot W)^d}, \quad a, b, c, d, e \geq 0$$

where  $J$  and  $K$  are multi-indices of length

$$|J| = a + b + e - 4, \quad |K| = c + d + e - 4,$$

to ensure an overall homogeneity degree of  $(-4, -4)$  in  $(W, Z)$ . We denote this function by  $[a, b, e, c, d, J, K]$ . For fixed  $a, b, c, d, e$  we consider



the subspace  $S(a, b, c, d, e) \subseteq V_D$  spanned by all functions satisfying these conditions on  $J$  and  $K$ . We use the diagram

$$\begin{array}{c} a \backslash \quad e \quad / c \\ b \quad \quad \quad d \end{array}$$

to denote this subspace modulo  $B_D$ , that is, the image of  $S(a, b, c, d, e)$  under the projection  $V_D \rightarrow V_D/B_D$ . Letting  $M = \mathbb{CP}^{3*} \times \mathbb{CP}^3$  and  $D \subset M$  be the variety defined by the scalar product diagram, we have

$$H^6(M - D, \mathbb{C}) = V_D/B_D = \sum_{a, b, e, c, d \geq 0} \begin{array}{c} a \backslash \quad e \quad / c \\ b \quad \quad \quad d \end{array}.$$

We can show that many of the subspaces described by these diagrams are contained in the span of the others. For example

(5.1) PROPOSITION. Let  $a \geq 2$ .

(a) If  $c, d \neq 0$  then

$$\begin{array}{c} a \backslash \quad e \quad / c \\ b \quad \quad \quad d \end{array} \subseteq \begin{array}{c} a-1 \backslash \quad e \quad / c \\ b \quad \quad \quad d \end{array} \oplus \begin{array}{c} a-1 \backslash \quad e+1 \quad / c-1 \\ b \quad \quad \quad d \end{array} \oplus \begin{array}{c} a-1 \backslash \quad d+1 \quad / e \\ b \quad \quad \quad d-1 \end{array}$$

(b) if  $c = 0, d \neq 0$ , then

$$\begin{array}{c} a \backslash \quad e \quad / 0 \\ b \quad \quad \quad d \end{array} \subseteq \begin{array}{c} a-1 \backslash \quad e \quad / 0 \\ b \quad \quad \quad d \end{array} \oplus \begin{array}{c} a-1 \backslash \quad e+1 \quad / 0 \\ b \quad \quad \quad e \end{array} \oplus \begin{array}{c} a-1 \backslash \quad e+1 \quad / 0 \\ b \quad \quad \quad d-1 \end{array}$$

In both cases, if  $a + b + e = 4$  then omit the first space in the sum of the right.

Note that in all subspaces on the right

- (1)  $a$  is reduced by 1,
- (2)  $b$  is unchanged,
- (3) the total sum of indices  $a + b + e + c + d$  is reduced by 1 in (a).

PROOF. Construct  $M^\alpha \in \text{span}(C^\alpha, D^\alpha)$  such that  $M.A \neq 0$ ,  $M.B = 0$ , that is, take  $M^\alpha = (B.D)C^\alpha - (B.C)D^\alpha$ . Now (writing  $f \equiv g$  if  $f$  and  $g$  differ by a divergence) we have

$$\frac{\partial}{\partial Z^\alpha} \{M^\alpha[a-1, b, e, c, d, J, K]\} \equiv 0.$$

The divergence produces four terms (from derivatives of  $A^J$ ,  $Z.A$ ,  $Z.B$ ,  $Z.W$  respectively)

$$\begin{aligned} t_1 &= [a-1, b, e, c, d, 0, K] M^\alpha \frac{\partial}{\partial Z^\alpha} (Z^J), \\ t_2 &= (-a+1)(M.A)[a, b, e, d, c, J, K], \\ t_3 &= (-b)(M.B)[a-1, b+1, e, c, d, J, K] = 0, \\ t_4 &= (-e)(B.D)[a-1, e+1, c-1, d, J, K] \\ &\quad - (-e)(B.C)[a-1, b, e+1, c, d-1, J, K]. \end{aligned}$$

Since  $M.A \neq 0$  we get  $t_2 \equiv -(t_1 + t_3 + t_4)$ . For varying  $J$  and  $K$ ,  $t_2$  ranges over a basis of  $S(a, b, c, d, e)$ , so we have proved (a). Note that if  $a + b + e = 4$ , then  $|J| = 0$  so  $t_1 = 0$ .

The result (b) follows from the same calculation, producing a slightly different  $t_4$ .

(5.2) PROPOSITION (moving indices from internal to external lines without decreasing total sum).

(a) if  $a \neq 0$ ,  $b \neq 0$ ,

$$\begin{array}{c} a \\ \backslash \end{array} \begin{array}{c} e \\ \text{---} \end{array} \begin{array}{c} c \\ / \end{array} \begin{array}{c} d \\ \backslash \end{array} \subseteq \bigoplus_{\hat{c}, \hat{d}} \begin{array}{c} a \\ \backslash \end{array} \begin{array}{c} 2 \\ \text{---} \end{array} \begin{array}{c} \hat{c} \\ / \end{array} \begin{array}{c} \hat{d} \\ \backslash \end{array},$$

where on the RHS  $\hat{c} \geq c$ ,  $\hat{d} \geq d$  and  $\hat{c} + \hat{d} + 2 = c + d + e$ . If  $c = 0$  on the LHS then  $\hat{c} = 0$  on the RHS.

(b) if  $a \neq 0$ ,

$$\begin{array}{c} a \\ \backslash \end{array} \begin{array}{c} e \\ \text{---} \end{array} \begin{array}{c} c \\ / \end{array} \begin{array}{c} d \\ \backslash \end{array} \subseteq \bigoplus_{\hat{c}, \hat{d}} \begin{array}{c} a \\ \backslash \end{array} \begin{array}{c} 3 \\ \text{---} \end{array} \begin{array}{c} \hat{c} \\ / \end{array} \begin{array}{c} \hat{d} \\ \backslash \end{array},$$

with the same conditions on  $\hat{c}$ ,  $\hat{d}$  as in (a).

(c)

$$\begin{array}{c} 0 \\ \backslash \end{array} \begin{array}{c} e \\ \text{---} \end{array} \begin{array}{c} c \\ / \end{array} \begin{array}{c} d \\ \backslash \end{array} \subseteq \bigoplus_{\hat{c}, \hat{d}} \begin{array}{c} 0 \\ \backslash \end{array} \begin{array}{c} 4 \\ \text{---} \end{array} \begin{array}{c} \hat{c} \\ / \end{array} \begin{array}{c} \hat{d} \\ \backslash \end{array},$$

with same conditions on  $\hat{c}$ ,  $\hat{d}$  as in (a).

PROOF. A general basis function of  $S(a, b, c, d, e)$  is  $[a, b, e, c, d, J, K]$ , with  $|J| = a + b + e - 4$ . Let  $L$  be a sub-index of  $J$  with  $|L| = e - 2$ , which always exists since  $a, b \geq 1$ . Write  $J = LM$ . Now, setting  $u = 1/(W.Z)^2$  so  $(\partial/\partial W)_L \cdot u = \text{const} \cdot Z^L/(W.Z)^e$  we get  $[a, b, e, c, d, J, K] = [a, b, 0, c, d, M, K](\partial/\partial W)_L u$ . By Proposition 4.2 this differs by a divergence from

$$u \cdot \left( -\frac{\partial}{\partial W} \right)_L [a, b, 0, c, d, M, K].$$

Each differentiated term has  $c, d$  increased to  $\hat{c}, \hat{d}$ , but all have the same  $W$ -homogeneity  $\hat{c} + \hat{d} = c + d + |L| = c + d + e - 2$ . Thus we get

$$\begin{array}{c} a \\ \backslash \end{array} \begin{array}{c} e \\ \text{---} \end{array} \begin{array}{c} c \\ / \end{array} \begin{array}{c} d \\ \backslash \end{array} \subseteq \bigoplus_{\hat{c}, \hat{d}} \begin{array}{c} a \\ \backslash \end{array} \begin{array}{c} 2 \\ \text{---} \end{array} \begin{array}{c} \hat{c} \\ / \end{array} \begin{array}{c} \hat{d} \\ \backslash \end{array}, \quad \hat{c} + \hat{d} + 2 = c + d + e.$$

The proof of (b) and (c) are similar using  $u = 1/(W.Z)^3$ ,  $u = 1/(W.Z)^4$  respectively.

## (5.3) PROPOSITION.

$$(a) \quad \begin{array}{c} 0 \\ \diagdown \end{array} \begin{array}{c} 4 \\ \diagup \end{array} \begin{array}{c} c \\ \diagdown \end{array} \begin{array}{c} d \\ \diagup \end{array} \subseteq 0 \quad \text{for any } c, d.$$

$$(b) \quad \begin{array}{c} 0 \\ \diagdown \end{array} \begin{array}{c} 3 \\ \diagup \end{array} \begin{array}{c} c \\ \diagdown \end{array} \begin{array}{c} 0 \\ \diagup \end{array} \subseteq \begin{array}{c} 0 \\ \diagdown \end{array} \begin{array}{c} 3 \\ \diagup \end{array} \begin{array}{c} 1 \\ \diagdown \end{array} \begin{array}{c} 0 \\ \diagup \end{array} \quad c \geq 1.$$

$$(c) \quad \begin{array}{c} 0 \\ \diagdown \end{array} \begin{array}{c} 3 \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array} \begin{array}{c} 1 \\ \diagup \end{array} = \begin{array}{c} 0 \\ \diagdown \end{array} \begin{array}{c} 3 \\ \diagup \end{array} \begin{array}{c} 1 \\ \diagdown \end{array} \begin{array}{c} 0 \\ \diagup \end{array} = \begin{array}{c} 1 \\ \diagdown \end{array} \begin{array}{c} 3 \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array} \begin{array}{c} 1 \\ \diagup \end{array} = \begin{array}{c} 1 \\ \diagdown \end{array} \begin{array}{c} 3 \\ \diagup \end{array} \begin{array}{c} 1 \\ \diagdown \end{array} \begin{array}{c} 0 \\ \diagup \end{array}.$$

PROOF.

(a) For any  $k$ 

$$[0, 0, 4, c, d, 0, K] = \frac{\partial}{\partial Z^\alpha} \left( \frac{C^\alpha}{(-3)} [0, 0, 3, c+1, d, 0, K] \right)$$

(b) Taking  $\partial/\partial W_\alpha(B_\alpha[0, 1, 3, c-1, 0, 0, K])$  and using (a) gives

$$\begin{array}{c} 0 \\ \diagdown \end{array} \begin{array}{c} 3 \\ \diagup \end{array} \begin{array}{c} c \\ \diagdown \end{array} \begin{array}{c} 0 \\ \diagup \end{array} \subseteq \begin{array}{c} 0 \\ \diagdown \end{array} \begin{array}{c} 3 \\ \diagup \end{array} \begin{array}{c} c-1 \\ \diagdown \end{array} \begin{array}{c} 0 \\ \diagup \end{array}, \quad c \geq 2.$$

(c) To prove

$$\begin{array}{c} 1 \\ \diagdown \end{array} \begin{array}{c} 3 \\ \diagup \end{array} \begin{array}{c} 1 \\ \diagdown \end{array} \begin{array}{c} 0 \\ \diagup \end{array} = \begin{array}{c} 0 \\ \diagdown \end{array} \begin{array}{c} 3 \\ \diagup \end{array} \begin{array}{c} 1 \\ \diagdown \end{array} \begin{array}{c} 1 \\ \diagup \end{array},$$

use the divergence

$$\frac{\partial}{\partial W_\alpha} \left( \frac{(C.B)A_\alpha - (C.A)B_\alpha}{(Z.A)(Z.B)(Z > W)^2(W > C)} \right).$$

Others are similar.

We now use the previous propositions to reduce the general basis function  $[a, b, e, c, d, J, K]$ , modulo divergences to two special cases. Note that Propositions 5.1(a), 5.3(b) enable the reduction of the total sum  $a + b + c + d + e$  while the other redistribute the values of these parameters. All propositions above have exact analogues with the roles of  $a, b$ , and  $c, d$ , etcetera, interchanged.

**The reduction process.** If  $a, b \geq 1$ , use Proposition 5.1 repeatedly to reduce both to 1:

$$\begin{array}{c} a \\ \diagdown \end{array} \begin{array}{c} e \\ \diagup \end{array} \begin{array}{c} c \\ \diagdown \end{array} \begin{array}{c} d \\ \diagup \end{array} \subseteq \bigoplus_{\hat{c}, \hat{d}, \hat{e}} \begin{array}{c} 1 \\ \diagdown \end{array} \begin{array}{c} \hat{e} \\ \diagup \end{array} \begin{array}{c} \hat{c} \\ \diagdown \end{array} \begin{array}{c} \hat{d} \\ \diagup \end{array}.$$

Proposition 5.2(a) now gives

$$\begin{array}{c} 1 \\ \diagdown \end{array} \begin{array}{c} e \\ \diagup \end{array} \begin{array}{c} c \\ \diagdown \end{array} \begin{array}{c} d \\ \diagup \end{array} \subseteq \bigoplus_{\hat{c}, \hat{d}} \begin{array}{c} 1 \\ \diagdown \end{array} \begin{array}{c} 2 \\ \diagup \end{array} \begin{array}{c} \hat{c} \\ \diagdown \end{array} \begin{array}{c} \hat{d} \\ \diagup \end{array}.$$

Then using Proposition 5.1, on the  $c, d$  side, if one of  $c, d$ , say  $c$ , is  $> 1$

$$\begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} 2 \\ \diagdown \end{array} \begin{array}{c} c \\ \diagup \end{array} \begin{array}{c} d \\ \diagdown \end{array} \subseteq \begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} 2 \\ \diagdown \end{array} \begin{array}{c} c-1 \\ \diagup \end{array} \begin{array}{c} d \\ \diagdown \end{array} \oplus \begin{array}{c} 0 \\ \diagup \end{array} \begin{array}{c} 3 \\ \diagdown \end{array} \begin{array}{c} c-1 \\ \diagup \end{array} \begin{array}{c} d \\ \diagdown \end{array} \oplus \begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} e \\ \diagdown \end{array} \begin{array}{c} c-1 \\ \diagup \end{array} \begin{array}{c} d \\ \diagdown \end{array}.$$

Iterating this on the first term of the RHS, we can reduce  $c$  to 1, and then  $d$  to 1, modulo spaces with  $a$  or  $b$  zero.

Hence we have achieved

$$\begin{array}{c} a \\ \diagup \end{array} \begin{array}{c} e \\ \diagdown \end{array} \begin{array}{c} c \\ \diagup \end{array} \begin{array}{c} d \\ \diagdown \end{array} \subseteq \begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} 2 \\ \diagdown \end{array} \begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} 1 \\ \diagdown \end{array} \oplus \{\text{spaces with at least one of } a, b, c, d \text{ zero}\}.$$

To do the remaining spaces, suppose, say,  $d = 0$ ,  $a, b, c \geq 1$ . Then

$$\begin{array}{c} a \\ \diagup \end{array} \begin{array}{c} e \\ \diagdown \end{array} \begin{array}{c} c \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array} \subseteq \bigoplus_{\hat{c}, \hat{e}} \begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} \hat{e} \\ \diagdown \end{array} \begin{array}{c} \hat{c} \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array} \quad (\text{Proposition 5.1, iterated}),$$

$$\begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} e \\ \diagdown \end{array} \begin{array}{c} c \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array} \subseteq \bigoplus_{\hat{c}} \begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} 2 \\ \diagdown \end{array} \begin{array}{c} \hat{c} \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array} \quad (\text{Proposition 5.2(a)}).$$

Now Proposition 5.1(a), on the  $c, d$  side, gives

$$\begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} 2 \\ \diagdown \end{array} \begin{array}{c} c \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array} \subseteq \begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} 2 \\ \diagdown \end{array} \begin{array}{c} c-1 \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array} \oplus \begin{array}{c} 0 \\ \diagup \end{array} \begin{array}{c} 3 \\ \diagdown \end{array} \begin{array}{c} c-1 \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array} \oplus \begin{array}{c} 0 \\ \diagup \end{array} \begin{array}{c} 3 \\ \diagdown \end{array} \begin{array}{c} c-1 \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array}.$$

Iterating this on the first term of RHS and using Proposition 5.3(b), (c) to reduce the other RHS terms, gives,

$$\begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} 2 \\ \diagdown \end{array} \begin{array}{c} c \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array} \subseteq \begin{array}{c} 0 \\ \diagup \end{array} \begin{array}{c} 3 \\ \diagdown \end{array} \begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array}.$$

Putting all this together gives

$$\begin{array}{c} a \\ \diagup \end{array} \begin{array}{c} e \\ \diagdown \end{array} \begin{array}{c} c \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array} \subseteq \begin{array}{c} 0 \\ \diagup \end{array} \begin{array}{c} 3 \\ \diagdown \end{array} \begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array}.$$

Similarly, if  $a$  or  $b$  or  $c$  was zero, with  $d \neq 0$ , we get the same reduction (using also Proposition 5.3(c)) and likewise for two of  $a, b, c, d$  not on the same vertex, being zero. Also Propositions 5.2(c), 5.3(a) give

$$\begin{array}{c} 0 \\ \diagup \end{array} \begin{array}{c} e \\ \diagdown \end{array} \begin{array}{c} c \\ \diagup \end{array} \begin{array}{c} d \\ \diagdown \end{array} = 0.$$

Thus for any  $a, b, c, d, e$ ,

$$\begin{array}{c} a \\ \diagup \end{array} \begin{array}{c} e \\ \diagdown \end{array} \begin{array}{c} c \\ \diagup \end{array} \begin{array}{c} d \\ \diagdown \end{array} \subseteq \begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} 2 \\ \diagdown \end{array} \begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} 1 \\ \diagdown \end{array} \oplus \begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} 3 \\ \diagdown \end{array} \begin{array}{c} 1 \\ \diagup \end{array} \begin{array}{c} 0 \\ \diagdown \end{array}.$$

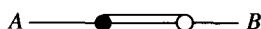
Both RHS spaces are one dimensional, since the sum of indices at each vertex is 4 (so the numerator of any function in  $S(1, 1, 1, 1, 2)$  or  $S(0, 1, 0, 1, 3)$  is constant). This gives

$$\dim H^6(M - D, \mathbb{C}) = 2$$

providing two contours for the scalar product diagram.

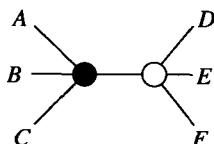
This example, written out in detail, illustrates how this algebraic technique can be used to investigate any twistor diagram: we look for procedures which reduce the total sum of indices on all lines, and others which redistribute the values. Used together these enable reduction of the infinite dimensional space of the rational homogeneous functions to a finite dimensional space modulo divergences.

An easy exercise along these lines shows that



has only one contour.

As a further example, consider the diagram



with  $A, B, C, D, E, F$  in general position.

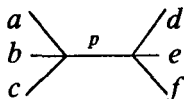
We can invent a reducing procedure, exactly analogous to Proposition 5.1, by setting (compare with the proof of Proposition 5.1)

$$K^{\alpha\beta\gamma} = d^{[\alpha} E^{\beta} F^{\gamma]}$$

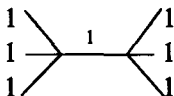
and

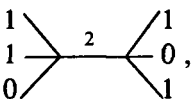
$$M^{\alpha} = K^{\alpha\beta\gamma} B_{\beta} C_{\gamma}$$

so  $M \in \text{span}(D, E, F)$  with  $M.A \neq 0$ ,  $M.B = M.C = 0$ . Proposition 5.2 also has a direct analogue for the present case. If one of  $a, b, c$  and one of  $d, e, f$  have been reduced to zero, we can continue the reduction using the scalar product results. This readily enables reduction of a general space

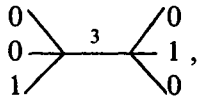


to the following 19 cases

(a)  (1 case),

(b)  and all permutations of 0, 1,

1 on each end (9 cases),

(c)  , and all permutations of 0, 1,

1 on each end (9 cases)

Now, all 9 cases in (c) are equivalent. For example, the divergence

$$\frac{\partial}{\partial W_\alpha} \left( \frac{(F.C)B_\alpha - (F.B)C_\alpha}{(Z.B)(Z.C)(Z.W)^2(W.F)} \right)$$

shows that

$$\begin{array}{c} 0 \\ 1 \\ 3 \end{array} \begin{array}{c} \diagup \\ \hline \diagdown \end{array} \begin{array}{c} 0 \\ 0 \\ 1 \end{array} = \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \begin{array}{c} \diagup \\ \hline \diagdown \end{array} \begin{array}{c} 0 \\ 0 \\ 1 \end{array}.$$

The others are similar. Also (b) can be reduced to 4 cases: the divergence

$$\frac{\partial}{\partial Z^\alpha} \left( \frac{K^{\alpha\beta\gamma} A_\beta B_\gamma}{(Z.A)(Z.B)(Z.W)(W.D)(W.E)(W.F)} \right)$$

shows that

$$\begin{array}{c} 1 \\ 1 \\ 0 \end{array} \begin{array}{c} \diagup \\ \hline \diagdown \end{array} \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \subseteq \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \begin{array}{c} \diagup \\ \hline \diagdown \end{array} \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \oplus \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \begin{array}{c} \diagup \\ \hline \diagdown \end{array} \begin{array}{c} 1 \\ 1 \\ 0 \end{array},$$

that is, for the three spaces obtained by fixing one end and permuting the other end, any one is in the span of the other two. Thus without loss of generality, we can impose  $d = 1$ , and similarly  $a = 1$ .

Hence everything has been reduced to the 6 cases

$$\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \begin{array}{c} \diagup \\ \hline \diagdown \end{array} \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \\ \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \begin{array}{c} \diagup \\ \hline \diagdown \end{array} \begin{array}{c} 1 \\ 0 \\ 1 \end{array}, \quad \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \begin{array}{c} \diagup \\ \hline \diagdown \end{array} \begin{array}{c} 1 \\ 1 \\ 0 \end{array}, \quad \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \begin{array}{c} \diagup \\ \hline \diagdown \end{array} \begin{array}{c} 1 \\ 0 \\ 1 \end{array}, \quad \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \begin{array}{c} \diagup \\ \hline \diagdown \end{array} \begin{array}{c} 1 \\ 1 \\ 0 \end{array} \\ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \begin{array}{c} \diagup \\ \hline \diagdown \end{array} \begin{array}{c} 1 \\ 0 \\ 0 \end{array}$$

and we have shown that the number of contours is not larger than six.

It is not clear how to prove that no further reduction is possible, which at present, provides a limitation to the usefulness of this method, especially for diagrams with many external lines. However the methods of algebraic topology, referred to in the introduction, also tend to become intractable for these larger diagrams. We may conjecture the following interpretation of the

final six diagrams. 'Ones' on the two external lines ( $A$ ,  $B$  say) indicate that the contour separates the poles at  $Z.A = 0$  and  $Z.B = 0$  (that is, 'pinches' if  $A$  and  $B$  are moved into coincidence). 'Zero' on  $A$  and 'one' on  $B$  indicates that  $A$  can be moved into coincidence with  $B$ , in the confines of the contour.

## References

- [1] M. G. Eastwood, 'Some comments on the topology of twistor diagrams', *Twistor Newsletter* **14** (1982).
- [2] M. G. Eastwood, R. Penrose and R. O. Wells, Jr., 'Cohomology and massless fields', *Comm. Math. Phys.* **78** (1981) 305–351.
- [3] M. L. Ginsberg, 'Scattering theory and the geometry of multi-twistor spaces', *Trans. Amer. Math. Soc.* **276** (1983) 789–815.
- [4] M. L. Ginsberg, *A cohomological approach to scattering theory* (D. Phil. thesis, Oxford University, 1980).
- [5] P. Griffiths and J. Harris, *Principles of algebraic geometry* (John Wiley and Sons, New York 1978).
- [6] A. P. Hodges, 'Twistor Diagrams', *Physica* **114A** (1982) 157–175.
- [7] A. P. Hodges, 'Twistor diagrams and massless Moller scattering', *Proc. Royal Soc. London* **A385** (1983) 207–228.
- [8] S. A. Huggett and M. A. Singer, 'Relative cohomology and projective twistor diagrams', preprint, Mathematical Institute, Oxford, 1988.
- [9] S. A. Huggett, *Sheaf cohomology in twistor diagrams* (D. Phil. thesis, Oxford University, 1980).
- [10] R. Penrose and W. Rindler, *Spinors and spacetime*, vol. 2 (Cambridge University Press, Cambridge, 1986).
- [11] R. Penrose, 'Twistor theory, its aims and achievements', in *Quantum Gravity, and Oxford Symposium* (eds. C. J. Isham, R. Penrose and D. W. Sciama) (Oxford University Press, Oxford, 1975).
- [12] G. A. J. Sparling, 'Homology and twistor theory', in *Quantum Gravity, and Oxford Symposium* (eds. C. J. Isham, R. Penrose and D. W. Sciama) (Oxford University Press, Oxford 1975).

School of Information Science and Technology  
 The Flinders University of South Australia  
 G.P.O. BOX 2100 Adelaide SA 5001  
 Australia