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ON THE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION

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Abstract

Let $N_n(w)$ be the number of real roots of the random algebraic equation $\sum_{\nu=0}^{n} a_\nu \xi_\nu(w) x^\nu = 0$, where the $\xi_\nu(w)$'s are independent, identically distributed random variables belonging to the domain of attraction of the normal law with mean zero and $P\{\xi_\nu(w) \neq 0\} > 0$; also the a_ν's are nonzero real numbers such that $(k_n/t_n) = O(\log n)$ where $k_n = \max_{0\leq\nu\leq n} |a_\nu|$ and $t_n = \min_{0\leq\nu\leq n} |a_\nu|$. It is shown that for any sequence of positive constants $(\varepsilon_n,\ n \geq 0)$ satisfying $\varepsilon_n \to 0$ and $\varepsilon_n^2 \log n \to \infty$ there is a positive constant μ so that

$$\Pr\left\{\inf_{n>n_0} N_n(w)/\log n < \varepsilon_n\right\} < \mu(\varepsilon_{n_0} \log n_0)^{-1}$$

for all n_0 sufficiently large.

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1. Introduction

Let $N_n(w)$ be the number of real roots of the algebraic equation

$$f(x, w) = \sum_{\nu=0}^{n} \xi_\nu(w) x^\nu = 0; \quad x \in \mathbf{R} \tag{1.1}$$

where the $\xi_\nu(w)$'s are independent, identically distributed real-valued random variables. Samal [7] has considered the general case when the $\xi_\nu(w)$'s are independent random variables identically distributed with expectation zero, the variance and the third absolute moment finite and nonzero. He has shown that $N_n(w) > \varepsilon_n \log n$ outside an exceptional set whose measure tends to zero as n tends to infinity, where $\varepsilon_n \to 0$ but $\varepsilon_n \log n \to \infty$.

Mishra et al. [4] consider the equation

$$f(x, w) = \sum_{\nu=0}^{n} a_\nu \xi_\nu(w) x^\nu = 0 \tag{1.2}$$

in which the $\xi_\nu(w)$'s are independent, identically distributed random variables belonging to the domain of attraction of the normal law with $P\{\xi_\nu(w) \neq 0\} > 0$ and a_ν's are nonzero real numbers such that

$$k_n = \max_{0 \le \nu \le n} |a_\nu|, \; t_n = \min_{0 \le \nu \le n} |a_\nu|, \; k_n/t_n = O(\log n). \tag{1.3}$$

They show that when $n > n_0$,

$$N_n(R, w) > (\mu \log n)/\log\left\{\frac{k_n}{t_n} \log n\right\} \tag{1.4}$$

outside a set of measure at most

$$\mu'/\left\{\log\left(\frac{k_n}{t_n}\log n\right) \cdot (\log n)^{1-\varepsilon}\right\} \tag{1.5}$$

for $0 < \varepsilon < 1$ and positive constants μ and μ'.

Mishra et al. [5] consider the polynomial equation (1.2) under the conditions (1.3) and prove that there exists a positive integer n_0 such that for $n > n_0$ and positive constants C and C',

$$N_n(R, w) > C\left\{\log n/\log\left(\frac{k_n}{t_n}\log\log n\right)\right\}^{1/2} \tag{1.6}$$

outside a set of measure at most

$$C'\left\{\log\left(\frac{k_{n_0}}{t_{n_0}}\log\log n_0\right)/\log n_0\right\}^{(1-\varepsilon)/2}; \qquad 0 < \varepsilon < 1. \tag{1.7}$$

The result (1.4) and (1.5) is of the form

$$\Pr\left\{N_n(R, w)/\log n < \mu/\log\left(\frac{k_n}{t_n}\log n\right)\right\} \to 0,$$

while the result contained in (1.6) and (1.7) is of the form:

$$\Pr\left\{\inf_{n > n_0} N_n(R, w)/\log n < c/\left\{\log n \log\left(\frac{k_n}{t_n}\log\log n\right)\right\}^{1/2}\right\} \to 0.$$

The latter result is called the 'strong result' and may be referred to as the strong-version or 0-version of the former.

Mishra et al. [6] solve the same problem, obtaining for $n > n_0$,

$$N_n(R, w) > \varepsilon_n \log n \tag{1.8}$$

outside an exceptional set of measure at most

$$\mu/\{\varepsilon_n \log n + (k_n/t_n)^\beta \exp(-\mu' \beta/\varepsilon_n)\}, \tag{1.9}$$

$(0 < \beta < 2 - \varepsilon, 0 < \varepsilon < 2)$, provided that $\lim_{n\to\infty}(k_n/t_n)$ is finite.

Earlier Samal and Pratihari [10] had obtained the lower bound (1.8) with an exceptional set of measure smaller than (1.9) in case the $\xi_\nu(w)$'s are independent and identically distributed random variables with common characteristic function $\exp(-C|t|^\alpha)$; C being a positive constant and $\alpha \geq 1$. Samal and Pratihari [8] have proved the 0-version of their theorems in [10] with refinement of their exceptional set and they have extended this result to the general case in [9] when the $\xi_\nu(w)$'s are independent, identically distributed random variables with mean zero and the variance and the third absolute moment finite and nonzero. They have obtained the lower bound (1.8) outside an exceptional set of measure at most $\mu/(\varepsilon_{n_0} \log n_0)$ for $n > n_0$, n_0 being sufficiently large and μ a positive constant. It is apparent that Mishra, Nayak and Pattanayak are not aware of [8, 9, 10].

In this paper our object is to prove the following theorem.

THEOREM. *Let $N_n(w)$ be the number of real roots of the equation $f(x, w) = \sum_{\nu=0}^n a_\nu \xi_\nu(w) x^\nu = 0$ of degree n, where the coefficients $\xi_\nu(w)$ are independent, identically distributed random variables belonging to the domain of attraction of the normal law with mean zero and* $\Pr\{\xi_\nu(w) \neq 0\} > 0$. *Let the a_ν's be nonzero real numbers such that $k_n/t_n = O(\log n)$, where $k_n = \max_{0\leq\nu\leq n}|a_\nu|$, $t_n = \min_{0\leq\nu\leq n}|a_\nu|$. Then, for any sequence of positive constants $(\varepsilon_n, n \geq 0)$ satisfying $\varepsilon_n \to 0$ and $\varepsilon_n^2 \log^{\cdot} n \to \infty$, there is a positive constant μ so that*

$$\Pr\left\{\inf_{n>n_0} N_n(w)/\log n < \varepsilon_n\right\} < \mu(\varepsilon_{n_0} \log n_0)^{-1}$$

for all n_0 sufficiently large.

This theorem gives the strong result of Mishra et al. [5] as a particular case. Choosing $\varepsilon_n = c/\{\log n \log(\frac{k_n}{t_n} \log\log n)\}^{1/2}$ in our theorem, their lower bound (1.6) is obtained. Moreover, for this choice of ε_n our exceptional set becomes smaller than theirs (1.7). Of course, for such choice of ε_n, $\varepsilon_n^2 \log n$

tends to zero instead of ∞, but $(\alpha\varepsilon_n \log n)^2$ tends to infinity. It will be seen in the sequel that k appearing in (2.8) is a positive integer tending to infinity.

Throughout this paper [x] denotes the greatest integer not exceeding x, $V(\eta)$ the variance of the random variable η. We assume that all inequalities are satisfied for n sufficiently large. Positive constants are denoted by μ's.

2. Proof of the theorem

Since the $\xi_\nu(w)$'s belong to the domain of attraction of the normal law, their characteristic function is given by (cf. Ibragimov and Linnik [3, page 91])

$$\phi(t) = \exp\left\{-\frac{t^2}{2}h(t)\right\} \tag{2.1}$$

where $h(t)$ is a slowly varying function as $t \to 0$ with the property that

$$h(t) = \operatorname{Re} h(t)\{1 + o(1)\}. \tag{2.2}$$

Let

$$h_1(t) = \begin{cases} \operatorname{Re} h(t) & \text{if } V(\xi_\nu) = \infty, \\ \sigma^2 & \text{if } V(\xi_\nu) = \sigma^2 < \infty, \end{cases}$$

which is a slowly varying function in a neighbourhood of the origin. By (2.2), $h(t) = h_1(t)\{1 + o(1)\}$ in both the cases as $t \to 0$.

2.1. Take absolute constants A and B such that $A > 1$ and $0 < B < 1$. Choose

$$\beta_n = \left(\frac{t_n}{k_n}\right) \exp\{C_1/(\varepsilon_n^2 \log n)\} \tag{2.3}$$

where C_1 is a constant to be chosen later. Let, for constants $d_1 > 1$, $e = \exp(1)$,

$$M_n = \left[\frac{d_1^2(\sqrt{2}+1)^2}{16}\beta_n^2\left(\frac{k_n}{t_n}\right)^2 \cdot \left(\frac{Ae}{B}\right)\right] + 1 \tag{2.4}$$

so that

$$\mu_1\{(k_n/t_n)\beta_n\}^2 \le M_n \le \mu - 2\{(k_n/t_n)\beta_n\}^2 \tag{2.5}$$

Let

$$\phi(x) = x^x \tag{2.6}$$

and k be an integer determined by

$$\phi(8k+7)M_n^{8k+7} \le n < \phi(8k+11)M_n^{8k+11}. \tag{2.7}$$

The first inequality of (2.7) gives

$$(8k+7)\{\log(8k+7)+\log M_n\} \le \log n$$

and ultimately

$$k \le \mu''(\log n)/\log\left(\frac{k_n}{t_n}\beta_n\right).$$

The second inequality of (2.7) gives

$$\begin{aligned}\log n &< (8k+11)\{\log(8k+11)+\log M_n\}\\ &< (8k+11)^2+(8k+11)\log M_n\\ &< \mu k^2 \log M_n,\end{aligned}$$

so that $k > \mu'\{(\log n)/\log(k_n\beta_n/t_n)\}^{1/2}$. Thus, from (2.7) we have

$$\frac{\mu_1}{\sqrt{C_1}}\varepsilon_n \log n \le K \le \frac{\mu_2}{C_1}(\varepsilon_n \log n)^2. \tag{2.8}$$

We consider $f(x_m, w) = U_m(w) + R_m(w)$ at the points

$$x_m = \left\{1 - \frac{1}{\phi(4m+1)M_n^{4m}}\right\}^{1/2} \tag{2.9}$$

for $m = [k/2]+1$, $[k/2]+2$, $[k/2]+3, \dots, k$, where

$$U_m(w) = \sum_1 a_\nu \xi_\nu(w) x_m^\nu \quad \text{and} \quad R_m(w) = \left(\sum_2 + \sum_3\right) a_\nu \xi_\nu(w) x_m^\nu,$$

the index ν ranging from $\nu_1 + 1 = \phi(4m-1)M_n^{4m-1} + 1$ to $\nu_2 = \phi(4m+3)M_n^{4m+3}$ in $\sum_1$, from 0 to ν_1 in $\sum_2$ and from ν_2+1 to n in $\sum_3$. So

(2.10)

$$f(x_{2m}, w) = U_{2m}(w) + R_{2m}(w);\ f(x_{2m+1}, w) = U_{2m+1}(w) + R_{2m+1}(w)$$

where $U_{2m}(w)$ and $U_{2m+1}(w)$ are independent. Let V_m be given by the relation

$$\frac{1}{V_m^2}\sum_{\nu=\nu_1+1}^{\nu_2} a_\nu^2 x_m^{2\nu} h_1(a_\nu x_m^\nu \theta/V_m) = 1 \tag{2.11}$$

where θ is a small positive number to be chosen later. Ibragimov and Maslova [2] show that normalising constants such as V_m exist under conditions of our theorem for θ sufficiently small.

If $V(\xi_\nu(w)) = \sigma^2 < \infty$, then

$$V_m^2 = \sigma^2 \sum_{\nu=\nu_1+1}^{\nu_2} a_\nu^2 x_m^{2\nu} \geq \sigma^2 t_n^2 \sum_{\nu=\nu_1+1}^{\nu_2} x_m^{2\nu}$$
$$> \sigma^2 t_n^2 \phi(4m+1) M_n^{4m} (B/Ae)$$

or,

$$\phi(4m+1) M_n^{4m} < (Ae/B) \frac{V_m^2}{\sigma^2 t_n^2}. \tag{2.12}$$

If $V(\xi_\nu(w)) = \infty$, then we have by (2.2) $\lim_{t\to 0} h_1(t) = \infty$ so that we choose θ such that $h_1(t) > 1$ for $|t| < \theta$. Hence, in this case we have

$$\phi(4m+1) M_n^{4m} < (Ae/B)(V_m^2/t_n^2). \tag{2.13}$$

2.2. We give here three lemmas to be used in the proof.

LEMMA 1. $|\sum_2 a_\nu \xi_\nu(w) x_m^\nu| < (m\beta_n) W_m$ *except for a set of measure at most* $\mu/(m\beta_n)^{2-\varepsilon}$ *for* $\varepsilon > 0$, *where*

$$W_m^2 = \sum_2 a_\nu^2 x_m^{2\nu} h_1(a_\nu x_m^\nu \theta / W_m). \tag{2.14}$$

LEMMA 2. $|\sum_3 a_\nu \xi_\nu(w) x_m^\nu| < (m\beta_n) Z_m$ *except for a set of measure at most* $\mu/(m\beta_n)^{2-\varepsilon}$ *for* $\varepsilon > 0$, *where*

$$Z_m^2 = \sum_3 a_\nu^2 x_m^{2\nu} h_1(a_\nu x_m^\nu \theta / Z_m). \tag{2.15}$$

These lemmas are proved in the same way as in Mishra et al. [4].

LEMMA 3. $|R_m(w)| < V_m$ *except for a set of measure at most* $\mu/(m\beta_n)^{2-\varepsilon}$ *for* $m = m_0,\ m_0+1,\ m_0+2, \dots, k;\ m_0 = [k/2]+1$.

PROOF. (CASE I.) Let $V(\xi_\nu(w)) = \infty$ Then by Lemmas 1 and 2, $|R_m| < m\beta_n(W_m + Z_m)$ for any m, except for a set of measure at most $\mu/(m\beta_n)^{2-\varepsilon}$, $\varepsilon > 0$. That is

$$|R_m| < m\beta_n \left(\left\{ \sum_2 a_\nu^2 x_m^{2\nu} h_1(a_\nu x_m^\nu \theta / W_m) \right\}^{1/2} + \left\{ \sum_3 a_\nu^2 x_m^{2\nu} h_1(a_\nu x_m^\nu \theta / Z_m) \right\}^{1/2} \right)$$
$$< m\beta_n k_n d \left(\left\{ \sum_2 x_m^{2\nu} \right\}^{1/2} + \left\{ \sum_3 x_m^{2\nu} \right\}^{1/2} \right)$$

where $d = \max_{0\le\nu\le\eta}(\{h_1(a_\nu x_m^\nu\theta/W_m)\}^{1/2}, \{h_1(a_\nu x_m^\nu\theta/Z_m)\}^{1/2})$.

Clearly $d > 1$ since θ is small. Again, we can choose θ so that h_1 are bounded (cf. Mishra et al. [6, page 23]). Hence d is bounded above. Let d_1 be a positive constant such that $d \le d_1$. Then

$$|R_m| < m\beta_n k_n d_1 \left(\left\{\sum_2 x_m^{2\nu}\right\}^{1/2} + \left\{\sum_3 x_m^{2\nu}\right\}^{1/2}\right).$$

Again

$$\sum_{\nu=0}^{\phi(4m-1)M_n^{4m-1}} x_m^{2\nu} < \phi(4m-1)M_n^{4m-1} + 1$$

$$< 2\phi(4m-1)M_n^{4m-1} < \frac{2\phi(4m+1)M_n^{4m}}{16m^2 M_n}. \tag{2.16}$$

Also

$$\sum_{\nu=\phi(4m+3)M_n^{4m+3}+1}^{n} x_m^{2\nu} < \sum_{\nu=\phi(4m+3)M_n^{4m+3}+1}^{\infty} x_m^{2\nu}$$

$$= \frac{x^{2\{\phi(4m+3)M_n^{4m+3}+1\}}}{1-x_m^2}$$

$$< \phi(4m+1)M_n^{4m}\left\{1 - \frac{1}{\phi(4m+1)M_n^{4m}}\right\}^{\phi(4m+3)M_n^{4m+3}}.$$

Now $\phi(4m+3)M_n^{4m+3} > \phi(4m+1)M_n^{4m}(4m+1)^2M_n^2$ so that

(2.17)

$$\sum_3 x^{2\nu} < \phi(4m+1)M_n^{4m}\exp\{-(4m+1)^2M_n^2\} < \phi(4m+1)M_n^{4m}/(16m^2M_n).$$

Therefore, using (2.16), (2.17) and (2.13) we get

$$|R_m| < \frac{(\sqrt{2}+1)}{4} d_1\beta_n k_n\{\phi(4m+1)M_n^{4m}\}^{1/2}/M_n^{1/2} < V_m.$$

(Case II.) When $V(\xi_\nu(w)) = \sigma^2 < \infty$, we have

$$|R_m| < m\beta_n k_n\sigma\left\{\left(\sum_2 x_m^{2\nu}\right)^{1/2} + \left(\sum_3 x_m^{2\nu}\right)^{1/2}\right\}$$

$$< \frac{(\sqrt{2}+1)}{4}\sigma\beta_n k_n\{\phi(4m+1)M_n^{4m}\}^{1/2}/M_n^{1/2} < V_m.$$

Hence $|R_m| < V_m$ in both cases and for $m = m_0,\ m_0 + 1, \ldots, k$ except for a set of measure at most $\mu/(m\beta_n)^{2-\varepsilon}$.

2.3. We define events E_m as the sets of w for which $U_{2m}(w) > V_{2m}$ and $U_{2m+1}(w) < -V_{2m+1}$ and the events F_m as the sets of w for which $U_{2m}(w) < -V_{2m}$ and $U_{2m+1}(w) > V_{2m+1}$.

Let S_m^+, S_m^- be the sets of w in which $U_m(w) > V_m$ and $U_m(w) < -V_m$ respectively. Hence $E_m \cup F_m = (S_{2m}^+ \cap S_{2m+1}^-) \cup (S_{2m}^- \cap S_{2m+1}^+)$. Since the two sets within the braces on the right hand side are disjoint and since $U_{2m}(w)$ and $U_{2m+1}(w)$ are independent random variables, we have

$$\begin{aligned}(2.18)\qquad P = P(E_m \cup F_m) &= P(S_{2m}^+)P(S_{2m+1}^-) + P(S_{2m}^-)P(S_{2m+1}^+)\\ &= P\left(U_{2m} > V_{2m}\right) P\left(U_{2m+1} < -V_{2m+1}\right)\\ &\quad + P\left(U_{2m} < -V_{2m}\right) P\left(U_{2m+1} > V_{2m+1}\right) = \delta_m \text{ (say).}\end{aligned}$$

Let $G_m(x)$ and $g_m(t)$ be respectively the distribution function and the characteristic function of (U_m/V_m). Then

$$g_m(t) = \exp\left\{-\frac{t^2}{2}\cdot\frac{1}{V_m^2}\sum_{\nu=\nu_1+1}^{\nu_2} a_\nu^2 x_m^{2\nu} h(a_\nu x_m^\nu t/V_m)\right\}.$$

Let $F(x) = (1/\sqrt{2\pi})\int_{-\infty}^{x}\exp(-u^2/2)\,du$. As in [6], as $m \to \infty$, $g_m(t) \to \exp(-t^2/2)$ in any bounded interval of t-values. Hence

$$\sup_x |G_m(x) - F(x)| = o(1).$$

So $|G_{2m}(-1) - f(-1)| < \varepsilon$ and $|G_{2m+1}(-1) - F(-1)| < \varepsilon$; $\varepsilon > 0$. Thus, from (2.18), we get

$$P = \delta_m > 2\{F(-1) - \varepsilon\}\{1 - F(1) - \varepsilon\} = \delta \text{ (say).}$$

Obviously $\delta_m > \delta > 0$ for large values of m.

2.4. Let η_m be a random variable such that it takes values 1 on $E_m \cup F_m$ and zero elsewhere. In other words,

$$\eta_m = \begin{cases} 1 \text{ with probability } \delta_m \\ 0 \text{ with probability } 1 - \delta_m. \end{cases}$$

The η_m's are thus random variables with $E(\eta_m) = \delta_m$ and $V(\eta_m) = \delta_m - \delta_m^2 < 1$.

Let ρ_m be defined as follows:

$$\rho_m = \begin{cases} 0 & \text{if } |R_{2m}| < V_{2m} \text{ and } |R_{2m+1}| < V_{2m+1} \\ 1 & \text{elsewhere,} \end{cases}$$

where (2.10) holds.

Let $\theta_m = \eta_m - \eta_m \rho_m$. Now $\theta_m = 1$ only if $\eta_m = 1$ and $\rho_m = 0$, which implies the occurrence of one of the events:

(i) $U_{2m} > V_{2m}$, $|R_{2m}| < V_{2m}$; $|U_{2m+1}| < -V_{2m+1}$, $|R_{2m+1}| < V_{2m+1}$.
(ii) $U_{2m} < -V_{2m}$, $|R_{2m}| < V_{2m} U_{2m+1} > V_{2m+1}$, $|R_{2m+1}| < V_{2m+1}$.

It is obvious that (i) implies $f(x_{2m}) > 0$ and $f(x_{2m+1}) < 0$ and (ii) implies that $f(x_{2m}) < 0$ and $f(x_{2m+1}) > 0$. Thus, if $\theta_m = 1$, there is a root of the polynomial in the interval (x_{2m}, x_{2m+1}). Hence the number of roots in the interval (x_{2m_0}, x_{2k+1}) must exceed $\sum_{m=m_0}^{k} \theta_m$ where $m_0 = [k/2] + 1$.

2.5. We have

$$\left|\sum_{m=m_0}^{k} \{\theta_m - E(\eta_m)\}\right| \leq \left|\sum_{m=m_0}^{k} \{\eta_m - E(\eta_m)\}\right| + \sum_{m=m_0}^{k} \rho_m. \tag{2.19}$$

Let $A(w)$ be the set of w for which

$$\sup_{k-m_0+1 \geq k_0} \frac{1}{k - m_0 + 1} \left|\sum_{m=m_0}^{k} \{\theta_m - E(\eta_m)\}\right| > \varepsilon,$$

$B(w)$ be the set of w for which

$$\sup_{k-m_0+1 \geq k_0} \frac{1}{k - m_0 + 1} \left|\sum_{m=m_0}^{k} \{\eta_m - E(\eta_m)\}\right| > \varepsilon/2$$

and $C(w)$ be the set of w for which

$$\sup_{k-m_0+1 \geq k_0} \frac{1}{k - m_0 + 1} \sum_{m=m_0}^{k} \rho_m > \varepsilon/2.$$

Since $E(\rho_m) = 1$, $P(\rho_m = 1)$,

$$\begin{aligned} E(\rho_m) &= P\{(|R_{2m}| \geq V_{2m}) \cup (|R_{2m+1}| \geq V_{2m+1})\} \\ &\leq P(|R_{2m}| \geq V_{2m}) + P(|R_{2m+1}| \geq V_{2m+1}). \end{aligned}$$

Using Lemma 3 and (2.3) we have $E(\rho_m) < \mu/m^{2-\varepsilon}$. Therefore

$$\frac{1}{k - m_0 + 1} \sum_{m=m_0}^{k} E(\rho_m) \leq \frac{1}{k - m_0 + 1} \sum_{m=m_0}^{k} (\mu/m^{2-\varepsilon}) < \mu//m_0^{2-\varepsilon}$$

and so

$$\begin{aligned} P\{C(w)\} &< \sum_{k-m_0+1 \geq k_c} P\left\{\frac{1}{k - m_0 + 1} \sum_{m=m_0}^{k} \rho_m > \varepsilon/2\right\} \\ &< (2\mu'/\varepsilon) \sum_{k-m_0+1 \geq k_0} (1/m_0^{2-\varepsilon}). \end{aligned}$$

Here we need the strong law of large numbers in following form, which is a consequence of the Hajek-Renyi inequality (see [1]):

LEMMA 4. *Let* $\eta_1, \eta_2, \dots$ *be a sequence of independent random variables with* $V(\eta_i) < 1$ *for all* i. *Then, for each* $\varepsilon > 0$,

$$\Pr\left\{\sup_{k\geq k_0}\left|\frac{1}{k}\sum_{i=1}^{k}\{\eta_i - E(\eta_i)\}\right| \geq \varepsilon\right\} \leq \frac{D}{\varepsilon^2 k_0},$$

where D *is a positive constant.*

Applying Lemma 4, we have

$$P\{B(w)\} < 4D/(\varepsilon^2 k_0) = \mu_3/k_0.$$

From (2.19) it follows that $A(w) \subseteq B(w) \cup C(w)$. Therefore

$$P\{A(w)\} < \mu_3/k_0 + \mu_4 \sum_{k-m_0+1\geq k_0} (1/m_0^{2-\varepsilon}).$$

Hence

$$\sup_{k-m_0+1\geq k_0} \frac{1}{k-m_0+1}\left|\sum_{m=m_0}^{k}\{\theta_m - E(\eta_m)\}\right| < \varepsilon$$

outside the set $A(w)$ where $P\{A(w)\} < \mu_3/k_0 + \mu_4 \sum_{k-m_0+1\geq k_0}(1/m_0^{2-\varepsilon})$. Therefore

$$\frac{1}{k-m_0+1}\sum_{m=m_0}^{k}\theta_m > \frac{1}{k-m_0+1}\sum_{m=m_0}^{k}E(\eta_m) - \varepsilon$$

for all k such that $k - m_0 + 1 \geq k_0$. So that

$$\begin{aligned} N_n(w) &> \sum_{m=m_0}^{k}\theta_m > (k-m_0+1)(\delta-\varepsilon) \\ &= (k-[k/2])(\delta-\varepsilon) > k(\delta-\varepsilon)/2 > \frac{\mu_1(\delta-\varepsilon)}{2\sqrt{C_1}}\varepsilon_n \log n, \end{aligned}$$

for all k such that $k - m_0 + 1 \geq k_0$, that is, for all $n > n_0$. We have

$$\begin{aligned} P\{A(w)\} &< \mu_3/k_0 + \mu_4 \sum_{k\geq 2k_0-1}(1/m_0^{2-\varepsilon}) < \mu_3/k_0 + \mu_4\sum_{k\geq k_0}(1/k^{2-\varepsilon}) \\ &< \mu/k_0 < \left(\frac{\mu\sqrt{C_1}}{\mu_1}\right)/(\varepsilon_{n_0}\log n_0). \end{aligned}$$

Now the result follows by taking $C_1 = \mu_1^2(\delta-\varepsilon)^2/4$.

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