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A COMPUTER-AIDED ANALYSIS OF SOME FINITELY PRESENTED GROUPS

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Abstract

We answer some questions which arise from a recent paper of Campbell, Heggie, Robertson and Thomas on one-relator free products of two cyclic groups. In the process we show how publicly accessible computer programs can be used to help answer questions about finite group presentations.

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1. Introduction

We answer some questions which arise from a recent paper of Campbell, Heggie, Robertson and Thomas (1992) and in the process show how publicly accessible computer programs can be used to help answer questions about finite group presentations. Another discussion of the use of computers in investigating group presentations is given in Neubüser and Sidki (1988).

The paper by Campbell *et al.* is part of a project which investigates one-relator free products of two cyclic groups. In it they consider the groups, $G(\alpha, n)$, which have the following parameterised presentation:

$$\{ a, b : a^2 = b^n = ab^{-1}ab(abab^{-1})^{\alpha-1}ab^2ab^{-2} = 1 \}$$

where both α and n are positive integers. They also consider the preimages, $H(\alpha, n)$, which have the following parameterised presentation:

$$\{ a, b : a^2 = b^n, ab^{-1}ab(abab^{-1})^{\alpha-1}ab^2ab^{-2} = 1 \}.$$

Campbell *et al.* prove that the largest metabelian quotients of the groups $G(\alpha, n)$ are finite and determine their orders. For n odd and all α, they show that $G(\alpha, n)$ is finite and soluble, with soluble length at most 3. For $n \leq 7$, they prove that all groups $G(\alpha, n)$ are finite (except $G(1, 6)$) and soluble and they determine their orders and solubility lengths. Further they establish that

- $G(6, 8)$ is infinite and soluble with soluble length 4 or 5 and (implicitly) that it is polycyclic with Hirsch length (the number of infinite factors in a polycyclic series) between 5 and 15;
- $G(7, 8)$ is finite and soluble with order divisible by $2^{10} \times 3^3 \times 7 \times 17^2 \times 53$ and soluble length 4 or 5;
- $G(6, 10)$ is finite with order divisible by $2^9 \times 3^2 \times 5^3 \times 13 \times 1481^2$ and with a composition factor $\mathrm{PSU}(3, 4)$;
- $H(6, 10)$ is finite with order divisible by $2^9 \times 3^2 \times 5^3 \times 7 \times 13 \times 1481^2$.

We show that

- $G(6, 8)$ and $H(6, 8)$ have soluble length 5 and Hirsch length 5;
- $G(7, 8)$ and $H(7, 8)$ have soluble length 5;
- $G(7, 8)$ has order $2^{11} \times 3^5 \times 7 \times 17^2 \times 53$;
- $H(7, 8)$ has order $2^{15} \times 3^6 \times 7 \times 17^2 \times 53$;
- $G(6, 10)$ has order $2^9 \times 3^2 \times 5^3 \times 13 \times 1481^2$;
- $H(6, 10)$ has order $2^{10} \times 3^2 \times 5^3 \times 7 \times 13 \times 1481^2$.

In Section 2 we describe the computational tools used in our analysis. In Section 3, we provide some necessary background material from Campbell *et al.* (1992). In the remaining sections, we describe in some detail the computations which lead to the results given above and we remark on some other structural features of the groups. The order in which the groups are discussed below reflects the complexity of the methods used.

It should be observed that the large orders of these groups prevent the explicit computation of their elements on a computer. Our calculations are with one or two exceptions relatively minor in terms of the sort of resources now routinely available. A more detailed account of these calculations is given in a research report which is available from the authors.

We thank Campbell *et al.* for supplying us with a preprint of their paper.

2. The computational tools

We mention below some of the algorithms, implementations and programs used in carrying out the computations.

The system CAYLEY, described in Cannon (1984), provides access to the following:

• The Alford and Havas implementation of coset enumeration [TC] (see Cannon, Dimino, Havas and Watson, (1973)).

• The Havas and Newman (1980) implementation of a nilpotent quotient algorithm [NQ].

• The Havas and Sterling (1979) implementation of integer matrix diagonalisation [IMD].

The system SPAS, described in Felsch (1989), provides access to the following:

• The Havas (1974) implementation of the Reidemeister-Schreier algorithm [RS].

• The Aachen implementation of the Reduced Reidemeister-Schreier algorithm [RRS].

• The Tietze Transformation program [TZ] developed by Havas, Kenne, Richardson and Robertson (1984).

The IMD and NQ implementations are also available as stand-alone programs. Since these provide enhanced levels of control to the user, the stand-alone programs were used for the larger computations. We made significant use of the ability of the SPAS system to produce output files in a format suitable for input to both of these programs.

All coset enumerations were carried out in CAYLEY using the default HLT look-ahead method, unless otherwise specified. Structural information about certain groups was also computed using CAYLEY.

Given a description for a subgroup of finite index in a finitely presented group, the RS algorithm permits the construction of a presentation for this subgroup. The algorithm is implemented in both CAYLEY and SPAS. Typically, the presentation obtained as output from RS contains many redundant generators and relations. Tietze transformations may be applied to simplify this presentation by eliminating some of these. A Tietze Transformation program, implementing these transformations, is available in both CAYLEY and SPAS. The program may be run automatically or may be driven by the user. In cases where the presentation obtained is to be handed to NQ for further analysis, simplification must be continued at least until the number of defining generators is less than 256 – in order to meet a limitation imposed by the NQ implementation.

We used SPAS Version 2.4 running on a VAXStation 3100, and CAYLEY Version 3.8 running on a Sparc Server I+. Both stand-alone programs were run on the latter machine.

3. The presentations

We use the notation of Campbell *et al.* (1992) and the following information from their paper. Two new generators $e = abab^{-1}$ and $d = e^{b^{-1}}$ are introduced into the presentations for the $G(\alpha, n)$ to give

$$\{ a, b, d, e \ : \ a^2 = b^n = ab^{-1}abe^{\alpha-1}ab^2ab^{-2} = 1, e = abab^{-1}, \\ d^b = e, (eb)^n = [d, e^{\alpha-1}] = 1, e^b = e^\alpha d \}.$$

Let $N = \langle aba, b \rangle$; then N has index 2 in $G(\alpha, n)$ and has the following presentation:

$$\{ b, d, e : b^n = (eb)^n = [d, e^{\alpha-1}] = 1, d^b = e, e^b = e^\alpha d \}.$$

The derived group, G', of $G(\alpha, n)$ is generated by d and e, has index n in N and equals the derived group of N. Since $G' = \langle d, e \rangle$, it follows that $y = e^{\alpha-1}$ is central in G'. Let Y be the normal closure of y in N; then Y is central in G'. Campbell *et al.* (1992) show that G'/G'' is finite for all n and α. It follows that $Y/Y \bigcap G''$ is finite and so, by some further argument, $G(\alpha, n)$ is finite if (and only if) $\bar{N} = N/Y$ is finite. Clearly $G(\alpha, n)$ is soluble if $\bar{N}$ is soluble.

The group $H(\alpha, n)$ has a cyclic central subgroup generated by $z = a^2$, and $G(\alpha, n)$ is isomorphic to $H(\alpha, n)/\langle z \rangle$. It is easy to see that z has order $\alpha + 1$ modulo H', and so the order of $H(\alpha, n)$ is divisible by $(\alpha + 1) \times |G(\alpha, n)|$.

4. $G(6, 10)$ and $H(6, 10)$

Campbell *et al.* (1992) show that for $G = G(6, 10)$, the order of G/G'' is $2^3 \times 3 \times 5 \times 1481^2$ and G'/Y is PSU(3, 4) (which has order 62400). Hence $G/G'' \bigcap Y$ has order $2^9 \times 3^2 \times 5^3 \times 13 \times 1481^2$. We show this is the order of G by establishing that $G'' \bigcap Y$ is trivial. Clearly $YG'' = G'$, so $G''/G'' \bigcap Y$ is PSU(3, 4). Since PSU(3, 4) has trivial multiplicator (see Conway *et al.*, 1985), there is a normal subgroup M of G contained in G'' such that M is PSU(3, 4) and $M \bigcap Y$ is trivial. Since Y is abelian, $M = G''$ and hence $G'' \bigcap Y$ is trivial as required.

Although we cannot currently explicitly list the elements of G on a computer, it is possible to get quite a detailed picture of its structure. It follows from the previous paragraph that G is a subdirect product of G/Y and G/G''. The automorphism group of PSU(3, 4) is an extension of PSU(3, 4) by a cyclic group of order 4 (see Conway *et al.*, 1985). Hence the centre of G/Y has order 10 or 20. We show that it has order 10 by first using RS to get a presentation for the normal subgroup of index 5 in G/Y. This group, of order 62400×4, is sufficiently small to show, using CAYLEY, that it has a centre of order 2. Let X/Y be the centre of G/Y. Then G/X is the subgroup of index 2 in the automorphism group of PSU(3, 4) and G is a subdirect product of G/X and G/G''. Let B be the normal closure of b^2 in

G. It is easy to see that G/B is dihedral of order 24. Let A be the normal closure of $\langle [d, e], d^{1481}\rangle$. Then G/A is an extension of C_{1481}^2 by $C_{10} \times C_2$ and G is a subdirect product of G/X, G/B and G/A. We refrain from going further here.

We establish that $H(6, 10)$ has order $2^{10} \times 3^2 \times 5^3 \times 7 \times 13 \times 1481^2$ by the following steps:

1. We use IMD to verify that H/H' is $C_2 \times C_{70}$.

2. We now construct, using RS, a presentation for H'. We then use TZ to simplify this presentation. An IMD calculation on the resulting presentation shows that the torsion invariants of H'/H'' are 1481 and 17772, with prime decompositions 1481 and $2^2 \times 3 \times 1481$.

3. This calculation shows that H' has non-trivial p-quotients only for the primes 2, 3, 1481. The 2- and 3-quotients are clearly cyclic. We use the presentation constructed for H' as input to NQ to establish that the 1481-quotient is abelian. It follows that H'/H'' is the largest nilpotent quotient of H'. Hence, a similar argument to that used for $G(6, 10)$ shows that H'' is $\mathrm{PSU}(3, 4)$.

5. $G(7, 8)$ and $H(7, 8)$

In considering $G(7, 8)$, Campbell *et al.* (1991) show that $\bar{N}$ has soluble length 4 and the factors of its derived series are C_8, $C_3 \times C_3$, C_3 and C_2^6.

We establish our order and solubility claims as follows:

1. We use IMD to verify that G/G' is $C_2 \times C_8$.

2. We now construct, using RS, a presentation for G'. We then use TZ to simplify this presentation. An IMD computation shows that the torsion invariants of G'/G'' are 51 and 18921, with prime decompositions 3×17 and $3 \times 7 \times 17 \times 53$.

3. Let R be the normal closure of e^3 in G. It is easy to establish that $G'/G'''Y$ has exponent 3 and hence R/Y is abelian. Therefore R is nilpotent because Y is central in G'. Further, R has index $2^4 \times 3^3$ in G. We first construct, using RS, a presentation for R. We then use TZ to simplify this presentation. An IMD calculation on the resulting presentation shows that the torsion invariants of R/R' are 2, 2, 2, 2, 102 and 37842, with the last two invariants having prime decompositions $2 \times 3 \times 17$ and $2 \times 3 \times 7 \times 17 \times 53$. Thus, R has non-trivial p-quotients only for the primes 2, 3, 7, 17 and 53.

4. We use the presentation for R as input to NQ to construct a power-commutator presentation for the largest 2-quotient of R. This shows that

the largest 2-quotient is a 6-generator group having order 2^7 and nilpotency class 2. It is easy to show that this group is a central product of three copies of the quaternion group of order 8. Using NQ, we also establish that for both of 3 and 17 the largest p-quotient of R is C_p^2. Hence $G(7, 8)$ has order $2^{11} \times 3^5 \times 7 \times 17^2 \times 53$.

5. We add the relations $e^{12} = 1$, $[e^6, a] = 1$ and $[e^6, b] = 1$ to our presentation for G. The resulting presentation defines a group whose elements can be explicitly computed. It has order $2^{11} \times 3^3$ and soluble length 5. It is an extension of a central product of three copies of the quaternion group of order 8 by a non-abelian exponent-3 group of order 27 by $C_2 \times C_8$.

The group $H = H(7, 8)$ may be analysed in a similar manner to show that it has order $2^{15} \times 3^6 \times 7 \times 17^2 \times 53$ and soluble length 5. Let R be the normal closure of e^3 in H. The NQ calculations on the presentation of R establish the following:

- the largest 2-quotient has order 2^{11}, nilpotency class 2, and a commutator subgroup of order 2;
- the largest 3-quotient is an elementary abelian group of order 3^3;
- the largest 17-quotient is an elementary abelian group of order 17^2.

Hence, R has nilpotency class 2 and H has soluble length 5. Note that H is another example of a group of deficiency zero having soluble length 5 (see Kenne, 1990).

6. $G(6, 8)$ and $H(6, 8)$

Campbell *et al.* (1992) show that $G = G(6, 8)$ is an infinite group, that it has soluble length 4 or 5, and that the factors of the derived series of $\bar{N}'$ are C_5, C_2^4 and C_∞^5. We establish that both the soluble length and Hirsch length of G are 5.

1. We use IMD to verify that G/G' is $C_2 \times C_8$.

2. We now compute a presentation for G' using RS. An IMD calculation on the simplified presentation obtained from TZ shows that the torsion invariants of G'/G'' are 152 and 2280, with prime decompositions $2^3 \times 19$ and $2^3 \times 3 \times 5 \times 19$.

3. Let R be the inverse image of C_∞^5 in G/Y. Observe that R/Y is torsion-free abelian and that R has index $640 = 2^7 \times 5$ in N. Since Y is central in G', it follows that R is nilpotent.

Coset enumeration shows that the normal closure in N of the subgroup generated by e^5 and $(ed^2)^2$ is R. We construct, using RRS, a presentation for R and simplify it using TZ. The simplified presentation is on 216 generators with 946 relations of total length 8649. Routine IMD calculations on this presentation fail: integer overflow occurs because of its size. Hence, we use modular diagonalisation to show that R/R' is $C_{16} \times C_8 \times C_4 \times C_2 \times C_3 \times C_{19}^2 \times C_\infty^5$; so Y/R' is the torsion part.

4. Since R is nilpotent, we use the presentation on 216 generators as input to NQ to construct a power-commutator presentation for the largest exponent-7 central class 2 quotient of R. Since this quotient has order 7^{10}, it follows that Y is finite, that it is the torsion subgroup of R, and that the Hirsch length of G is 5.

5. We construct a power-commutator presentation for the largest exponent-2 central class 6 quotient of R. This presentation demonstrates that the 2-torsion subgroup of R is $C_{32} \times C_{16} \times C_4 \times C_2$ and the 2-torsion subgroup of R' is $C_2 \times C_2$. Since $R = YG'''$ and Y is central in G', it follows that G'''' is non-trivial and therefore the soluble length of G is 5.

6. Let p be an odd prime. Since $R/Y = (G'/Y)''$, it can be deduced that R/R^pY is a faithful $Z_p(G'/R)$ module. It is straight-forward to check that each faithful irreducible $Z_p(G'/R)$ module M has dimension 5 and that the wedge product $M \wedge M$ is a sum of two 5-dimensional irreducibles. Since $R'/(R')^p$ is a trivial G'/R module, it follows that R' is a 2-group. (We are indebted to Dr L.G. Kovács for a helpful discussion of the representation theory.)

7. Since Y is abelian, we deduce that it is $C_{32} \times C_{16} \times C_4 \times C_2 \times C_3 \times C_{19}^2$. It also follows that G'''' is $C_2 \times C_2$.

If we wish to establish simply that G is infinite, we can use the standard approach of finding a subgroup of small index with an infinite abelian quotient as described by Slattery (1985). An application of the CAYLEY implementation of the low index subgroup algorithm finds a subgroup of index 10 whose derived quotient is $C_2 \times C_4 \times C_\infty$ – for a description of this algorithm, see Neubüser (1982, §6).

We can also demonstrate easily that G/Y has soluble length 4 – we use RS repeatedly to compute presentations for the terms of the derived series of G/Y; the presentation obtained for $(G/Y)'''$ is visibly that of C_∞^5.

A similar investigation of $H(6, 8)$ establishes that:

1. The torsion invariants of H'/H'' are 2, 152 and 2280, with prime decompositions 2, $2^3 \times 19$ and $2^3 \times 3 \times 5 \times 19$.

2. Let Y be the normal closure of e^5 in H and let R be the inverse image of C_∞^5 in H/Y. Modular IMD calculations show that R/R' is $C_{16} \times C_8 \times C_4 \times C_4 \times C_3^2 \times C_{19}^2 \times C_\infty^5$; so Y/R' is the torsion part.

3. The commutator subgroup of R, or equivalently H'''', is $C_2 \times C_4$ and the 2-torsion subgroup of R is $C_{32} \times C_{32} \times C_4 \times C_4$. For each of 3 and 19

the p-quotient of R is abelian and has as a torsion subgroup C_p^2.

4. Since Y is abelian, we deduce that it is $C_{32} \times C_{32} \times C_4 \times C_4 \times C_3^2 \times C_{19}^2$.

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