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THE SPECTRA OF WEIGHTED MEAN OPERATORS ON bv_0

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Abstract

In a series of papers, the author has previously investigated the spectra and fine spectra for weighted mean matrices, considered as bounded operators over various sequence spaces. This paper examines the spectra of weighted mean matrices as operators over bv_0, the space of null sequences of bounded variation.

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Let x be a sequence, c_0 the spaces of null sequences. Then $bv_0 := bv \cap c_0$, where $bv = \{x|\Sigma_k|x_k - x_{k-1}| < \infty\}$. From [6, formula 119] for example, we have a matrix $A\colon bv_0 \to bv_0$ if and only if A has null columns and

(1) $$\|A\|_{bv_0} := \sup_r \sum_n \left|\sum_{k=0}^r a_{nk} - a_{n-1,k}\right| < \infty.$$

A weighted mean matrix is a lower triangular matrix $A = (a_{nk})$ with $a_{nk} = p_k/P_n$, where $p_0 > 0$, $p_n \geq 0$ for $n > 0$ and $P_n := \sum_{k=0}^n p_k$. $B(bv_0)$ will denote the set of bounded linear operators on bv_0, and $\sigma(A)$ will denote the spectrum of A for $A \in B(bv_0)$. The results of this paper are similar to those obtained in [1], but the proofs are different because of the bv_0 norm.

THEOREM 1. *Let A be a weighted mean matrix with $P_n \to \infty$. Then*

$$\sigma(A) \subset \left\{\lambda \,\middle|\, \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}.$$

PROOF. From Okutoyi [2, Lemma 2], $A \in B(bv_0)$. Since $I \in B(bv_0)$, $B = A - \lambda I \in B(bv_0)$. Let λ satisfy $|\lambda - 1/2| > 1/2$. This inequality is equivalent to $\alpha > -1$, where $-1/\lambda = \alpha + i\beta$. From Cass and Rhoades [1, Lemma 1], $D = B^{-1}$ has entries

$$d_{nn} = \frac{P_n}{p_n - \lambda P_n}$$

$$d_{nk} = \frac{(-1)^{n-k}\lambda^{n-k-1}p_k}{P_n}\prod_{j-k}^{n}\frac{P_j}{p_j - \lambda P_j}, \qquad k < n.$$

For $r < n$, it can be shown that

$$\sum_{k=0}^{r} d_{nk} = \frac{(-1)^{n+r}P_r\lambda^{n-r-1}}{(1-\lambda)P_n}\prod_{i=r+1}^{n}\frac{P_i}{p_i - \lambda P_i},$$

and hence that

$$\sum_{k=0}^{r} d_{nk} - d_{n-1,k} = \frac{(-1)^{n+r}P_r\lambda^{n-r-2}p_n}{P_nP_{n-1}}\prod_{i=r+1}^{n}\frac{P_i}{p_i - \lambda P_i}$$
$$= \frac{p_nP_r}{\lambda^2P_nP_{n-1}}\cdot\frac{1}{\prod_{i=r+1}^{n}(1 - c_i/\lambda)},$$

where $c_i := p_i/P_i$.

Since each row sum of B is $1-\lambda$, each row sum of D is $1/(1-\lambda)$, and, from (1),

$$\text{(2)}\quad \|D\|_{bv_0} = \sup_r \sum_{n>r}\left|\sum_{k=0}^{r} d_{nk} - d_{n-1,k}\right|$$
$$= \sup_r\left\{\sum_{k=0}^{r} d_{r+1,k} - d_{rk} + \sum_{n=r+2}^{\infty}\left|\frac{p_nP_r}{\lambda^2P_nP_{n-1}\prod_{i=r+1}^{n}(1 - c_i/\lambda)}\right|\right\}$$
$$\le \sup_r\left\{\frac{1}{|c_{r+1} - \lambda|} + \frac{1}{|\lambda|^2}P_r\sum_{n=r+1}^{\infty}\frac{p_n}{P_nP_{n-1}\prod_{i=r+1}^{n}(1+\alpha c_i)}\right\}.$$

CASE I. Suppose $\alpha \ge 0$. Then $|c_{r+1} - \lambda| = |\lambda|\,|1 - c_{r+1}/\lambda| \ge |\lambda|((1+\alpha c_{r+1}) \ge |\lambda|$, and

$$\|D\|_{bv_0}| \le \sup_r\left\{\frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\sum_{n=r+1}^{\infty}\left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right)\right\} < \infty.$$

CASE II. Suppose $-1 < \alpha < 0$. Then since $0 \le c_r \le 1$, $1 + c_{r+1}\alpha \ge 1 + \alpha > 0$, and it remains to show that the series in (2) converges.

Define

$$f(r) = P_r \sum_{n=r+1}^{\infty} \frac{p_n}{P_n P_{n-1} \prod_{i=r+1}^{n}(1+\alpha c_i)}$$
$$= P_r \prod_{i=0}^{r}(1+\alpha c_i) \sum_{n=r+1}^{\infty} p_n \Big/ \left[P_n P_{n-1} \prod_{i=0}^{n}(1+\alpha_i)\right].$$

Then

$$f(r) - f(r+1) = \frac{p_{r+1} P_r \prod_{i=0}^{r}(1+\alpha c_i)}{P_r P_{r+1} \prod_{i=0}^{r+1}(1+\alpha c_i)}$$
$$+ \left[P_r \prod_{i=0}^{r}(1+\alpha c_i) - P_{r+1} \prod_{i=0}^{r+1}(1+\alpha c_i)\right]$$
$$\times \sum_{n=r+2}^{\infty} p_n / P_{n-1} P_n \prod_{i=0}^{n}(1+\alpha c_i)$$
$$= \frac{c_{r+1}}{1+\alpha c_{r+1}} - (1+\alpha) p_{r+1} \prod_{i=0}^{r}(1+\alpha c_i) \sum_{n=r+2}^{\infty} c_n / P_{n-1} \prod_{i=0}^{n}(1+\alpha c_i).$$

Define

$$g(r) = [f(r) - f(r+1)]/p_{r+1} \prod_{i=0}^{r}(1+\alpha c_i)$$
$$= \frac{c_{r+1}}{p_{r+1} \prod_{i=0}^{r+1}(1+\alpha c_i)} = (1+\alpha) \sum_{n=r+2}^{\infty} c_n / P_{n-1} \prod_{i=0}^{n}(1+\alpha c_i).$$

Then

$$g(r) - g(r+1) = \frac{1}{P_{r+1} \prod_{i=0}^{r+1}(1+\alpha c_i)} - \frac{1}{P_{r+2} \prod_{i=0}^{r+2}(1+\alpha c_i)}$$
$$- (1+\alpha) \frac{c_{r+2}}{P_{r+1} \prod_{i=0}^{r+1}(1+\alpha c_i)}$$
$$= \frac{1}{\prod_{i=0}^{r+2}(1+\alpha c_i)} \left[\frac{1+\alpha c_{r+2}}{P_{r+1}} - \frac{1}{P_{r+2}} - \frac{(1+\alpha)c_{r+2}}{P_{r+1}}\right]$$
$$= \frac{1}{\prod_{i=0}^{r+2}(1+\alpha c_i)} \left[\frac{1}{P_{r+1}} - \frac{1}{P_{r+2}} - \frac{c_{r+2}}{P_{r+1}}\right] = 0.$$

Therefore g is a constant function and

$$g(0) = \frac{c_1}{p_1(1+\alpha c_0)(1+\alpha c_1)} = (1+\alpha) \sum_{n=2}^{\infty} c_n / P_{n-1} \prod_{i=0}^{n}(1+\alpha c_i).$$

But

$$\sum_{n=2}^{\infty} c_n/P_{n-1}\prod_{i=0}^{n}(1+\alpha c_i) = \sum_{n=2}^{\infty} 1/P_{n-1}\prod_{i=0}^{n}(1+\alpha c_i) - \sum_{n=2}^{\infty} 1/P_n\prod_{i=0}^{n}(1+\alpha c_i)$$

$$= \frac{1}{P_1(1+\alpha c_0)(1+\alpha c_1)(1+\alpha c_2)}$$

$$+ \sum_{n=2}^{\infty} \frac{1}{P_n\prod_{i=0}^{n}(1+\alpha c_i)}\left[\frac{1}{1+\alpha c_{n+1}} - 1\right].$$

Therefore

$$g(0) = \frac{c_1}{p_1(1+\alpha c_0)(1+\alpha c_1)} - \frac{(1+\alpha)}{P_1(1+\alpha c_0)(1+\alpha c_1)(1+\alpha c_2)}$$

$$-(1+\alpha)(-\alpha)\sum_{n=2}^{\infty} c_{n+1}\Big/ P_n\prod_{i=0}^{n+1}(1+\alpha c_i)$$

$$= (-\alpha)\left[\frac{1-c_2}{P_1(1+\alpha c_0)(1+\alpha c_1)(1+\alpha c_2)}\right.$$

$$\left. -(1+\alpha)\sum_{n=3}^{\alpha} c_n/P_{n-1}\prod_{i=0}^{n}(1+\alpha c_i)\right]$$

$$= (-\alpha)\left[\frac{1-c_2}{P_1(1+\alpha c_0)(1+\alpha c_1)(1+\alpha c_2)} + g(0) - \frac{c_1}{p_1(1+\alpha c_0)(1+\alpha c_1)}\right.$$

$$\left. + \frac{(1+\alpha)c_2}{P_1(1+\alpha c_0)(1+\alpha c_1)(1+\alpha c_2)}\right]$$

$$= (-\alpha)\left[g(0) + \frac{1+\alpha c_2}{P_1(1+\alpha c_0)(1+\alpha c_1)(1+\alpha c_2)} - \frac{c_1}{p_1(1+\alpha c_0)(1+\alpha c_1)}\right]$$

$$= (-\alpha)g(0),$$

and $(1+\alpha)g(0)=0$, which implies that $g(0)=0$.

Therefore f is also a constant function and

$$\begin{aligned}
f(0) &= P_0\sum_{n=1}^{\infty} c_n/P_{n-1}\prod_{i=1}^{n}(1+\alpha c_i)\\
&= P_0\left[\sum_{n=1}^{\infty} /P_{n-1}\prod_{i=1}^{n}(1+\alpha c_i)-\sum_{n=1}^{\infty}1/P_n\prod_{i=1}^{n}(1+\alpha c_i)\right]\\
&= P_0\left[\frac{1}{P_0(1+\alpha c_1)}+\sum_{n=1}^{\infty}\frac{1}{P_n\prod_{i=1}^{n+1}(1+\alpha c_i)}(1-(1+\alpha c_{n+1}))\right]\\
&= \frac{1}{1+\alpha c_1}-\alpha P_0\sum_{n=1}^{\infty} c_{n+1}/P_n\prod_{i=1}^{n+1}(1+\alpha c_0)\\
&= \frac{1}{1+\alpha c_1}-\alpha P_0\sum_{n=2}^{\infty} c_n/P_{n-1}\prod_{i=1}^{n}(1+\alpha c_i)\\
&= \frac{1}{1+\alpha c_1}-\alpha\left[f(0)-\frac{c_1P_0}{P_0(1+\alpha c_1)}\right],
\end{aligned}$$

or

$$(1+\alpha)f(0)=\frac{1}{1+\alpha c_1}+\frac{\alpha c_1}{1+\alpha c_1}=1.$$

Therefore $f(0)=1/(1+\alpha)$ and D has finite norm.

Set $\delta=\overline{\lim}\, c_n$, $\gamma=\underline{\lim}\, c_n$.

THEOREM 2. *Let A be a weighted mean method with $P_n\to\infty$. Then*

$$\sigma(A)\geq\{\lambda|\,|\lambda-(2-\delta)^{-1}\leq(1-\delta)/(2-\delta)\}\cup S,$$

where $S=\overline{\{c_n|n\geq 0\}}$.

PROOF. Let $B=A-\lambda I$. Fix λ satisfying $|\lambda-(2-\delta)^{-1}|<(1-\delta)/(2-\delta)$ and $\lambda\neq c_n$ for any n.

We may write

$$\begin{aligned}
(3)\quad 1-c_i/\lambda &= (1-p_i/\lambda P_i)\frac{P_i}{P_{i-1}}\cdot\frac{P_{i-1}}{P_i}\\
&= \left(\frac{P_{i-1}+p_i}{P_{i-1}}-\frac{p_i}{\lambda P_{i-1}}\right)\cdot\frac{P_{i-1}}{P_i}=\left[1+\left(1-\frac{1}{\lambda}\right)\frac{p_i}{P_{i-1}}\right]\frac{P_{i-1}}{P_i}.
\end{aligned}$$

We have from (2) that

$$(4)\qquad \|D\|_{bv_0}\geq\sup_r\frac{P_r}{|\lambda|^2}\sum_{n=r+1}^{\infty}P_rc_n/P_{n-1}|\prod_{i=r+1}^{n}(1-c_i/\lambda)|$$

we have from Cass and Rhoades [1, Theorem 2], that the condition on λ implies

$$\left|1+(1-1/\lambda)\frac{P_i}{P_{i-1}}\right| \le 1$$

for all r sufficiently large.

Thus, substituting (3) into (4) yields

$$\|D\|_{bv_0} \ge \sup_r \frac{P_r}{|\lambda|^2} \sum_{n=r+2}^{\infty} P_r \frac{c_n}{P_{n-1}} \cdot \frac{P_n}{P_r},$$

which diverges. Therefore $\lambda \in \sigma(A)$.

If $\lambda = c_n$ for any n, then clearly $\lambda \in \sigma(A)$. Since the spectrum is closed, the proof is complete.

COROLLARY 1. *Let A be a weighted mean method with $P_n \to \infty$, $\delta = 0$. Then*

$$\sigma(A) = \{\lambda | \lambda - 1/2| \le 1/2\}.$$

To prove Corollary 1, combine Theorems 1 and 2, noting that S is already contained in the disc.

Okutoyi [2, Theorem 2.2] is a special case of Corollary 1.

THEOREM 3. *Let A be a weighted mean method with $P_n \to \infty$, $\gamma > 0$. Then*

$$\sigma(A) \subseteq \{\lambda | |\lambda - (2-\gamma)| \le (1-\gamma)/(2-\gamma)\} \cup S.$$

PROOF. We have from Cass and Rhoades [1, Theorem 3] that if λ satisfies $|\lambda - (2-\gamma)^{-1}| > (1-\gamma)/(2-\gamma)$ and $\lambda \ne c_n$ for any n, then $|1+(1-1/\lambda)p_n/P_{i-1}| \ge m > 1$ for all i sufficiently large and $p_i/P_{i-1} < \delta(1-\delta)+1$. Therefore, for all r sufficiently large,

$$\begin{aligned} P_r \sum_{n=r+2}^{\infty} p_n/P_{n-1}P_r \prod_{i=r+1}^{n} |1+(1-1/\lambda)p_i/P_{i-1}| \\ \le \sum_{n=r+2}^{\infty} \frac{p_n}{P_{n-1}} m^{n-r} \\ \le \left(\frac{\delta}{1-\delta}+1\right) \sum_{n=r+2}^{\infty} m^{n-r} < \infty, \end{aligned}$$

and $\|D\|_{bv_0} < \infty$.

COROLLARY 2. *Let A be a weighted mean method with $P_n \to \infty$, $\gamma = \lim c_n > 0$. Then*

$$\sigma(A) = \{\lambda | |\lambda - 1/(2-\gamma)| \leq (1-\gamma)/(2-\gamma)\} \cup E,$$

where $E = \{c_n | c_n < \gamma/(2-\gamma)\}$.

PROOF. Combine Theorem 2 and 3, use the fact that $S - E$ is already contained in the disc, and that E is a finite set.

Given an $A \in B(bv_0)$, $bv_{0_A} := \{x | Ax \in bv_0\}$.

THEOREM 4. *Let A be a weighted mean method with $P_n \to \infty$. Then $bv_{0_A} = bv_0$ if and only if $\theta := \underline{\lim}\, p_{n+1}/P_n > 0$.*

PROOF. From Okutoyi [2, Lemma 2], $A \in B(bv_0)$. If $\theta > 0$, then $p_{n+1}/P_n \geq \theta/2$ for all n sufficiently large. For each n we may write $c_{n+1} = (p_{n+1}/P_n)/(1+p_{n+1}/P_n)$. The function $f(y) := y/(1+y)$ is monotone increasing in y, so that, for all $n \geq N$, $c_{n+1} \geq \theta/(2+\theta)$, and the diagonal entries of A are nonzero for $n \geq N$.

If A has any zero diagonal entries for $n < N$, replace this entry with a 1, and call the new matrix B. Since $p_k = 0$ for some k implies $a_{nk} = 0$ for all n $p_{nk} = 0$ for all $n > k$ and $b_{kk} = 1$. Thus $bv_{0_B} = bv_{0_A}$.

Since B is a triangle it has a unique two sided inverse.

Now let N denote the largest integer for which $p_N = 0$. Then B agrees with A for all columns $k > N$. In column N, B^{-1} contains a finite number of nonzero entries. The number is 3 if $p_{N-1} \neq 0$ and 4 if $p_{N_1} = 0$. Let M denote the largest number of nonzero entries in a column of B^{-1} for values of $k < N$. Then

$$\begin{aligned}\|B^{-1}\|_{bv_0} &= \sup_r \sum_n \left| \sum_{k=0}^{r} (b_{nk}^{-1} - b_{n-1,k}^{-1}) \right| \\ &\leq \sup_{r \leq m+N} \sum_{n=0}^{m+N} \left| \sum_{k=0}^{r} (b_{nk}^{-1} - b_{n-1,k}^{-1}) \right| + \sup_{r > m+N} \sum_{n > m+N} \left| \sum_{k=0}^{r} (b_{nk}^{-1} - b_{n-1,k}^{-1}) \right|.\end{aligned}$$

For values of $n > m + N$, the row sums of B^{-1} are one. Therefore, if $r \geq n$, $\sum_{k=0}^{r} (b_{nk}^{-1} - b_{n-1,k}^{-1}) = 0$, and we need only consider $n > r$.

On the other hand, if $n > r + 2$, the corresponding rows of B^{-1} will contain only the two nonzero term b_{nn}^{-1} and $b_{n,n-1}^{-1}$, so that, again the inner summation is zero.

Thus

$$\begin{aligned}\sum_{n=m+N+1}^{\infty}&\left|\sum_{k=0}^{r}(b_{nk}^{-1}-b_{n-1,k}^{-1})\right| \\ &= \left|\sum_{k=0}^{r}(b_{r+1,k}^{-1}-b_{rk}^{-1})\right| + \left|\sum_{k=0}^{r}(b_{r+1,k}^{-1}-b_{rk}^{-1})\right| \\ &= |1-1/c_{r+1}-1| + |0-1+b_{r+1,r+1}^{-1}| \\ &= \frac{1}{c_{r+1}}\left|\frac{1}{c_{r+1}}-1\right| \le \frac{2}{c_{r+1}} \le \frac{2(2+\theta)}{\theta},\end{aligned}$$

and $\|B^{-1}\|_{bv_0} < \infty$.

Since B^{-1} has null columns, $B^{-1} \in B(bv_0)$ and $bv_{0_B} = bv_0$.

Suppose $\theta = 0$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\lim_k p_{n_k+1}/P_{n_k} = 0$.

Case I. Suppose $p_n = 0$ for at most a finite number of values of n. Let N denote the largest value of n for which $p_n = 0$. Again let M denote the largest number of nonzero entries in any column of B^{-1} for $k > N$. Choose m so that $n_m > N + M$.

Since $c_{n+1} = (p_{n+1}/.P_n)/(1 + p_{n+1}/P_n)$, $\lim_k c_{n_k} + 1 = 0$ and

$$\begin{aligned}\|B^{-1}\| &\ge \sup_{r=n_j}\sum_{n}\left|\sum_{k=0}^{r}(b_{n_k}^{-1}-b_{n-1,k}^{-1})\right| \\ &\ge \sup_{j\ge m}\left|\sum_{k=0}^{r}(b_{n_j+1,k}^{-1}-b_{n_j,k}^{-1})\right| \\ &= \sup_{j\ge m}\frac{1}{c_{n_j+1}} = \infty.\end{aligned}$$

Case II. Suppose $p_n = 0$ for infinitely many values of n. Let $\{n_k\}$ denote this set. Then $\{n_k\}$ will contain either an infinite number of even integers or an infinite number of odd integers. If it contains an infinite number of both, discard the even integers. Call the resulting sequence $\{n_r\}$. Define a sequence x by $x_k = 1$ for $k = n_r$, $x_k = 0$ otherwise. Then $x \notin bv_0$, but $Ax = 0 \in bv_0$. Therefore $bv_{0_A} \ne bv_0$.

In [1] examples were provided to show that, if $\delta = \overline{\lim}\, c_n > \underline{\lim}\, c_n = \gamma$, then the spectrum could consist of either an oval, two ovals tangent at a point, or two distinct ovals. The weighted mean matrices used for these examples were defined by $c_0 = 1$, $c_{2n} = 1/p$, $c_{2n-1} = 1/q$, $n > 0$, when $1 < p < q$.

If A denotes such a matrix then, as in [1], it can be shown that

$$\sigma(A) = \{\lambda \,|\, |\lambda|^2(p-1)(q-1) \geq |1-p\lambda|\,|1-q\lambda|\}.$$

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