

## THE ISOMETRIES OF $H^p(K)$

PEI-KEE LIN

(Received 11 July 1989)

Communicated by S. Yamamuro

### Abstract

Let  $1 \leq p < \infty$ ,  $p \neq 2$  and let  $K$  be any complex Hilbert space. We prove that every isometry  $T$  of  $H^p(K)$  onto itself is of the form

$$(TF)(z) = U(F \circ \phi(z)) \cdot (d\phi/dz)^{1/p} \quad (F \in H^p(K), |z| < 1),$$

where  $U$  is a unitary operator on  $K$  and  $\phi$  is a conformal map of the unit disc onto itself.

1980 *Mathematics subject classification* (Amer. Math. Soc.) (1985 Revision): 46 E 15, 46 E 30.

*Keywords and phrases*: Hardy space, isometry, Hilbert space, conformal map.

### 1. Introduction

Let  $D$  be the open unit disc in the complex plane and let  $E$  be any complex Banach space. Then the Banach space  $H^p(E)$ ,  $1 \leq p \leq \infty$ , consists of all  $F : D \rightarrow E$  such that  $\langle F, e^* \rangle$  belongs to the Hardy class  $H^p$  for all  $e^* \in E^*$ , and the norm of  $F$  is given by

$$\|F\|_p = \lim_{r \rightarrow 1^-} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \|F(re^{i\theta})\|^p d\theta \right\} \quad \text{if } 1 \leq p < \infty,$$
$$\|F\|_\infty = \operatorname{ess\,sup}_{|z| < 1} \|F(z)\|.$$

A complex Banach space  $E$  is said to have the *analytic Radon-Nikodym property*, if for each  $F \in H^p(E)$ ,  $F(e^{i\theta}) = \lim_{r \rightarrow 1^-} F(re^{i\theta})$  exists almost everywhere (for more detail see [1] and [4]). (It is known that the  $L_p$ -spaces,

---

This research was supported in part by NSF.

© 1991 Australian Mathematical Society 0263-6115/91 \$A2.00 + 0.00

$1 \leq p < \infty$ , have the analytic Radon-Nikodym property.) If  $E$  has the analytic Radon-Nikodym property and if  $F \in H^p(E)$ , then the norm of  $F$  is

$$\|F\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \|F(e^{i\theta})\|^p d\theta \right\}^{1/p} \quad \text{if } 1 \leq p < \infty,$$

$$\|F\|_\infty = \operatorname{ess\,sup}_{0 \leq \theta \leq 2\pi} \|F(e^{i\theta})\|.$$

The linear isometries of  $H^p$  were first studied by deLeeuw, Rudin, and Wermer [6]. They proved that if  $T$  is a surjective isometry on  $H^1$  (respectively,  $H^\infty$ ), then there are a conformal map  $\phi$  of the unit disk onto itself and a unimodular complex number  $b$  such that

$$Tf = b \cdot (d\phi/dz) \cdot f \circ \phi \quad (\text{respectively, } Tf = b \cdot f \circ \phi).$$

Later, F. Forelli [7] extended this result to  $H^p$  for  $p \neq 2$ . He proved that

**THEOREM A.** *If  $p \neq 2$  and if  $T$  is a linear isometry of  $H^p$  onto  $H^p$ , then there is a conformal map  $\phi$  of the unit disk onto itself and a unimodular complex number  $b$  such that*

$$Tf = b \cdot (d\phi/dz)^{1/p} \cdot f \circ \phi.$$

The isometries of the vector valued  $H^p$  function spaces were studied by M. Cambern. He [2] showed that if  $K$  is a complex Hilbert space and if  $T$  is a surjective isometry on  $H^\infty(K)$ , then there are a conformal map  $\phi$  of the unit disc onto itself and a unitary operator  $U$  on  $K$  such that for any  $F \in H^\infty(K)$  and any  $z \in D$ ,

$$TF(z) = U(F \circ \phi(z)).$$

Recently, M. Cambern and K. Jarosz [3] proved a similar result holds on  $H^1(K)$  if  $K$  is a finite dimensional complex Hilbert space. In this article, we extend this result to  $H^p(K)$ ,  $1 \leq p < \infty$ . The main result of this article is the following theorem.

**MAIN THEOREM.** *Let  $1 \leq p < \infty$ ,  $p \neq 2$ , and let  $K$  be any complex Hilbert space. If  $T: H^p(K) \rightarrow H^p(K)$  is a surjective isometry, then there exist a unitary operator  $U$  on  $K$ , and a conformal map  $\phi$  from the disc onto the disc such that*

$$(TF)(z) = U(F \circ \phi(z)) \cdot (d\phi/dz)^{1/p}(z) \quad (F \in H^p(K), |z| < 1).$$

Let  $\phi$  be a conformal map of the unit disk onto itself, and let  $U$  be a unitary operator on a complex Hilbert space  $K$ . If

$$TF(z) = U(F \circ \phi(z)) \cdot (d\phi/dz)^{1/p}(z)$$

for all  $F \in H^p(K)$ , then  $T$  satisfies the following conditions.

- (a) If  $\langle e_1, e_2 \rangle = 0$ , then  $\langle T(fe_1)(e^{i\theta}), T(ge_2)(e^{i\theta}) \rangle = 0$  a.e. for all  $f, g \in H^p$ .
- (a') If  $\langle e_1, e_2 \rangle = 0$ , then  $\langle T(z^n e_1)(e^{i\theta}), T(z^m e_2)(e^{i\theta}) \rangle = 0$  a.e. for all  $n, m \geq 0$ .
- (b)  $|\langle T(fe)(e^{i\theta}), T(ge)(e^{i\theta}) \rangle| = \|T(fe)(e^{i\theta})\| \cdot \|T(ge)(e^{i\theta})\|$  a.e. (that is  $T(fe)(e^{i\theta})$  and  $T(ge)(e^{i\theta})$  are linearly dependent a.e.) for all  $e \in K$  and  $f, g \in H^p$ .
- (b')  $|\langle T(z^n e)(e^{i\theta}), T(z^m e)(e^{i\theta}) \rangle| = \|T(z^n e)(e^{i\theta})\| \cdot \|T(z^m e)(e^{i\theta})\|$  a.e. for all  $e \in K$  and  $n, m \geq 0$ .
- (c) For any  $e \in K$ , there is  $e' \in K$  such that  $T(H^p e) = H^p e'$ . Moreover, if  $T(H^p e_1) = H^p e_3$ ,  $T(H^p e_2) = H^p e_4$ , and  $\langle e_1, e_2 \rangle = 0$ , then  $\langle e_3, e_4 \rangle = 0$ .

Clearly, (a) implies (a'), (b) implies (b'), and (c) implies (a) and (b). Since  $\{z^n: n \geq 0\}$  spans  $H^p$ , (a') implies (a) and (b') implies (b). By Theorem A, one can show that if  $T$  is a surjective isometry on  $H^p(K)$  which satisfies (c), then  $T$  satisfies the conclusion of the Main Theorem (see Section 3). Hence, we only need to show every surjective isometry on  $H^p(K)$  satisfies (c). However, we do not know any direct proof. In Section 2, we establish the following proposition.

**PROPOSITION 1.** *Suppose that  $1 \leq p < \infty$ , and  $p \neq 2$ . If  $K$  is a complex Hilbert space and if  $T$  is a linear isometry of  $H^p(K)$  onto  $H^p(K)$ , then  $T$  satisfies (a').*

If  $1 \leq p < 2$ , we provide a direct proof of Proposition 1. But we do not know whether there is a direct proof if  $2 < p < \infty$ . In this case, we first show that  $T$  satisfies (b'), and then use (b) to prove Proposition 1. In Section 3, we use the conclusion of Proposition 1 to show that every surjective isometry on  $H^p(K)$  satisfies (c), and then give the proof of the Main Theorem.

## 2. The proof of Proposition 1

In this section, we will assume that (i)  $1 \leq p < \infty$  and  $p \neq 2$ , (ii)  $K$  is a complex Hilbert space, and (iii)  $T$  is a surjective isometry of  $H^p(K)$  onto itself. Before proving Proposition 1, we need the following fact.

**FACT 1.** Suppose that  $1 \leq p < 2$ . If  $f_1, f_2$  are any two positive functions in  $L^p$  such that  $f_2 > 0$  a.e. and  $\|f_1\|_p = 1 = \|f_2\|_p$ , then

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{f_1^2(t)}{f_2^{2-p}(t)} dt \geq 1.$$

Moreover, equality holds if and only if  $f_1 = f_2$  a.e.

**PROOF OF PROPOSITION 1** WHEN  $1 \leq p < 2$ . Let  $e_1$  and  $e_2$  be any two nonzero vectors in  $K$ . If  $\|re_2\| < \|e_1\|$ , then

$$\begin{aligned} \|e_1 + re^{ix}e_2\|^p &= (\|e_1\|^2 + r^2\|e_2\|^2 + r(e^{-ix}\langle e_1, e_2 \rangle + e^{ix}\langle e_2, e_1 \rangle))^p \\ &= \|e_1\|^p \left( 1 + \frac{r}{\|e_1\|^2} (r\|e_2\|^2 + (e^{-ix}\langle e_1, e_2 \rangle + e^{ix}\langle e_2, e_1 \rangle)) \right)^{p/2} \\ &= \|e_1\|^p \sum_{j=0}^{\infty} \binom{p/2}{j} \left( \frac{r}{\|e_1\|^2} \right)^j (r\|e_2\|^2 + (e^{-ix}\langle e_1, e_2 \rangle + e^{ix}\langle e_2, e_1 \rangle))^j \\ &= \sum_{j=0}^{\infty} \binom{p/2}{j} \|e_1\|^{p-2j} r^j (r\|e_2\|^2 + (e^{-ix}\langle e_1, e_2 \rangle + e^{ix}\langle e_2, e_1 \rangle))^j, \end{aligned}$$

and so

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \|e_1 + re^{ix}e_2\|^p dx \\ (*) \quad &= \sum_{j=0}^{\infty} \sum_{l=0}^{\lfloor \frac{j}{2} \rfloor} \binom{p/2}{j} \binom{j}{2l} \binom{2l}{l} r^{2j-2l} \|e_1\|^{p-2j} \|e_2\|^{2j-4l} |\langle e_1, e_2 \rangle|^{2l}. \end{aligned}$$

This implies

$$\begin{aligned} &\lim_{r \rightarrow 0} \frac{1}{r^2} \left( \frac{1}{2\pi} \int_0^{2\pi} \|e_1 + re^{ix}e_2\|^p dx - \|e_1\|^p \right) \\ &= \frac{p}{2} \left( \|e_1\|^{p-2} \|e_2\|^2 + \frac{p-2}{2} \|e_1\|^{p-4} |\langle e_1, e_2 \rangle|^2 \right). \end{aligned}$$

Let  $F = T(z^m e_1)$  and  $G = T(z^n e_2)$ . Then  $\|F(e^{i\theta})\| \neq 0$  a.e. By Fatou's Lemma and the Fubini Theorem,

$$\begin{aligned}
& \frac{p}{4\pi} \int_0^{2\pi} \|F(e^{i\theta})\|^{p-2} \|G(e^{i\theta})\|^2 + \frac{p-2}{2} \|F(e^{i\theta})\|^{p-4} |\langle F(e^{i\theta}), G(e^{i\theta}) \rangle|^2 d\theta \\
& \leq \liminf_{r \rightarrow 0} \frac{1}{4r^2\pi^2} \int_0^{2\pi} \int_0^{2\pi} \|F(e^{i\theta}) + re^{ix}G(e^{i\theta})\|^p - \|F(e^{i\theta})\|^p dx d\theta \\
& = \liminf_{r \rightarrow 0} \frac{1}{4r^2\pi^2} \int_0^{2\pi} \int_0^{2\pi} \|F(e^{i\theta}) + re^{ix}G(e^{i\theta})\|^p - \|F(e^{i\theta})\|^p d\theta dx \\
& = \liminf_{r \rightarrow 0} \frac{1}{2r^2\pi} \int_0^{2\pi} \|F + re^{ix}G\|_p^p - \|F\|_p^p dx \\
& = \liminf_{r \rightarrow 0} \frac{1}{2r^2\pi} \int_0^{2\pi} \|z^m e_1 + re^{ix} z^n e_2\|_p^p - \|z^m e_1\|_p^p dx \\
& = \liminf_{r \rightarrow 0} \frac{((1+r^2)^{p/2} - 1)}{r^2} = \frac{p}{2}.
\end{aligned}$$

But  $\|F\|_p = \|z^m e_1\|_p = 1 = \|z^n e_2\|_p = \|G\|_p$ . By Fact 1, we have  $\|F(e^{i\theta})\| = \|G(e^{i\theta})\|$  a.e. and  $\langle F(e^{i\theta}), G(e^{i\theta}) \rangle = 0$  a.e.

Now, we assume  $2 < p < \infty$ , and we need the following lemma.

**LEMMA 2.** *Let  $m \neq n$ , and let  $e$  be any nonzero element in  $K$ . If  $F = T(z^m e)$  and  $G = T(z^n e)$ , then  $\|F(e^{i\theta})\| = \|G(e^{i\theta})\|$  a.e., and*

$$|\langle F(e^{i\theta}), G(e^{i\theta}) \rangle| = \|F(e^{i\theta})\| \cdot \|G(e^{i\theta})\| \quad \text{a.e.}$$

**PROOF.** By (\*), there exists  $A > 0$  such that for any two nonzero vectors  $e_1$  and  $e_2$  in  $K$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} \|e_1 + re^{ix}e_2\|^p - \|e_1\|^p dx \leq Ar^2 \|e_1\|^{p-2} \|e_2\|^2$$

whenever  $\|e_1\| > 2r\|e_2\|$ . On the other hand, if  $\|e_1\| \leq 2r\|e_2\|$ , then

$$\frac{1}{2\pi} \int_0^{2\pi} \|e_1 + re^{ix}e_2\|^p - \|e_1\|^p dx \leq (3^p + 1)r^p \|e_2\|^p.$$

So

$$\begin{aligned}
& \frac{1}{2\pi r^2} \int_0^{2\pi} \|F(e^{i\theta}) + re^{ix}G(e^{i\theta})\|^p - \|F(e^{i\theta})\|^p dx \\
& \leq \max(A, 3^p + 1) (\|F(e^{i\theta})\|^{p-2} \|G(e^{i\theta})\|^2 + \|G(e^{i\theta})\|^p)
\end{aligned}$$

for all  $0 < r < 1$ . By the dominated convergence theorem,

$$\begin{aligned}
 & \frac{p}{4\pi} \int_0^{2\pi} \|F(e^{i\theta})\|^{p-2} \|G(e^{i\theta})\|^2 + \frac{p-2}{2} \|F(e^{i\theta})\|^{p-4} |\langle F(e^{i\theta}), G(e^{i\theta}) \rangle|^2 d\theta \\
 &= \lim_{r \rightarrow 0} \frac{1}{4r^2\pi^2} \int_0^{2\pi} \int_0^{2\pi} \|F(e^{i\theta}) + re^{ix}G(e^{i\theta})\|^p - \|F(e^{i\theta})\|^p dx d\theta \\
 &= \lim_{r \rightarrow 0} \frac{1}{4r^2\pi^2} \int_0^{2\pi} \int_0^{2\pi} \|F(e^{i\theta}) + re^{ix}G(e^{i\theta})\|^p - \|F(e^{i\theta})\|^p d\theta dx \\
 &= \lim_{r \rightarrow 0} \frac{1}{2r^2\pi} \int_0^{2\pi} \|F + re^{ix}G\|_p^p - \|F\|_p^p dx \\
 &= \lim_{r \rightarrow 0} \frac{1}{2r^2\pi} \int_0^{2\pi} \|z^m e + re^{ix}z^n e\|_p^p - \|z^m e\|_p^p dx \\
 &= \lim_{r \rightarrow 0} \frac{1}{4r^2\pi^2} \int_0^{2\pi} \int_0^{2\pi} |1 + re^{ix+i(n-m)\theta}|^p - 1 d\theta dx \\
 &= \lim_{r \rightarrow 0} \frac{1}{4r^2\pi^2} \int_0^{2\pi} \int_0^{2\pi} |1 + re^{ix+i(n-m)\theta}|^p - 1 dx d\theta = \frac{p^2}{4}.
 \end{aligned}$$

This implies

$$p = \frac{1}{\pi} \int_0^{2\pi} \|F(e^{i\theta})\|^{p-2} \|G(e^{i\theta})\|^2 + \frac{p-2}{2} \|F(e^{i\theta})\|^{p-4} |\langle F(e^{i\theta}), G(e^{i\theta}) \rangle|^2 d\theta.$$

By Hölder inequality, we have  $\|F(e^{i\theta})\| = \|G(e^{i\theta})\|$  a.e. and

$$|\langle G(e^{i\theta}), F(e^{i\theta}) \rangle| = \|F(e^{i\theta})\|^2 \quad \text{a.e.}$$

**PROOF OF PROPOSITION 1 WHEN  $2 < p < \infty$ .** By Lemma 2, for any  $e \in K$ ,  $T(z^m e)(e^{i\theta})$  and  $T(z^n e)(e^{i\theta})$  are linearly dependent for almost all  $\theta \in [0, 2\pi]$ . Since  $\{z^n : n \geq 0\}$  spans  $H^p$ , for any  $e \in K$  and any  $f, g \in H^p$ ,  $T(fe)(e^{i\theta})$  and  $T(ge)(e^{i\theta})$  are linear dependent for almost all  $\theta \in [0, 2\pi]$ . Hence, for each  $e \in K \setminus \{0\}$ ,  $T|_{H^p_e}$  induces an isometry from  $H^p$  into  $L^p$ . By the proof of [7, Theorem 1], there is a function  $h_e$  such that

$$(1) |h_e(e^{i\theta})| = 1 \quad \text{a.e.},$$

$$(2) \text{ for each } n \in \mathbb{N}, T(z^n e) = h_e^n T(1_D e).$$

Clearly, if  $h_e = h_{e'}$ , then  $h_e = h_{\alpha e + \beta e'}$  for all  $\alpha, \beta \in \mathbb{C}$ .

(1) Let  $e, e'$  be any two unit elements in  $K$ . We claim that  $h_e = h_{e'}$ . Since

$$h_e T(1_D e) + h_{e'} T(1_D e') = T(ze + ze') = h_{e+e'} T(1_D (e + e')),$$

and

$$h_e T(1_D e) + i h_{e'} T(1_D e') = T(ze + i z e') = h_{e+ie'} T(1_D(e + i e')),$$

for almost all  $\theta \in [0, 2\pi]$  we have

$$\begin{aligned} & \|T(1_D e)(e^{i\theta})\|^2 + \|T(1_D e')(e^{i\theta})\|^2 \\ & + h_e(e^{i\theta}) \bar{h}_{e'}(e^{i\theta}) \langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta}) \rangle \\ & + h_{e'}(e^{i\theta}) \bar{h}_e(e^{i\theta}) \langle T(1_D e')(e^{i\theta}), T(1_D e)(e^{i\theta}) \rangle \\ & = \|T(1_D e)(e^{i\theta})\|^2 + \|T(1_D e')(e^{i\theta})\|^2 + \langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta}) \rangle \\ & + \langle T(1_D e')(e^{i\theta}), T(1_D e)(e^{i\theta}) \rangle \end{aligned}$$

and

$$\begin{aligned} & \|T(1_D e)(e^{i\theta})\|^2 + \|T(1_D e')(e^{i\theta})\|^2 \\ & - i h_e(e^{i\theta}) \bar{h}_{e'}(e^{i\theta}) \langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta}) \rangle \\ & + i h_{e'}(e^{i\theta}) \bar{h}_e(e^{i\theta}) \langle T(1_D e')(e^{i\theta}), T(1_D e)(e^{i\theta}) \rangle \\ & = \|T(1_D e)(e^{i\theta})\|^2 + \|T(1_D e')(e^{i\theta})\|^2 - i \langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta}) \rangle \\ & + i \langle T(1_D e')(e^{i\theta}), T(1_D e)(e^{i\theta}) \rangle. \end{aligned}$$

So

$$\langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta}) \rangle = h_e(e^{i\theta}) \bar{h}_{e'}(e^{i\theta}) \langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta}) \rangle.$$

Replacing  $e'$  by  $e' + re$  for some  $r \in \mathbb{R}$  if necessary, we may assume that

$$\langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta}) \rangle \neq 0 \quad \text{a.e.}$$

Therefore,  $h_e = h_{e'}$  a.e., and if  $F \in H^p(K)$ , then  $T(z^n F) = h_e^n T(F)$ .

(2) Since  $T$  is an onto mapping, there is  $F \in H^p(K)$  such that  $T(F) = 1_D e$ . So  $T(zF) = h_e 1_D e$  and  $h_e \in H^\infty$ .

(3) By (2) there exist two inner functions  $h$  and  $h'$  such that for any  $g \in H^\infty$  and  $F \in H^p(K)$ ,  $T(gF) = g \circ h \cdot T(F)$  and  $T^{-1}(gF) = g \circ h' \cdot T^{-1}(F)$ . From  $TT^{-1}(F) = F = T^{-1}T(F)$ , we find

$$g \circ h' \circ h \cdot F = g \cdot F = g \circ h \circ h' \cdot F.$$

So  $h \circ h' = I = h' \circ h$  and  $h$  is a conformal map of the unit disk onto itself.

(4) Since  $h$  is an onto conformal mapping, for any  $f \in H^\infty$  there is  $g \in H^\infty$  such that  $f = g \circ h$ . Hence, if  $e, e'$  are any two unit vectors in  $K$ ,

and  $g$  is any function in  $H^\infty$ , then we have

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \|T(1_D e)(e^{i\theta})\|^p d\theta \\ &= \|fT(1_D e)\|_p^p = \|T(ge)\|_p^p \\ &= \|T(ge')\|_p^p = \|fT(1_D e')\|_p^p \\ &= \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \|T(1_D e')(e^{i\theta})\|^p d\theta. \end{aligned}$$

Since  $\{|g|: g \in H^\infty\}$  spans real  $L^\infty$ ,  $\|T(1_D e)(e^{i\theta})\| = \|T(1_D e')(e^{i\theta})\|$  a.e.

(5) Now, suppose that  $\langle e, e' \rangle = 0$ . Then there exists a measure zero subset  $A$  of  $[0, 2\pi]$  such that if  $\theta \notin A$  and  $\alpha \in \mathbb{Q}$ , then

$$\|\cos \alpha T(1_D e)(e^{i\theta}) + \sin \alpha T(1_D e')(e^{i\theta})\| = \|T(1_D e)(e^{i\theta})\|.$$

By continuity,

$$\|\cos \alpha T(1_D e)(e^{i\theta}) + \sin \alpha T(1_D e')(e^{i\theta})\| = \|T(1_D e)(e^{i\theta})\|$$

whenever  $\alpha \in \mathbb{R}$  and  $\theta \notin A$ . So we have  $\langle T(1_D e)(e^{i\theta}), T(1_D e')(e^{i\theta}) \rangle = 0$ .

□

**REMARK 1.** Let  $T: H^p(K) \rightarrow H^p(K)$  be an onto isometry. If  $e_1$  and  $e_2$  are linearly independent, then  $T(fe_1)(e^{i\theta})$  and  $T(fe_2)(e^{i\theta})$  are linearly independent for almost for  $\theta$ .

### 3. Proof of Theorem

Before proving the Main Theorem, we need another lemma.

**LEMMA 3.** For any unit vector  $e$  in  $K$ , there exists a unit vector  $e'$  such that  $T(H^p e) = H^p e'$ .

**PROOF.** Let  $\{e_j: j \in J\}$  be an orthonormal basis of  $K$ . For any unit vector  $e \in K$ , there exist  $F_j \in T(H^p e_j)$ , such that  $1_D e = \sum_{j \in J} F_j$ . Clearly,  $F_j = 0$  except for countably many  $j$ . Hence, we may assume that  $J$  is countable and  $F_j(e^{i\theta})$ 's are orthogonal. So for any  $\theta \in [0, 2\pi]$ , we have

$$(i) \sum_{j \in J} \langle F_j(e^{i\theta}), e \rangle = 1_D,$$

$$(ii) \sum_{j \in J} \|F_j(e^{i\theta})\|^2 = 1.$$

(1) For each  $j \in J$ ,  $\langle F_j(e^{i\theta}), e \rangle$  is an analytic function and for any  $\theta \in [0, 2\pi]$



$$1 = \left\| \sum_{j \in J} \overline{\text{sgn}(\langle F_j(e^{i\theta}), e \rangle)} F_j(e^{i\theta}) \right\| \geq \sum_{j \in J} |\langle F_j(e^{i\theta}), e \rangle| \geq 1.$$

So  $\langle F_j(e^{i\theta}), e \rangle$  is a non-negative constant function for each  $j \in J$ .

(2) If  $\langle F_k(e^{i\theta}), e \rangle = 0$ , then

$$1 \leq \left\| \sum_{j \neq k} F_j \right\|_p \leq \left\| \sum_{j \in J} F_j \right\|_p = 1.$$

The second inequality holds if and only if  $F_k \neq 0$ . So we must have  $F_k = 0$  if  $\langle F_k(e^{i\theta}), e \rangle = 0$ .

(3) Let  $e'$  be a nonzero element in  $K$  and  $k$  be a fixed element in  $J$ . We claim that if there exists an  $f \in H^p$  such that  $m\{\theta: \|T(fe_k)(e^{i\theta}) - e'\| < 1/n\} > 0$  for every  $n \in \mathbb{N}$ , then  $1_D e' \in T(H^p e_k)$ . With loss of generality, we may assume  $\|e'\| = 1$ . Since there exist  $F_j \in T(H^p e_j)$  such that  $\sum_{j \in J} F_j = 1_D e'$ , by Proposition 1, there exists a measurable set  $A$  such that

(iii)  $m(A) = 0$ ,

(iv) if  $\theta \notin A$ , then  $\{F_j(e^{i\theta}): j \in J\}$  (respectively  $\{T(fe_k)(e^{i\theta})\} \cup \{F_j(e^{i\theta}): j \neq k\}$ ) is orthogonal.

Hence, if  $\theta \notin A$ , then there exist  $1 \geq a \geq 0$ ,  $b \in \mathbb{C}$  and  $z, y \in K$  which satisfy

(v)  $\langle z, e' \rangle = 0 = \langle y, e' \rangle$ ,

(vi)  $F_k(e^{i\theta}) = ae' + z$ ,  $\sum_{j \neq k} F_j(e^{i\theta}) = (1-a)e' - z$ , and  $T(fe_k)(e^{i\theta}) = be' + y$ .

So we have

$$b(1-a) - \langle z, y \rangle = \left\langle \sum_{j \neq k} F_j(e^{i\theta}), F_k(e^{i\theta}) \right\rangle = 0,$$

$$a(1-a) - \|z\|^2 = \langle T(fe_k)(e^{i\theta}), f_k(e^{i\theta}) \rangle = 0.$$

If  $\|be' + y - e'\| < 1/n$ , then  $|b| \geq 1 - 1/n$ ,  $\|y\| \leq 1/n$ ,

$$\frac{\|z\|}{n} \geq |\langle z, y \rangle| = |b(1-a)| \geq \left(1 - \frac{1}{n}\right)(1-a),$$

$$\|z\| \geq (n-1)(1-a), \quad a(1-a) = \|z\|^2 \geq (n-1)^2(1-a)^2.$$

So we have  $a \geq (n-1)^2/n^2$ . But  $\langle F_k, e' \rangle$  is a constant function, so  $\langle F_k, e' \rangle \equiv 1$ . By (1) and (2),  $F_j \equiv 0$  for all  $j \neq i$ , and  $F_k \equiv 1$ .

Suppose that there exist  $e'_1, e'_2 \in K$  and  $f_1, f_2 \in H^p$  such that

$$m\{\theta: \|T(f_1 e)(e^{i\theta}) - e'_1\| < 1/n\} > 0$$

(respectively,  $m\{\theta: \|T(f_2 e)(e^{i\theta}) - e'_2\| < 1/n\} > 0$ ) for all  $n \in \mathbb{N}$ . By (3),  $1_D e'_1$  and  $1_D e'_2$  are in  $T(H^p e)$ . But  $T^{-1}$  is a surjective isometry from  $H^p(K)$  onto  $H^p(K)$ . By Remark 1,  $e'_1$  and  $e'_2$  are linearly dependent. And we have proved the lemma.

**PROOF OF THEOREM.** Let  $e$  be any unit vector in  $K$ . By Lemma 3, there exists a unit vector  $e'$  such that  $T(fe) = \langle T(fe), e' \rangle e'$ . We define the operator (it may not be linear)  $U$  by  $U(ce) = ce'$  for all  $c \in \mathbb{C}$ .

By Lemma 3, the restriction of  $T$  to  $H^p e_1$  is a surjective isometry from  $H^p e_1$  into  $H^p U(e_1)$ . Hence, there exist a conformal map  $\phi_1$  of the disc onto itself, and a unimodular complex number  $b_1$  such that  $T(fe_1) = b_1 \cdot (d\phi_1/dz)^{1/p} \cdot f \circ \phi_1 \cdot U(e_1)$ . (Replacing  $U(e_1)$  by  $b_1 U(e_1)$ , we may assume that  $b_1 = 1$ .) If  $e_2$  is any other vector in  $K$ , then there exist a conformal map  $\phi_2$  of the disc onto itself, and a unimodular complex number  $b_2$  such that  $T(fe_2) = b_2 \cdot (d\phi_2/dz)^{1/p} \cdot f \circ \phi_2 \cdot U(e_2)$ . We claim that  $\phi_2 = \phi_1$ . Clearly, this is true if  $e_1$  and  $e_2$  are linearly dependent. So we may assume that  $e_1$  and  $e_2$  are linearly independent. By Lemma 3,

$$\begin{aligned} & (d\phi_1/dz)^{1/p} \cdot f \circ \phi_1 \cdot U(e_1) + b_2 \cdot (d\phi_2/dz)^{1/p} \cdot f \circ \phi_2 \cdot U(e_2) \\ &= T(f(e_1 + e_2)) = \left\langle T(f(e_1 + e_2)), U\left(\frac{e_1 + e_2}{\|e_1 + e_2\|}\right) \right\rangle U\left(\frac{e_1 + e_2}{\|e_1 + e_2\|}\right). \end{aligned}$$

Since  $U(e_1)$  and  $U(e_2)$  are linearly independent (by Remark 1), we have  $(d\phi_1/dz)^{1/p} f \circ \phi_1$  and  $(d\phi_2/dz)^{1/p} f \circ \phi_2$  are linearly dependent. Let  $f = 1$ . Then we have  $(d\phi_2/dz) = d_1(d\phi_1/dz)$  or  $\phi_2 = d_1\phi_1 + d_2$  for some  $d_1, d_2 \in \mathbb{C}$ . But  $\phi_1$  and  $\phi_2$  are conformal maps from the unit disc onto itself. This implies  $|d_1| = 1$  and  $d_2 = 0$ . Let  $f = z + 1$ . We have  $(d\phi_1/dz)^{1/p}(\phi_1 + 1)$  and  $d_1(d\phi_1/dz)^{1/p}(d_1\phi_1 + 1)$  are linearly dependent. But  $\phi_1 \neq 1$ . So  $d_1$  must be 1.

Replace  $U(ce_2)$  by  $b_2 \cdot c \cdot U(e_2)$ . Then we have  $T(fe) = (d\phi_1/dz)^{1/p} \cdot f \circ \phi_1 \cdot U(e)$  for any  $f \in H^p$  and  $\|e\| = 1$ . Hence, for any  $a, b \in \mathbb{C}$

$$\begin{aligned} & (d\phi_1/dz)^{1/p} \cdot a \cdot U(e_1) + (d\phi_1/dz)^{1/p} \cdot b \cdot U(e_2) \\ &= T(ae_1 + be_2) = (d\phi_1/dz)^{1/p} \cdot (\|ae_1 + be_2\|) \cdot U\left(\frac{ae_1 + be_2}{\|ae_1 + be_2\|}\right) \\ &= (d\phi_1/dz)^{1/p} \cdot U(ae_1 + be_2). \end{aligned}$$

This implies that  $U$  is a linear isometry. Since  $T$  is an onto mapping,  $U$  must be an onto mapping. So  $U$  is a unitary operator.

### References

- [1] A. V. Bukhvalov and A. A. Danilevich, 'Boundary properties of analytic and harmonic functions with values in Banach space', *Math. Notes* **31** (1982), 104–110.
- [2] M. Cambern, 'The isometries of  $H^\infty(K)$ ', *Proc. Amer. Math. Soc.* **36** (1972), 173–178.
- [3] M. Cambern and K. Jarosz, 'The isometries of  $H^1_{\mathcal{X}}$ ', preprint.
- [4] G. A. Edgar, 'Analytic martingale convergence', *J. Funct. Anal.* **69** (1986), 268–280.
- [5] K. Hoffman, '*Banach spaces of analytic functions*' (Prentice-Hall Series in Modern Analysis, Prentice-Hall, Englewood Cliffs, N.J., 1962).
- [6] K. deLeeuw, W. Rudin and J. Wermer, 'The isometries of some function spaces', *Proc. Amer. Math. Soc.* **11** (1960), 694–698.
- [7] F. Forelli, 'The isometries of  $H^p$ ', *Canad. J. Math.* **16** (1964), 721–728.

Department of Mathematics  
Memphis State University  
Memphis, Tennessee 38152  
U.S.A.