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ON COUNTABLY COMPACT QUASI-PSEUDOMETRIZABLE SPACES

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Abstract

We prove the following results: (1) A quasi-metrizable space is compact if and only if every compatible quasi-metric has a quasi-metric left d-sequential completion. (2) A quasi-pseudometrizable space is countably compact if and only if every compatible quasi-pseudometric is pointwise bounded. (3) A quasi-pseudometrizable space is compact if and only if every compatible quasi-pseudometric is precompact.

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1. Introduction

It is well-known that for a metrizable space (X, T) the following are equivalent:

(A) (X, T) is (countably) compact;
(B) every metric compatible with T is complete;
(C) every metric compatible with T is (pointwise) bounded;
(D) every metric compatible with T is precompact.

This paper extends these characterizations to quasi-pseudometrizable spaces. However that extension presents some peculiarities. In fact while a countably compact quasi-pseudometrizable space can be characterized by means of conditions of type (B) and (C) (in the "pointwise bounded" case), a compact quasi-pseudometrizable space is characterized by using a condition of type (D). In the light of these facts it seems appropriate to recall that while every countably quasi-metrizable space is compact (see [3, page 40]), compactness and countable compactness are not equivlaent in quasi-pseudometrizable spaces [2], [5].

Terms and concepts which are not defined are used as in [3]. The letter $\mathbb{N}$ will denote the set of all positive integer numbers.

A quasi-pseudometric on a set X is a non-negative real-valued function d on $X \times X$ such that, for all $x, y, z \in X$: (i) $d(x, x) = 0$; (ii) $d(x, y) \leq d(x, z) + d(z, y)$. If d satisfies the additional condition (iii) $d(x, y) = 0$ implies $x = y$, then, d is called a quasi-metric on X.

A quasi-(pseudo) metric space is a pair (X, d) such that X is a non-empty set and d is a quasi-(pseudo) metric on X.

Each quasi-pseudometric d on X induces a topology $T(d)$ which has as a base the family of d-balls $\{B_d(x, r): x \in X, r > 0\}$ where $B_d(x, r) = \{y \in X: d(x, y) < r\}$. A topological space (X, T) is called quasi-(pseudo) metrizable if there exists a quasi-(pseudo) metric d on X such that $T = T(d)$. In this case we say that d is compatible with T.

By using the metrization lemma [4, page 185] the authors have presented [9] a characterization of countably compact quasi-pseudometrizable spaces which extends the Niemytzki-Tychonoff theorem for these spaces. In Section 2 of this paper we shall prove (Theorem 1) a more general result, without using the metrization lemma. In this way we generalize the corresponding result for quasi-metric spaces proved by Fletcher and Lindgren [3, Theorem 7.35] and at the same time provide a simple proof. Furthermore, under very general conditions, it follows that notions of completeness for quasi-pseudometric spaces which are generally distinct, actually coincide when the topology is countably compact. Our method of proof also permits us to state (Theorem 2) a characterization of compact quasi-metrizable spaces in terms of several kinds of completion. Finally, in Section 3 we prove (Theorem 3) that a quasi-pseudometrizable space is countably compact if and only if every compatible quasi-pseudometric is pointwise bounded and we deduce that a quasi-pseudometrizable space is compact if and only if every compatible quasi-pseudometric is precompact.

2. Complete quasi-pseudometric spaces and countable compactness

Intuitively, the following concepts (see [7]) correspond to strongest and weakest notions for a sequence to be "Cauchy".

DEFINITION 1. Let (X, d) be a quasi-pseudometric space. A sequence $\{x_n\}_{n \in N}$ in X is said to be *left d-Cauchy* if for each $\varepsilon > 0$ there are a point $x \in X$ and a $k \in \mathbb{N}$ such that $d(x, x_n) < \varepsilon$ for all $n \geq k$. The sequence $\{x_n\}_{n \in \mathbf{N}}$ is called *symmetric d-Cauchy* (d-Cauchy in [7]) if for each $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq k$.

DEFINITION 2. We say that a quasi-pseudometric d on a set X is *left d-sequentially complete* if every left d-Cauchy sequence in X converges to a point in X (with respect to the topology $T(d)$) and we say that d is *left d-weakly sequentially complete* if every left d-Cauchy sequence in X has a $T(d)$-cluster point in X. Similarly, we say that d is *symmetric d-sequentially complete* if every symmetric d-Cauchy sequence in X converges to a point in X (with respect to the topology $T(d)$).

A quasi-pseudometric space (X, d) is called *left d-(weakly) sequentially complete* if the quasi-pseudometric d is left d-(weakly) sequentially complete. Similarly, (X, d) is called *symmetric d-sequentially complete* if the quasi-pseudometric d is symmetric d-sequentially complete.

A filter $\mathscr{F}$ on a quasi-uniform spce $(X, \mathscr{U})$ is called a *$\mathscr{U}$-Cauchy filter* [3, page 47] if for each $V \in \mathscr{U}$ there is an $x \in X$ such that $V(x) \in \mathscr{F}$. Similarly, the filter $\mathscr{F}$ is called a *symmetric $\mathscr{U}$-Cauchy filter* if for each $V \in \mathscr{U}$ there is an $F \in \mathscr{F}$ such that $F \times F \subset V$. We say that a quasi-uniform space $(X, \mathscr{U})$ is *(symmetric)* $\mathscr{U}$-complete if each (symmetric) $\mathscr{U}$-Cauchy filter on X converges to some point in X (with respect to the topology $T(\mathscr{U})$) and $(X, \mathscr{U})$ is called *$\mathscr{U}$-weakly complete* if each $\mathscr{U}$-Cauchy filter on X has a $T(\mathscr{U})$-cluster point in X.

DEFINITION 3. Let (X, d) be a quasi-pseudometric space and let $\mathscr{U}(d)$ be the quasi-uniformity generated by d (see [3, page 3]). Then d and (X, d) are called *left d-(weakly) complete* if the quasi-uniform space $(X, \mathscr{U}(d))$ is $\mathscr{U}(d)$-(weakly) complete.

The following implications are clearly satisfied for a quasi-pseudometric space (X, d):

(a) left d-(weakly) complete implies left d-(weakly) sequentially complete and left d-(sequentially) complete implies left d-(weakly) sequentially complete.

(b) left d-sequentially complete implies symmetric d-sequentially complete.

In [6, Examples 1 and 2] H. P. Künzi gives an example of a left d-weakly complete quasi-metric space that is not left d-sequentially complete and an example of a left d-sequentially complete quasi-metric space that is not left d-weakly complete. He also gives [6, Example 3] an example of a left d-weakly complete and left d-sequentially complete quasi-metric space that is not left d-complete. Consequently, the converse implications of (a) are not satisified. On the other hand it is well-known that every sequence which converges with respect to $T(d)$ is left d-Cauchy but, unfortunately, $T(d)$-convergence does not imply symmetric d-Cauchyness. However, if (X, d) is symmetric d-sequentially complete we have the following equivalences:

LEMMA 1. *For a quasi-pseudometrizable space (X, d) the following are equivalent.*

(i) *(X, d) is symmetric d-sequentially complete.*
(ii) *Every symmetric d-Cauchy sequence in X has a $T(d)$-cluster point.*
(iii) *The quasi-uniform space $(X, \mathscr{U}(d))$ is symmetric $\mathscr{U}(d)$-complete.*
(iv) *Every symmetric $\mathscr{U}(d)$-Cauchy filter in X has a $T(d)$-cluster point.*

PROOF. (i) $\Rightarrow$ (ii). Obvious.
(ii) $\Rightarrow$ (i). [7, Theorem 1(iii)].
(iii) $\Rightarrow$ (iv). Obvious.
(iv) $\Rightarrow$ (ii). Obvious.
(i) $\Rightarrow$ (iii). Let $\mathscr{F}$ be a symmetric $\mathscr{U}(d)$-Cauchy filter on X. Then there exists a decreasing sequence $\{F_n\}_{n\in\mathbb{N}}$ of subsets of X such that, for all $n \in \mathbb{N}$, we have $F_n \in \mathscr{F}$ and $F_n \times F_n \subset \{(x, y) : d(x, y) < 2^{-n}\}$. For each $n \in \mathbb{N}$ choose $x_n \in F_n$. Then $\{x_n\}_{n\in\mathbb{N}}$ is a symmetric d-Cauchy sequence and, thus, it is $T(d)$-convergent to a point $x \in X$. Given $k \in \mathbb{N}$ there exists $n_0 > k$ such that $d(x, x_n) < 2^{-(k+1)}$ for all $n \geq n_0$. Hence, for every $y \in F_{k+1}$, we have $d(x, y) \leq d(x, x_{n_0}) + d(x_{n_0}, y) < 2^{-(k+1)} + 2^{-(k+1)}$ since $x_{n_0} \in F_{k+1}$. This implies that $B_d(x, 2^{-k}) \in \mathscr{F}$ and, hence, $\mathscr{F}$ is $T(d)$-convergent to x.

In the light of the above lemma we shall use, in the following, the term "symmetric d-complete" instead of "symmetric d-sequentially complete" for a symmetric d-sequentially complete quasi-pseudometric space (X, d).

COROLLARY 1. *Every left d-weakly sequentially complete quasi-pseudometric space is symmetric d-complete.*

Note that [7, Example 4] shows that, in general, the converse implication of (b) is not satisfied.

LEMMA 2. *Let d be a quasi-pseudometric on a set X such that $d \leq 1$ and supose that there exists a sequence $\{x_m\}_{m\in\mathbb{N}}$ in X that has no $T(d)$-cluster point in X. If for each $n \in \mathbb{N}$ we define $A_n = \{x_m : m \geq n\}$ and*

$$e_n(x, y) = \begin{cases} d(x, y) & \text{if } x, y \in X - \overline{A}_n, \\ 1 & \text{if } x \in X - \overline{A}_n,\ y \in \overline{A}_n, \\ 0 & \text{if } x \in \overline{A}_n, \end{cases}$$

then the real-valued function e, *defined on* $X \times x$ *as*

$$e(x, y) = \sup\{2^{-n} e_n(x, y) : n \in \mathbb{N}\},$$

is a quasi-pseudometric on X *such that* $T(d) = T(e)$. *Moreover,* $\{x_m\}_{m\in\mathbb{N}}$ *is a symmetric* e-*Cauchy sequence.*

Proof. Since the sequence $\{x_m\}_{m\in\mathbb{N}}$ has no $T(d)$-cluster point, we deduce that $X = \bigcup_{n=1}^{\infty}(X - \overline{A}_n)$. It is straightforward to verify that each e_n satisfies the triangle inequality and, therefore, e is a quasi-pseudometric on X. Assume $\{y_m\}_{m\in\mathbb{N}}$ is a $T(e)$-convergent sequence to a point $y \in X$. Then, for each $n \in \mathbb{N}$, $e_n(y, y_m) \to 0$ as $m \to \infty$. For the given y, let k be the smallest index for which $y \in X - \overline{A}_k$. Then, $e_k(y, y_m) \to 0$ gives $e_k(y, y_m) < 1$ eventually, so that $\{y_m\}_{m\in\mathbb{N}}$ is eventually in $X - \overline{A}_k$ by definition of e_k. Hence, there is $j \in \mathbb{N}$ such that, if $m \geq j$, $d(y, y_m) = e_k(y, y_m)$. Thus, $\{y_m\}_{m\in N}$ is $T(d)$-convergent to y. Conversely, assume $\{y_m\}_{m\in\mathbb{N}}$ is $T(d)$ convergent to y. For the given y let k be the smallest index for which $y \in X - \overline{A}_k$. Then, there is $j \in N$ such that, if $m \geq j$, $e_n(y, y_m) = d(y, y_m)$ for all $n \geq k$. Consequently, $e(y, y_m) = 2^{-k} d(y, y_m)$ for $m \geq j$. This proves that $\{y_m\}_{m\in\mathbb{N}}$ is $T(e)$-convergent to y and, hence, $T(e) = T(d)$.

Finally, $\{x_m\}_{m\in\mathbb{N}}$ is a symmetric e-Cauchy sequence since given $\varepsilon > 0$ let $j \in \mathbb{N}$ be such that $2^j \varepsilon > 1$. Let n, $m \geq j$. Then, for $k = 1, 2, \ldots, j$, we have $e_k(x_n, x_m) = 0$ since $x_n, x_m \in A_k$. For $k > j$, $e_k(x_n, x_m) \leq 1$ so that, for n, $m \geq j$, $e(x_n, x_m) < 2^{-j} < \varepsilon$.

Theorem 1. *For a quasi-pseudometrizable space* (X, T) *the following are equivalent.*

(i) (X, T) *is countably compact.*

(ii) *Every quasi-pseudometric* d *on* X *compatible with* T *is left* d-*complete.*

(iii) *Every quasi-pseudometric* d *on* X *compatible with* T *is left* d-*weakly sequentially complete.*

(iv) *Every quasi-pseudometric* d *on* X *compatible with* T *is symmetric* d-*complete.*

Proof. (i) $\Rightarrow$ (ii). [6, Proposition 5].

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (iv). Corollary 1.

(iv) $\Rightarrow$ (i). Suppose that there exists a sequence $\{x_n\}_{n\in N}$ in X that has no T-cluster point. Let d be a quasi-pseudometric on X compatible with T satisfying $d \leq 1$. By Lemma 2, there exists a quasi-pseudometric e on X compatible with T such that $\{x_m\}_{m\in\mathbb{N}}$ is a symmetric e-Cauchy sequence. Hence, the quasi-pseudometric e is not symmetric e-complete. This concludes the proof.

COROLLARY [3, page 178]. *A quasi-metrizable space (X, T) is compact if, and only if, every compatible quasi-metric d is left d-weakly (sequentially) complete.*

Let $(X, \mathscr{U})$ be a T_1 quasi-uniform space. A quasi-uniform space $(Y, \mathscr{V})$ is called a *(weak) $\mathscr{U}$-completion* of $(X, \mathscr{U})$ if $(Y, \mathscr{V})$ is a $\mathscr{V}$-(weakly) complete T_1 quasi-uniform space that has a dense subspace quasi-unimorphic (relative to $\mathscr{U}$ and $\mathscr{V}$) to $(X, \mathscr{U})$ [3]. The concept of *symmetric completion* is defined in the analogous manner where $(Y, \mathscr{V})$ is symmetric $\mathscr{V}$-complete.

Let (X, d) be a quasi-metric subspace of a quasi-metric space $(\hat{X}, \hat{d})$. We say that $(\hat{X}, \hat{d})$ is a *quasi-metric left d-(weak) sequential completion* of (X, d) if X is dense in $(\hat{X}, T(\hat{d}))$ and $(\hat{X}, \hat{d})$ is left $\hat{d}$-(weakly) sequentially complete. The concepts of *quasi-metric left d-completion, quasi-metric left d-weak completion* and *quasi-metric symmetric d-completion* are defined in the obvious manner where $(\hat{X}, \hat{d})$ is left $\hat{d}$-complete, left $\hat{d}$-weakly complete and symmetric $\hat{d}$-complete, respectively.

It is interesting to recall that a Hausdorff quasi-metric space need not have a quasi-metric symmetric d-completion [8, Theorem 2.6] (a similar situation occurs for quasi-uniform spaces [1], [3]).

In [3, Theorem 3.43 and Proposition 3.46] are proved the two following results.

(1) A T_1 quasi-uniform space $(X, \mathscr{U})$ has a weak $\mathscr{U}$-completion if and only if whenever $\mathscr{F}$ is a $\mathscr{U}$-Cauchy filter on X and $x \in X$ is a $T(\mathscr{U}^{-1})$-cluster point of $\mathscr{F}$, then x is a $T(\mathscr{U})$-cluster point of $\mathscr{F}$.

(2) A T_1 topological space (X, T) is compact if and only if every quasi-uniformity $\mathscr{U}$ on X compatible with T has a weak $\mathscr{U}$-completion.

In [6, Proposition 8] H. P. Künzi obtains an analogue of (1) for quasi-metric spaces. The following result provides an analogue of (2) for quasi-metric spaces.

THEOREM 2. *For a quasi-metrizable space (X, T) the following are equivalent.*

(i) *(X, T) is compact.*

(ii) *Every quasi-metric d on X compatible with T has a quasi-metric left d-completion.*

(iii) *Every quasi-metric d on X compatible with T has a quasi-metric left d-weak sequential completion.*

(iv) *Every quasi-metric* d *on* X *compatible with* T *has a quasi-metric symmetric* d*-completion.*

PROOF. (i) $\Rightarrow$ (ii). Theorem 1.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (iv). Obvious.

(iv) $\Rightarrow$ (i). Suppose that (X, T) is not compact. Then there exists a sequence $\{x_m\}_{m\in\mathbf{N}}$ in X that has no cluster point. Let d be a quasi-metric compatible with T such that $d \leq 1$. Consider the quasi-pseudometric e constructed in Lemma 2 and note that, in this case, e is a quasi-metric. Moreover $T(e) = T$ and $\{x_m\}_{m\in\mathbf{N}}$ is a symmetric e-Cauchy sequence. We shall show that, for each $x \in X$, $e(x_m, x) \to 0$. Let k be the smallest index for which $x \in X - \overline{A}_k$. Therefore, for $n \geq k$, $e_n(x_m, x) = d(x_m, x)$ when $x_m \in X - \overline{A}_n$, and $e_n(x_m, x) = 0$ when $x_m \in \overline{A}_n$. Given $\varepsilon > 0$ let $j > k$ be such that $2^j \varepsilon > 1$. Hence, for $m \geq j$, $e(x_m, x) \leq 2^{-(j+1)} < \varepsilon$. Assume $(\hat{X}, \hat{e})$ is a quasi-metric symmetric e-completion of (X, e). Then $\{x_m\}_{m\in\mathbf{N}}$ is a symmetric $\hat{e}$-Cauchy sequence and, therefore, there is a point $p \in \hat{X}$ satisfying $\hat{e}(p, x_m) \to 0$. Since $\hat{e}(x_m, x) \to 0$ we deduce $p = x \in X$ because $\hat{e}$ is a quasi-metric. This implies $e(x, x_m) \to 0$ and, consequently, the sequence $\{x_m\}_{m\in\mathbf{N}}$ is $T(e)$-convergent to x, a contradiction. The proof is complete.

We conclude this section with some remarks concerning different notions of completion of a quasi-metric space.

We first note that in the above theorems the conditions 'left d-', left d-weak(ly)' and 'symmetric d' can be replaced by other notions of completenss in the sense of [7]. For instance, it follows immediately from Theorem 2 that a quasi-metrizable space (X, T) is compact if and only if every quasi-metric d on X compatible with T has a quasi-metric left K-completion (see [7, pages 132 and 134]). On the other hand, H. P. Künzi [6] has shown that every quasi-metric space having a quasi-metric left d-weak sequential completion has a base of countable order. His method of proof actually shows a more general result, namely, every quasi-metric space having a quasi-metric left K-completion has a base of countable order. It follows from this observation that the Sorgenfrey line is a quasi-metric space that does not admit compatible quasi-metrics that have a quasi-metric left K-completion because this space does not have a base of countable order.

In [6] Künzi has also given an example of a quasi-metric space (X, d) that has a quasi-metric left d-weak sequential completion but does not have a quasi-metric left d-weak completion. In order to complete this jigsaw puzzle we give here a simple example of a left d-weakly complete quasi-metric space that does not admit any quasi-metric left d-sequential completion.

EXAMPLE 1 Let X be the set of all non-negative integer numbers and define a quasi-metric d on X as follows.

$$\begin{array}{ll} d(0, x) = 1 & \text{if } x \neq 0, \\ d(1, x) = 1 & \text{if } x \neq 1, \\ d(n, 0) = d(n, 1) = 1/n & \text{if } n \in X - \{0, 1\}, \\ d(n, x) = 1 & \text{if } n \neq x \text{ and } n, x \in X - \{0, 1\}, \\ d(x, x) = 0 & \text{for all } x \in X. \end{array}$$

Then $T(d)$ is the discrete topology on X. The sequence $\{x_m\}_{m\in N}$ such that $x_{2n} = 0$ and $x_{2n+1} = 1$ for all $n \in \mathbb{N}$, is a left d-Cauchy sequence that is not $T(d)$-convergent. Assume $(\hat{X}, \hat{d})$ is a quasi-metric left d-sequential completion of (X, d). Then $\hat{d}(p, x_n) \to 0$ for some $p \in \hat{X}$ and, hence, $p = 0$ and $p = 1$, a contradiction. On the other hand it is clear that every $\mathscr{U}(d)$-Cauchy filter on X has a $T(d)$-cluster point.

3. Pointwise bounded quasi-pseudometrics and countable compactness

A quasi-pseudometric d on a set X is called *bounded* if there is constant $M > 0$ such that $d(x, y) \leq M$ for every $x, y \in X$.

The following example shows that there exists a quasi-metric space (X, d) such that $(X, T(d))$ is a compact (and metrizable) space but d is not bounded.

EXAMPLE 2. Let X be the set of all non-negative integer numbers and define a quasi-metric d on X as follows:

$$\begin{array}{ll} d(0, x) = 1/x & \text{for all } x \in X - \{0\}, \\ d(x, y) = x & \text{for all } x \neq 0 \text{ and } y \neq x, \\ d(x, x) = 0 & \text{for all } x \in X. \end{array}$$

It is obvious that d is not bounded. However $T(d)$ is a Hausdorff compact space and hence it is metrizable. Note that, still, given $x \in X$ there exists a constant $M_x > 0$ such that, for all $y \in X$, $d(x, y) \leq M_x$. This fact suggests the next definition.

DEFINITION 4. We say that a quasi-pseudometric d on a set X is *pointwise bounded* if for some $x \in X$ there exists a constant $M_x > 0$ such that, for all $y \in X$, $d(x, y) \leq M_x$.

It is easy to see that boundedness and pointwise boundedness are equivalent for a pseudometric.

Lemma 3. *Let d be a quasi-pseudometric on a set X such that $d \leq 1$ and suppose that there exists a sequence $\{x_m\}_{m \in \mathbb{N}}$ in X that has no $T(d)$-cluster point in X. If for each $n \in \mathbb{N}$ we define $A_n = \{x_m : m \geq n\}$ and*

$$e_n(x, y) = \begin{cases} d(x, y) & \text{if } x, y \in X - \overline{A}_n, \\ 2^{2n} & \text{if } x \in X - \overline{A}_n, y \in \overline{A}_n, \\ 0 & \text{if } x \in \overline{A}_n, \end{cases}$$

then the real-valued function e, defined on $X \times X$ as

$$e(x, y) = \sup\{2^{-n} e_n(x, y) : n \in \mathbb{N}\},$$

is a quasi-pseudometric on X such that $T(d) = T(e)$. Moreover, e is not pointwise bounded.

Proof. A slight modification of the proof of Lemma 2 permits us to conclude that e is a quasi-pseudometric on X compatible with $T(d)$. Now let $x \in X$ and let k be the smallest index for which $x \in X - \overline{A}_k$. For each $m \geq k$ we have $e_m(x, x_m) = 2^{2m}$ and, hence, $e(x, x_m) \geq 2^m$ for $m \geq k$. Thus e is not pointwise bounded.

Theorem 3. *A quasi-pseudometrizable space (X, T) is countably compact if and only if every quasi-pseudometric on X compatible with T is pointwise bounded.*

Proof. Let (X, T) be a countably compact quasi-pseudometrizable space and let d be a compatible quasi-pseudometric on X. Suppose that there are a point $x \in X$ and a sequence $\{x_m\}_{m \in \mathbb{N}}$ in X satisfying $d(x, x_m) \geq m$ for all $m \in \mathbb{N}$. Since $\{x_m\}_{m \in \mathbb{N}}$ has a subsequence $\{x_{m(k)}\}_{k \in \mathbb{N}}$ which is $T(d)$-convergent to some pont $y \in X$, we obtain, from $d(x, x_{m(k)}) \leq d(x, y) + d(y, x_{m(k)})$, a contradiction. The converse follows immediately from Lemma 3.

A quasi-pseudometric on a set X is called *precompact* if for each $\varepsilon > 0$ there exists a finite subset F of X such that $d(F, x) < \varepsilon$ for all $x \in X$.

We omit the easy proof of the following observation

Lemma 4. *Every precompact quasi-pseudometric is pointwise bounded.*

Theorem 4. *A quasi-pseudometrizable space (X, T) is compact if and only if every quasi-pseudometric on X compatible with T is precompact.*

Proof. Suppose that every quasi-pseudometric compatible with T is precompact. By Lemma 4 and Theorem 4, (X, T) is countably compact.

Choose a compatible quasi-pseudometric d on X. By Theorem 1, d is left d-weakly complete and, therefore, the quasi-uniform space $(X, \mathscr{U}(d))$ is precompact and $\mathscr{U}(d)$-weakly complete. It follows from [3, Theorem 3.24] that (X, T) is compact. The converse is well-known.

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