

## SPECTRALITY OF ELEMENTARY OPERATORS

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### Abstract

Spectrality and prespectrality of elementary operators  $x \mapsto \sum_{i=1}^n a_i x b_i$ , acting on the algebra  $\mathcal{B}(\mathcal{H})$  of all bounded linear operators on a separable infinite-dimensional complex Hilbert space  $\mathcal{H}$ , or on von Neumann-Schatten classes in  $\mathcal{B}(\mathcal{H})$ , are treated. In the case when  $(a_1, a_2, \dots, a_n)$  and  $(b_1, b_2, \dots, b_n)$  are two  $n$ -tuples of commuting normal operators on  $\mathcal{H}$ , the complete characterization of spectrality is given.

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### 1. Introduction

Let  $\mathcal{B}(\mathcal{H})$  be the  $C^*$ -algebra of all bounded linear operators on a separable infinite-dimensional complex Hilbert space  $\mathcal{H}$ . For any two  $n$ -tuples  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  and  $\mathbf{b} = (b_1, b_2, \dots, b_n)$  of commuting elements of  $\mathcal{B}(\mathcal{H})$  one defines an elementary operator on  $\mathcal{B}(\mathcal{H})$  to be a bounded operator of the form

$$(1) \quad Rx = \sum_{i=1}^n a_i x b_i$$

( $x \in \mathcal{B}(\mathcal{H})$ ). The operator  $R$  depends on its coefficients  $a_i$  and  $b_i$ , and so the notation  $R = R_{\mathbf{a}, \mathbf{b}}$  is usually used. However, the non-uniqueness of the representation (1) of  $R$  is only up to certain linear transformations of the coefficients (see Fong and Sourour [9]).

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The spectral, range inclusion, compactness, and Fredholm properties of  $R$  have already been studied in great detail, and also in the case when  $R$  is restricted to some symmetrically normed ideals of  $\mathcal{B}(\mathcal{H})$  (see Fialkow [7], [8] and the references given there). For example, it is known (Fialkow [8], Mathieu [15]) that the spectrum of  $R = R_{\mathbf{a}, \mathbf{b}}$  is given by

$$(2) \quad \sigma(R) = \{\lambda \cdot \mu; \lambda \in \sigma(\mathbf{a}), \mu \in \sigma(\mathbf{b})\}$$

where  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ ,  $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ ,  $\lambda \cdot \mu = \lambda_1\mu_1 + \lambda_2\mu_2 + \dots + \lambda_n\mu_n$  and  $\sigma(\mathbf{a})$  and  $\sigma(\mathbf{b})$  are the joint spectra of the  $n$ -tuples  $\mathbf{a}$  and  $\mathbf{b}$ , respectively, in the sense of Harte [11] (see also [12, Chapter 11]). Probably best understood are the so-called generalized inner derivations  $D_{a,b}$  defined for each pair  $a, b \in \mathcal{B}(\mathcal{H})$  by  $D_{a,b}x = ax - xb$  ( $x \in \mathcal{B}(\mathcal{H})$ ) and the elementary multiplications  $M_{a,b}$  defined by  $M_{a,b}x = axb$  ( $x \in \mathcal{B}(\mathcal{H})$ ).

In [13] and [14] spectrality and prespectrality (in the sense of Dunford [6]) of generalized inner derivations acting on von Neumann-Schatten classes in  $\mathcal{B}(\mathcal{H})$  were characterized in terms of the spectra of coefficients  $a$  and  $b$ . This paper, dealing with general elementary operators, is an extension of [13]. Like there, we shall consider only the case where the coefficients are normal operators in  $\mathcal{B}(\mathcal{H})$ , although a straightforward generalization to the case of spectral coefficients is possible (see the concluding remarks).

In Section 2 we shall first give a geometrical classification of elementary operators on  $\mathcal{B}(\mathcal{H})$  related to mutual geometrical position of joint spectra,  $\sigma(\mathbf{a})$  and  $\sigma(\mathbf{b})$ , and according to which elementary operators are classified into operators of the first and operators of the second kind. Although very simple, this classification is crucial for the spectrality of elementary operators on  $\mathcal{B}(\mathcal{H})$ . It will be shown in subsequent sections that only elementary operators of the first kind can be spectral (they are spectral under some natural additional conditions) while elementary operators of the second kind never have this property. Section 3 is devoted to some preparatory results connected with elementary operators having normal coefficients. The main result, that is, the characterization of spectrality and prespectrality of elementary operators with normal coefficients, is given in Section 4 together with some final comments.

## 2. A classification of elementary operators

Let  $R = R_{\mathbf{a}, \mathbf{b}}$  be an elementary operator acting on the algebra  $\mathcal{B}(\mathcal{H})$ . Recall that the joint spectra  $\sigma(\mathbf{a})$  and  $\sigma(\mathbf{b})$  are nonvoid compact subsets in  $\mathbb{C}^n$  since it is assumed that each of the  $n$ -tuples,  $\mathbf{a}$  and  $\mathbf{b}$ , consists of commuting elements of the algebra (see Harte [11], [12]).

2.1. DEFINITION. We say that  $R_{\mathbf{a}, \mathbf{b}}$  is

(i) of the *first kind* if there exist finite decompositions of joint spectra  $\sigma(\mathbf{a})$  and  $\sigma(\mathbf{b})$  into nonvoid and pairwise disjoint closed subsets  $A_1, A_2, \dots, A_r$  and  $B_1, B_2, \dots, B_s$ , respectively, such that on each rectangle  $A_i \times B_j$ ,  $i \in \{1, 2, \dots, r\}$ ,  $j \in \{1, 2, \dots, s\}$ , the function  $(\lambda, \mu) \mapsto \lambda \cdot \mu$  depends only on one variable,  $\lambda$  or  $\mu$ ;

(ii) of the *second kind* if there exist infinite subsets  $A \subset \sigma(\mathbf{a})$  and  $B \subset \sigma(\mathbf{b})$  such that the function  $(\lambda, \mu) \mapsto \lambda \cdot \mu$  is one-to-one on  $A \times B$ .

The above definition is unambiguous, that is, independent of concrete representation of the operator  $R_{\mathbf{a}, \mathbf{b}}$  in the form (1). Namely, we may suppose that the representation  $R_{\mathbf{a}, \mathbf{b}}$  of the non-zero elementary operator  $R$  is minimal in the sense that  $n$  is minimal, or equivalently, that the elements in both  $n$ -tuples form linearly independent sets (Mathieu [15, Proposition 4.6]). Then for any other minimal representation  $R_{\mathbf{a}', \mathbf{b}'}$  of the same operator  $R$  there exists a unique invertible complex  $n \times n$  matrix  $\Gamma = (\gamma_{jk})_{1 \leq j, k \leq n}$  such that

$$a'_j = \sum_{k=1}^n \gamma_{jk} a_k, \quad b'_k = \sum_{j=1}^n \gamma_{jk} b_j$$

(Mathieu [15, Corollary 4.7]). If we write  $\Gamma^{-t}$  for the inverse of the transpose of  $\Gamma$ , we have  $\mathbf{a}' = \Gamma \mathbf{a}$  and  $\mathbf{b}' = \Gamma^{-t} \mathbf{b}$ . This yields the bijections  $\lambda \mapsto \lambda' = \Gamma \lambda$  and  $\mu \mapsto \mu' = \Gamma^{-t} \mu$  from  $\sigma(\mathbf{a})$  onto  $\sigma(\mathbf{a}')$  and  $\sigma(\mathbf{b})$  onto  $\sigma(\mathbf{b}')$ , respectively, by the spectral mapping theorem (Harte [11]). Since we have  $\lambda' \cdot \mu' = \lambda \cdot \mu$ , it follows that the properties of the map  $(\lambda, \mu) \mapsto \lambda \cdot \mu$  from  $\sigma(\mathbf{a}) \times \sigma(\mathbf{b})$  to  $\mathbb{C}$ , pointed out in Definition 2.1, do not depend on changing a minimal representation of  $R$ . In a similar way the same can also be shown for other representations and so Definition 2.1 is correct.

Moreover, we shall show now that 2.1(i) and 2.1(ii) are the only two possibilities for an elementary operator. So, the class of elementary operators is divided into two (disjoint) subclasses: that of the first and that of the second kind. This classification is a consequence of the following purely combinatorial result on infinite matrices having finite rank.

2.2. LEMMA. Let  $A$  be an infinite matrix over  $\mathbb{C}$  with finite rank. Then one and only one of the following two possibilities holds.

(a) The matrix  $A$  splits into finitely many submatrices each of which has constant rows or constant columns.

(b) In the matrix  $A$  there exists an infinite submatrix with all entries different.

This result might be well known to specialists, but since we were not able

to find it in the available literature, we sketch an elementary proof of the lemma.

**PROOF OF LEMMA 2.2.** If the rank of the matrix  $A$  is equal to 1, the statement of the lemma is almost evident. Namely, in this case we can write  $A$  as the matricial product of a column  $C$  and a row  $R$ , that is,  $A = CR$ . If there are only finitely many different entries in  $C$  (or in  $R$ ), the matrix  $A$  splits (after making some appropriate permutations of rows or columns) into finitely many horizontal (vertical) strips with constant columns (rows). Otherwise, let  $C_1$  be an infinite subcolumn in  $C$  with different elements, and  $R_1$  be an infinite subrow in  $R$  with different elements. It is then easy to choose an infinite submatrix in  $C_1 R_1$  having all entries different.

We shall prove the lemma by induction on the rank of  $A$ . Let  $n > 1$  and suppose we have already proved the lemma for all matrices with rank less than  $n$ . Let  $A$  be an infinite matrix with rank equal to  $n$ . It is easy to show and also well known that we can write  $A$  in the form  $A = CR$  where  $C$  is a matrix with infinitely many rows and  $n$  linearly independent columns, while  $R$  is a matrix with infinitely many columns and  $n$  linearly independent rows.

Choose in the matrix  $C$  a maximal family of rows with the property that any subfamily of it, consisting of no more than  $n$  rows, is linearly independent. Do the same with the columns in the matrix  $R$ . Then, by maximality, every other row in  $C$  (column in  $R$ ) depends linearly already on  $n - 1$  rows (columns) in the chosen family.

If the chosen maximal family in  $C$  (or in  $R$ ) is finite, then, obviously, the matrix  $C$  (or  $R$ , respectively), and consequently also the matrix  $A$ , splits into finitely many submatrices of rank less than  $n$ . By inductive hypothesis, for each of these submatrices the lemma holds. Then, either in one of them there exists an infinite submatrix with different entries, or all submatrices, and hence also the whole matrix  $A$ , split into finitely many submatrices with constant rows or columns. Thus, we see that either (a) or (b) holds for the matrix  $A$ .

Suppose now that the chosen maximal family of rows in  $C$  is infinite and that the same is true for the chosen maximal family of columns in  $R$ . Let  $C_0$  (respectively,  $R_0$ ) be the submatrix in  $C$  (respectively, in  $R$ ) consisting of chosen rows (respectively, columns). Then in the infinite matrix  $A_0 = C_0 R_0$  every  $n \times n$  submatrix is the product of two invertible  $n \times n$  submatrices in  $C_0$  and  $R_0$  and, hence, no minor of order  $n$  in  $A_0$  is equal to zero. It follows that there are no two rows (columns) in  $A_0$  such that in both of them an element repeats infinitely many times on the same places. From this property of  $A_0$  we can now easily derive that there exists an infinite submatrix  $A_1$  in  $A_0$  such that any of its elements repeats only finitely many times in the same row or in the same column. Consequently, there exists also

an infinite submatrix  $A_2$  in  $A_1$  having all entries different, so that in this case (b) holds for the matrix  $A$ .

In fact we shall need the following classification of pairs of closed subsets in  $\mathbb{C}^n$  according to a given bilinear form.

**2.3. PROPOSITION.** *Let  $A$  and  $B$  be arbitrary nonvoid closed subsets in  $\mathbb{C}^n$  and let  $\varphi: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$  be a bilinear form. Then for the pair  $(A, B)$  one and only one of the following two possibilities holds.*

(i) *There exist finite decompositions  $A = \bigcup_{i=1}^r A_i$  and  $B = \bigcup_{j=1}^s B_j$  of these sets  $A$  and  $B$  into nonvoid disjoint closed subsets  $A_1, A_2, \dots, A_r$  and  $B_1, B_2, \dots, B_s$  respectively, such that for each  $i \in \{1, 2, \dots, r\}$  and  $j \in \{1, 2, \dots, s\}$  we have either*

$$(*) \quad \varphi(\lambda, \mu) = \varphi(\lambda, \mu') \quad \text{for every } \lambda \in A_i, \mu, \mu' \in B_j$$

or

$$(**) \quad \varphi(\lambda, \mu) = \varphi(\lambda', \mu) \quad \text{for every } \lambda, \lambda' \in A_i, \mu \in B_j$$

or both.

(ii) *There exist infinite subsets  $A' \subset A$  and  $B' \subset B$  such that the map  $\varphi$  is one-to-one on  $A' \times B'$ .*

**PROOF.** If  $A$  or  $B$  is a finite set, then we have (i). So, we can suppose that both sets,  $A$  and  $B$ , are infinite. Take any dense countable subset  $\{\lambda_1, \lambda_2, \dots\} \subset A$  and any dense countable subset  $\{\mu_1, \mu_2, \dots\} \subset B$  and form the infinite matrix  $(a_{ij})_{i,j \in \mathbb{N}}$  where  $a_{ij} = \varphi(\lambda_i, \mu_j)$ . This matrix is certainly of finite rank and therefore it has one and only one of the properties (a) and (b) of Lemma 2.2.

If (b) holds, we clearly have (countable) infinite subsets  $A' \subset A$  and  $B' \subset B$  such that  $\varphi$  is one-to-one on  $A' \times B'$ . Thus, (ii) is valid.

On the other hand, if (a) holds for the matrix  $(a_{ij})_{i,j \in \mathbb{N}}$ , there exist finite decompositions  $\mathbb{N} = \bigcup_{i=1}^r M_i$  and  $\mathbb{N} = \bigcup_{j=1}^s N_j$  of positive integers such that for every  $i \in \{1, 2, \dots, r\}$  and  $j \in \{1, 2, \dots, s\}$  we have either

$$\varphi(\lambda_k, \mu_l) = \varphi(\lambda_k, \mu_{l'}) \quad \text{for } k \in M_i, l, l' \in N_j$$

or

$$\varphi(\lambda_k, \mu_l) = \varphi(\lambda_{k'}, \mu_l) \quad \text{for } k, k' \in M_i, l \in N_j$$

or both. Let  $A_i$  be the closure of  $\{\lambda_k; k \in M_i\}$  and  $B_j$  be the closure of  $\{\mu_l; l \in N_j\}$ , for each pair  $(i, j)$ . Clearly,  $A_1, A_2, \dots, A_r$  and  $B_1, B_2, \dots, B_s$  are nonvoid closed subsets in  $A$  and  $B$ , respectively, with the property that  $A = \bigcup_{i=1}^r A_i$  and  $B = \bigcup_{j=1}^s B_j$ . Further, it is possible to show that the sets  $A_1, A_2, \dots, A_r$  (and, in the same way, the sets

$B_1, B_2, \dots, B_s$ ) can be taken to be pairwise disjoint (otherwise join two intersecting sets together to get a new one with the same property). Since  $\varphi$  is continuous in both arguments, it is easy to see that  $(*)$  or  $(**)$  holds for each pair  $(i, j)$ , and we have proved (i).

**REMARK.** By requiring that the decomposition in 2.3(i) is minimal, that is, that  $r$  and  $s$  are minimal, we can show that this decomposition is unique.

Taking  $\varphi: (\lambda, \mu) \mapsto \lambda \cdot \mu$  and recalling Definition 2.1, we have as a consequence of Proposition 2.3 the following classification of elementary operators.

**2.4. THEOREM.** *Every elementary operator is either of the first or of the second kind.*

**2.5. EXAMPLE.** From Definition 2.1 and Theorem 2.4 it is clear that a generalized inner derivation  $D_{a,b}$  and an elementary multiplication  $M_{a,b}$  are of the first kind if and only if at least one of the spectra,  $\sigma(a)$  or  $\sigma(b)$ , is finite. This follows from the fact that in this case  $\lambda \cdot \mu$  is equal to  $\lambda - \mu$  or  $\lambda\mu$  for  $\lambda \in \sigma(a)$  and  $\mu \in \sigma(b)$ . Namely, if  $\sigma(a)$  or  $\sigma(b)$  is finite, we easily find finite decompositions of  $\sigma(a)$  and  $\sigma(b)$  as in Definition 2.1(i). On the other hand, if both spectra,  $\sigma(a)$  and  $\sigma(b)$ , are infinite, it is always possible to choose infinite subsets  $\{\lambda_1, \lambda_2, \dots\} \subset \sigma(a)$  and  $\{\lambda_1, \lambda_2, \dots\} \subset \sigma(b)$  such that the numbers  $\lambda_i - \mu_j$  ( $i, j = 1, 2, \dots$ ) are all different and the same is true for the numbers  $\lambda_i \mu_j$  ( $i, j = 1, 2, \dots$ ). Hence, by Definition 2.1,  $D_{a,b}$  and  $M_{a,b}$  are of the second kind.

### 3. Elementary operators with normal coefficients

In this and subsequent sections we assume that all coefficients are normal operators.

**3.1.** Let  $\mathbf{a} = (a_1, a_1, \dots, a_n)$  be a normal  $n$ -tuple (consisting of commuting normal operators) and let  $C^*(\mathbf{a})$  be the commutative  $C^*$ -subalgebra in  $\mathcal{B}(\mathcal{H})$  generated by 1 and  $a_i$ ,  $i = 1, 2, \dots, n$ . It is well known that in this case the maximal ideal space of  $C^*(\mathbf{a})$  can be identified with the joint spectrum  $\sigma(\mathbf{a})$ . By the inverse of the Gelfand representation every element of  $C(\sigma(\mathbf{a}))$ , that is, every continuous function  $f$  on  $\sigma(\mathbf{a})$ , determines a unique element  $f(\mathbf{a})$  in  $C^*(\mathbf{a})$ . It is also known that the map  $f \mapsto f(\mathbf{a})$  can be extended from  $C(\sigma(\mathbf{a}))$  to all bounded Borel functions on  $\sigma(\mathbf{a})$ . Furthermore, there exists a unique spectral measure  $p$ , defined on the  $\sigma$ -algebra of

all Borel subsets in  $\sigma(\mathbf{a})$ , such that

$$f(\mathbf{a}) = \int_{\sigma(\mathbf{a})} f(\lambda) dp_{\lambda}$$

for every bounded Borel function  $f$  on  $\sigma(\mathbf{a})$  (see Rudin [17, Theorem 12.22]). We shall call  $p$  the spectral resolution of identity for the normal  $n$ -tuple  $\mathbf{a}$ .

Having now two normal  $n$ -tuples  $\mathbf{a}$  and  $\mathbf{b}$  with spectral resolutions of identity  $p$  and  $q$ , respectively, one can suggestively write elementary operator  $R_{\mathbf{a}, \mathbf{b}}$  in the “spectral” form

$$(3) \quad R_{\mathbf{a}, \mathbf{b}}x = \iint_{\sigma(\mathbf{a}) \times \sigma(\mathbf{b})} (\lambda \cdot \mu) dp_{\lambda} x dq_{\mu}$$

( $x \in \mathcal{B}(\mathcal{H})$ ). One may guess from this expression that spectral projections for  $R_{\mathbf{a}, \mathbf{b}}$  corresponding to any Borel subsets  $\delta \subset \mathbb{C}$  should be given by

$$x \mapsto \iint_{\lambda \cdot \mu \in \delta} dp_{\lambda} x dq_{\mu}$$

but, of course, the double integral is still to be defined. However, this is possible only for finite rank operators  $x$ .

For fixed  $\xi, \eta \in \mathcal{H}$  and every  $\zeta \in \mathcal{H}$  put  $(\xi \otimes \eta)\zeta = \langle \zeta, \eta \rangle \xi$ , where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathcal{H}$ . Let  $x$  be a finite rank operator on  $\mathcal{H}$  written in the form  $x = \sum_{i=1}^m \xi_i \otimes \eta_i$ . For each Borel subset  $\delta \subset \mathbb{C}$  and each  $\zeta, \theta \in \mathcal{H}$  define

$$m_{\zeta, \theta}(\delta) = \sum_{i=1}^m \iint_{\lambda \cdot \mu \in \delta} \langle dp_{\lambda} \xi_i, \theta \rangle \langle dq_{\mu} \zeta, \eta_i \rangle.$$

Here, of course, the double integral is nothing else but the value of the product measure of  $\langle p(\cdot) \xi_i, \theta \rangle$  and  $\langle q(\cdot) \zeta, \eta_i \rangle$  on the subset  $\{(\lambda, \mu); \lambda \cdot \mu \in \delta\} \subset \mathbb{C}^n \times \mathbb{C}^n$ .

Obviously,  $m_{\zeta, \theta}(\delta)$  depends linearly on  $\zeta$  and antilinearly on  $\theta$ . Since it is also bounded by

$$4 \left( \sum_{i=1}^m \|\xi_i\| \|\eta_i\| \right) \|\zeta\| \|\theta\|$$

(see Diestel, Uhl [3, page 4]), the map  $(\zeta, \theta) \mapsto m_{\zeta, \theta}(\delta)$  defines a bounded linear operator  $y \in \mathcal{B}(\mathcal{H})$ .

**3.2. DEFINITION.** For each finite rank operator  $x$  and each Borel subset  $\delta \subset \mathbb{C}$  let

$$(4) \quad \iint_{\lambda \cdot \mu \in \delta} dp_{\lambda} x dq_{\mu}$$

be the unique bounded linear operator  $y \in \mathcal{B}(\mathcal{H})$  such that  $\langle y\zeta, \theta \rangle = m_{\zeta, \theta}(\delta)$ .

The above definition is correct since it can be shown that  $m_{\zeta, \theta}(\delta)$  is independent of the concrete representation of  $x$  in the form  $x = \sum_{i=1}^m \xi_i \otimes \eta_i$ .

Thus, the double integral (4) is defined (in the weak sense) for any finite rank operator  $x$  and is, obviously, linear in  $x$ .

**3.3. REMARKS.** (a) Note that, unlike  $x$ , the value of the double integral (4) is in general not an operator of finite rank. However, it will follow from the next proposition that it always belongs to the Hilbert-Schmidt class.

(b) Note that (4) can be written in the form

$$\iint \chi_{\delta}(\lambda \cdot \mu) dp_{\lambda} x dq_{\mu}$$

where  $\chi_{\delta}$  is the characteristic function of the set  $\delta$ . In a similar way as above one can define also

$$\iint f(\lambda \cdot \mu) dp_{\lambda} x dq_{\mu}$$

for every Borel function  $f$  on  $\sigma(R_{\mathbf{a}, \mathbf{b}})$  (see (2)).

**3.4.** In the sequel we shall consider also restrictions of  $R_{\mathbf{a}, \mathbf{b}}$  to von Neumann-Schatten ideals  $\mathcal{E}_p(\mathcal{H})$ ,  $1 \leq p < \infty$ , in  $\mathcal{B}(\mathcal{H})$ . Recall that  $\mathcal{E}_p(\mathcal{H})$  consists of all compact operators  $x \in \mathcal{B}(\mathcal{H})$  such that

$$\|x\|_p = (\operatorname{tr}|x|^p)^{1/p} < \infty,$$

where  $\operatorname{tr}$  is the trace function and  $|x| = (x^*x)^{1/2}$ . Equivalently

$$\|x\|_p = \left( \sum_{i=1}^{\infty} \lambda_i^p \right)^{1/p} < \infty,$$

where  $\lambda_i$  are eigenvalues for  $|x|$  denumerated according to their multiplicities (see Gohberg and Krein [10, page 92]). Now  $\mathcal{E}_p(\mathcal{H})$  is a Banach space in the above norm, and its dual space may be identified with  $\mathcal{E}_q(\mathcal{H})$  where  $1/p + 1/q = 1$  (Gohberg and Krein [10, pages 129–132]). For  $p = 1$  one must take  $q = \infty$  and  $\mathcal{E}_{\infty}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$ . Duality is given by the bilinear form  $(x, y) \mapsto \operatorname{tr}(xy)$ ,  $x \in \mathcal{E}_p(\mathcal{H})$ ,  $y \in \mathcal{E}_q(\mathcal{H})$ .

Note that the Hilbert-Schmidt class  $\mathcal{E}_2(\mathcal{H})$  is in fact a Hilbert space with inner product  $\langle x, y \rangle = \operatorname{tr}(xy^*)$ .

**3.5. PROPOSITION.** *The restriction of the elementary operator  $R_{\mathbf{a}, \mathbf{b}}$  with normal commuting coefficients to  $\mathcal{E}_2(\mathcal{H})$  is a normal operator. If  $E_{\mathbf{a}, \mathbf{b}}$  is its*



resolution of identity, we have

$$E_{\mathbf{a}, \mathbf{b}}(\delta)x = \iint_{\lambda, \mu \in \delta} dp_{\lambda} x dq_{\mu}$$

for every Borel subset  $\delta \subset \mathbb{C}$  and every finite rank operator  $x$ .

PROOF. The normality of  $R_{\mathbf{a}, \mathbf{b}}|_{\mathcal{E}_2(\mathcal{H})}$  is a consequence of the equality

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j^* x b_j^* b_i = \sum_{i=1}^n \sum_{j=1}^n a_j^* a_i x b_i b_j^*$$

valid by Fuglede's theorem (Rudin [17, page 300]) for each  $x \in \mathcal{B}(\mathcal{H})$ , and the identity

$$(R_{\mathbf{a}, \mathbf{b}}|_{\mathcal{E}_2(\mathcal{H})})^* = R_{\mathbf{a}^*, \mathbf{b}^*}|_{\mathcal{E}_2(\mathcal{H})}$$

which can be easily verified using the definition of inner product in  $\mathcal{E}_2(\mathcal{H})$ . Here  $\mathbf{a}^* = (a_i^*, a_2^*, \dots, a_n^*)$  and  $\mathbf{b}^* = (b_1^*, b_2^*, \dots, b_n^*)$ .

By the spectral representation theorem (see Dunford and Schwartz [5, Theorems X.5.1 and X.5.3]) there exist a finite positive scalar measure  $m_{\mathbf{a}}$  on a measurable space  $(S, \Sigma_S)$  and a unitary operator  $u: \mathcal{H} \rightarrow L^2(m_{\mathbf{a}})$  such that for every bounded Borel function  $f$  on  $\sigma(\mathbf{a})$  the operator  $uf(\mathbf{a})u^{-1}$  is multiplication by a measurable function  $f \circ \alpha$  ( $\alpha: S \rightarrow \sigma(\mathbf{a})$ ) on the space  $L^2(m_{\mathbf{a}})$ . In the same way there exist another finite positive scalar measure  $m_{\mathbf{b}}$  on a measurable space  $(T, \Sigma_T)$  and a unitary operator  $v: \mathcal{H} \rightarrow L^2(m_{\mathbf{b}})$  such that for every bounded Borel function  $g$  on  $\sigma(\mathbf{b})$  the operator  $vg(\mathbf{b})v^{-1}$  is multiplication by a measurable function  $g \circ \beta$  ( $\beta: T \rightarrow \sigma(\mathbf{b})$ ) on the space  $L^2(m_{\mathbf{b}})$ .

Now, we can identify the Hilbert-Schmidt class  $\mathcal{E}_2(\mathcal{H})$  with the space  $L^2(m_{\mathbf{a}} \times m_{\mathbf{b}})$  since this Hilbert space is isometrically isomorphic to the Hilbert space of all Hilbert-Schmidt operators from  $L^2(m_{\mathbf{b}})$  to  $L^2(m_{\mathbf{a}})$ . In this interpretation the restriction of  $R_{\mathbf{a}, \mathbf{b}}$  to  $\mathcal{E}_2(\mathcal{H})$  obviously becomes multiplication by the function  $(s, t) \mapsto \alpha(s) \cdot \beta(t)$  ( $s \in S, t \in T$ ) on the space  $L^2(m_{\mathbf{a}} \times m_{\mathbf{b}})$  while its spectral resolution of identity  $E_{\mathbf{a}, \mathbf{b}}$  taken on the Borel subset  $\delta \subset \sigma(R_{\mathbf{a}, \mathbf{b}})$  becomes multiplication by the characteristic function  $(s, t) \mapsto \chi_{\delta}(\alpha(s) \cdot \beta(t))$  ( $s \in S, t \in T$ ). Hence, the proof of the proposition will be completed if we show that, for each finite rank operator  $x$ , the value of the double integral (4), represented as an element of  $L^2(m_{\mathbf{a}} \times m_{\mathbf{b}})$ , is exactly the representation of  $x$  in  $L^2(m_{\mathbf{a}} \times m_{\mathbf{b}})$  multiplied by the above characteristic function. But this follows directly from Definition 3.2 and we omit the details (compare the proof of [13, Proposition 3.2]).

In the next section we shall find conditions under which the resolution of identity  $E_{\mathbf{a}, \mathbf{b}}$  can be extended from  $\mathcal{E}_2(\mathcal{H})$  to  $\mathcal{E}_p(\mathcal{H})$  for  $p > 2$ .

#### 4. Main results

We shall need later the following specific result on prespectral operators and their restrictions to an invariant, not necessarily closed, subspace (for basic facts on prespectrality and spectrality see Dowson [4]).

Let a Banach space  $(\mathcal{X}_0, \|\cdot\|)$  be continuously imbedded into another Banach space  $(\mathcal{X}, \|\cdot\|)$ , so that we can write simply  $\mathcal{X}_0 \subset \mathcal{X}$ . If a bounded linear operator  $T \in \mathcal{B}(\mathcal{X})$  leaves invariant the subspace  $\mathcal{X}_0$ , then its restriction  $T|_{\mathcal{X}_0}$  to  $\mathcal{X}_0$  defines on  $\mathcal{X}_0$  a linear operator  $T_0$  which is bounded in the norm of  $\mathcal{X}_0$  by the closed graph theorem.

**4.1. PROPOSITION.** *Using the above notation, let  $T$  and  $T_0$  be prespectral operators on the spaces  $\mathcal{X}$  and  $\mathcal{X}_0$ , respectively, with canonical decompositions  $T = S + Q$  and  $T_0 = S_0 + Q_0$  and with the spectral resolutions of identity  $E$  of the class  $\Gamma$  and  $E_0$  of the class  $\Gamma_0$ . If  $\Gamma|_{\mathcal{X}_0} \subset \Gamma_0$ , then the subspace  $\mathcal{X}_0$  is invariant also for operators  $S$ ,  $Q$  and  $E(\delta)$ , for every Borel subset  $\delta \subset \mathbb{C}$ , and we have  $S_0 = S|_{\mathcal{X}_0}$ ,  $Q_0 = Q|_{\mathcal{X}_0}$  and  $E_0(\delta) = E(\delta)|_{\mathcal{X}_0}$ .*

**PROOF.** Let us define  $\mathcal{X}_1 = \mathcal{X} \oplus \mathcal{X}_0$  with  $\|(x, x_0)\| = \|x\| + \|x_0\|$ ,  $T_1 = T \oplus T_0$ ,  $S_1 = S \oplus S_0$ ,  $Q_1 = Q \oplus Q_0$ ,  $E_1(\delta) = E(\delta) \oplus E_0(\delta)$  for every Borel subset  $\delta$ , and  $\Gamma_1 = \Gamma \oplus \Gamma_0$ . Then  $\Gamma_1$  is a total subset in  $\mathcal{X}'_1$ , the dual of  $\mathcal{X}_1$ , and  $T_1$  is a prespectral operator on  $\mathcal{X}_1$  with the canonical decomposition  $T_1 = S_1 + Q_1$  and with the spectral resolution of identity  $E_1$  of the class  $\Gamma_1$ . Let  $A$  be the bounded linear operator on  $\mathcal{X}_1$ , defined by  $A(x, x_0)$  for  $x \in \mathcal{X}$ ,  $x_0 \in \mathcal{X}_0$ . Then  $A$  commutes with  $T_1$  and therefore by Dowson [4, Theorems 4.22 and 5.23]  $A$  commutes also with  $S_1$  and  $Q_1$ . Since  $A'\Gamma_1 \subset \Gamma_1$  because of  $\Gamma|_{\mathcal{X}_0} \subset \Gamma_0$ , it follows from the proof of [4, Theorem 6.6] that  $A$  commutes also with  $E_1(\delta)$  for every Borel subset  $\delta \subset \mathbb{C}$  (see also Ricker [16]). Now, the proposition is a consequence of the definition of  $A$ .

**4.2.** Let us consider more closely elementary operators  $R_{\mathbf{a}, \mathbf{b}}$  of the first kind. If

$$\sigma(\mathbf{a}) = \bigcup_{i=1}^r A_i, \quad \sigma(\mathbf{b}) = \bigcup_{j=1}^s B_j$$

are decompositions from Definition 2.1(i), let us choose fixed points  $\lambda_i \in A_i$  and  $\mu_j \in B_j$  for each  $i = 1, 2, \dots, r$  and  $j = 1, 2, \dots, s$ . Then for every

pair  $(i, j)$  we have by Definition 2.1(i)

$$(5') \quad \lambda \cdot \mu = \lambda \cdot \mu_j \quad \text{for } \lambda \in A_i, \mu \in B_j$$

or

$$(5'') \quad \lambda \cdot \mu = \lambda_i \cdot \mu \quad \text{for } \lambda \in A_i, \mu \in B_j.$$

Let  $I'$  consist of all those pairs  $(i, j)$  where  $(5')$  holds and let  $I''$  be the rest of the set  $I = \{(i, j); i = 1, 2, \dots, r, j = 1, 2, \dots, s\}$ . So  $I'$  and  $I''$  are disjoint sets and  $I = I' \cup I''$ .

Now suppose that  $\mathbf{a}$  and  $\mathbf{b}$  are normal  $n$ -tuples with spectral decompositions of identity  $p$  and  $q$ , respectively, and let  $p_i = p(A_i)$  for  $i = 1, 2, \dots, r$  and  $q_j = q(B_j)$  for  $j = 1, 2, \dots, s$  be spectral projections belonging to the above decompositions of the joint spectra  $\sigma(\mathbf{a})$  and  $\sigma(\mathbf{b})$ . Note that  $p_1, p_2, \dots, p_r$  are all nontrivial pairwise orthogonal projections on  $\mathcal{H}$  with the sum equal to 1, and the same is true for  $q_1, q_2, \dots, q_s$ .

It is easy to see from (3) that, in the above notation, the action of an elementary operator  $R_{\mathbf{a}, \mathbf{b}}$  of the first kind with normal coefficients on a finite rank operator  $x \in \mathcal{B}(\mathcal{H})$  can be written in the form

$$R_{\mathbf{a}, \mathbf{b}}x = \sum_{(i, j) \in I'} (\mu_j \cdot \mathbf{a}) p_i x q_j + \sum_{(i, j) \in I''} p_i x q_j (\lambda_i \cdot \mathbf{b}).$$

Here we have

$$\mu_j \cdot \mathbf{a} = \sum_{k=1}^n \mu_{jk} a_k \quad \text{and} \quad \lambda_i \cdot \mathbf{b} = \sum_{k=1}^n \lambda_{ik} b_k.$$

It is clear that the same formula holds in fact for every  $x \in \mathcal{B}(\mathcal{H})$ . Since according to decompositions

$$\mathcal{H} = \sum_{i=1}^r p_i \mathcal{H}, \quad \mathcal{H} = \sum_{j=1}^s q_j \mathcal{H}$$

we have also

$$\mathcal{E}_p(\mathcal{H}) = \sum_{i=1}^r \sum_{j=1}^s p_i \mathcal{E}_p(\mathcal{H}) q_j$$

for all  $p$ ,  $1 \leq p \leq \infty$ , and since for each pair  $(i, j)$  the subspace  $p_i \mathcal{E}_p(\mathcal{H}) q_j$  of  $\mathcal{E}_p(\mathcal{H})$  is invariant under left multiplications by (Borel) functions of  $\mathbf{a}$  and under right multiplications by (Borel) functions of  $\mathbf{b}$ , we see that the operator  $R_{\mathbf{a}, \mathbf{b}}$  is equal to a direct sum of left and right multiplications by normal operators. As is well known, the direct sum of prespectral operators is again a prespectral operator (see Dowson [4, Lemma 10.1]). Hence, the following proposition is not surprising.

**4.3. PROPOSITION.** *The elementary operator  $R_{\mathbf{a}, \mathbf{b}}$  of the first kind with normal coefficients, acting on the space  $\mathcal{E}_p(\mathcal{H})$  with  $1 \leq p \leq \infty$ , is a scalar-type prespectral operator of the class  $\Gamma \cong \mathcal{E}_p(\mathcal{H})$ ,  $1/p + 1/q = 1$ . Its spectral resolution of identity is given by*

$$E_{\mathbf{a}, \mathbf{b}}(\delta)x = \sum_{(i, j) \in I'} \chi_\delta(\mu_j \cdot \mathbf{a}) p_i x q_j + \sum_{(i, j) \in I''} p_i x q_j \chi_\delta(\lambda_i \cdot \mathbf{b})$$

for every  $x \in \mathcal{E}_p(\mathcal{H})$  and every Borel subset  $\delta \subset \mathbb{C}$ .

**PROOF.** The above expression obviously defines a bounded spectral measure  $E_{\mathbf{a}, \mathbf{b}}$  concentrated on  $\sigma(R_{\mathbf{a}, \mathbf{b}})$  and commuting with  $R_{\mathbf{a}, \mathbf{b}}$ .

For  $p = 2$  this is exactly the resolution of identity from Proposition 3.5 for the normal operator  $R_{\mathbf{a}, \mathbf{b}}$  acting on the Hilbert space  $\mathcal{E}_2(\mathcal{H})$ . Since the operators  $\mu_j \cdot \mathbf{a}$  and  $\lambda_i \cdot \mathbf{b}$  are normal, it follows that all maps

$$\delta \mapsto \chi_\delta(\mu_j \cdot \mathbf{a}), \quad \delta \mapsto \chi_\delta(\lambda_i \cdot \mathbf{b})$$

are bounded and countably additive in the weak (and also in the strong) operator topology. Consequently, they are countably additive also in the weak  $*$ -topology on the Banach space  $\mathcal{E}_p(\mathcal{H})$  as the dual of  $\mathcal{E}_q(\mathcal{H})$  (see, for example, Conway [2]) for  $p > 1$  and the dual of compact operators for  $p = 1$ . So, the map

$$\delta \mapsto \text{tr}((E_{\mathbf{a}, \mathbf{b}}(\delta)x)y)$$

is a scalar measure for every  $x \in \mathcal{E}_p(\mathcal{H})$  and  $y \in \mathcal{E}_q(\mathcal{H})$  and  $E_{\mathbf{a}, \mathbf{b}}$  is of the class  $\Gamma \cong \mathcal{E}_p(\mathcal{H})$ .

The other properties needed for  $R_{\mathbf{a}, \mathbf{b}}$  to be a prespectral operator follow directly from the definition of  $R_{\mathbf{a}, \mathbf{b}}$  and  $E_{\mathbf{a}, \mathbf{b}}$ .

Also, the operator  $R_{\mathbf{a}, \mathbf{b}}$  is of scalar type. For  $p > 2$  this is a consequence of Proposition 4.1 since its restriction to  $\mathcal{E}_2(\mathcal{H})$  is normal and hence spectral of scalar type. For  $p < 2$  one may use the obvious fact that the Banach adjoint of the operator  $R_{\mathbf{a}, \mathbf{b}}$  on the space  $C_p(\mathcal{H})$  is the operator  $R_{\mathbf{b}, \mathbf{a}}$  acting on the space  $\mathcal{E}_q(\mathcal{H})$  where  $1/p + 1/q = 1$ . Note that  $R_{\mathbf{a}, \mathbf{b}}$  is an elementary operator of the first kind with normal coefficients if and only if the same is true for  $R_{\mathbf{b}, \mathbf{a}}$ .

**4.4. REMARK.** Because of duality between  $\mathcal{E}_p(\mathcal{H})$  and  $\mathcal{E}_q(\mathcal{H})$  it is obvious that for  $1 \leq p < \infty$  the elementary operator  $R_{\mathbf{a}, \mathbf{b}}$  of the first kind with normal coefficients is in fact not only prespectral but also spectral (of scalar type). For  $p = \infty$  this is not true; in general we have only prespectrality of the class  $\Gamma \cong \mathcal{E}_1(\mathcal{H})$ . However, the next theorem covers this special case.

**4.5. THEOREM.** *An elementary operator  $R_{\mathbf{a}, \mathbf{b}}$  of the first kind with normal coefficients, acting on the space  $\mathcal{B}(\mathcal{H})$ , is a (scalar-type) spectral operator if and only if there exist decompositions*

$$\sigma(\mathbf{a}) = \bigcup_{i=1}^r A_i, \quad \sigma(\mathbf{b}) = \bigcup_{j=1}^s B_j$$

*which satisfy besides the condition (i) of Definition 2.1 also the following: the map  $(\lambda, \mu) \mapsto \lambda \cdot \mu$  is constant on every rectangle  $A_i \times B_j$  with projections  $p_i = p(A_i)$  and  $q_j = q(B_j)$  of infinite rank.*

**PROOF.** Let us suppose we have a decomposition with the property that  $\lambda \cdot \mu = \lambda_i \cdot \mu_j$  for  $\lambda \in A_i$  and  $\mu \in B_j$  and every pair  $(i, j) \in I$  where  $p_i$  and  $q_j$  are projections of infinite rank (see 4.2). Then for every  $x \in \mathcal{B}(\mathcal{H})$  and every Borel subset  $\delta \subset \mathbb{C}$  we have, according to Proposition 4.3, that

$$E_{\mathbf{a}, \mathbf{b}}(\delta)x = \sum_{i,j} \chi_\delta(\lambda_i \cdot \mu_j) p_i x q_j + \sum'_{i,j} \chi_\delta(\mu_j \cdot \mathbf{a}) p_i x q_j + \sum''_{i,j} p_i x q_j \chi_\delta(\lambda_i \cdot \mathbf{b}).$$

The first sum spreads over  $(i, j) \in I$  with infinite rank projections  $p_i$  and  $q_j$ , the second one over  $(i, j) \in I'$  where at least one of the projections,  $p_i$  or  $q_j$ , has finite rank, and the third one over  $(i, j) \in I''$  with the same property.

Obviously, each summand in the first sum is a countably additive operator function of  $\delta$ . Now, take any summand in the second sum. The map  $\delta \mapsto \chi_\delta(\mu_j \cdot \mathbf{a})$  is a countably additive operator function in the strong operator topology. Since  $p_i x q_j$  is of finite rank for each  $x \in \mathcal{B}(\mathcal{H})$ , the map

$$\delta \mapsto \chi_\delta(\mu_j \cdot \mathbf{a}) p_i x q_j$$

is countably additive in the norm topology. The same is true also for the map

$$\delta \mapsto p_i x q_j \chi_\delta(\lambda_i \cdot \mathbf{b})$$

since we have

$$p_i x q_j \chi_\delta(\lambda_i \cdot \mathbf{b}) = (\chi_\delta(\lambda_i \cdot \mathbf{b}) q_j x^* p_i)^*.$$

As a result we get countable additivity of the map

$$\delta \mapsto E_{\mathbf{a}, \mathbf{b}}(\delta)x$$

for every  $x \in \mathcal{B}(\mathcal{H})$  and hence  $R_{\mathbf{a}, \mathbf{b}}$  is a (scalar-type) spectral operator.

Let us now turn to the necessity of the given condition. Suppose that for each decomposition there is a rectangle  $A_i \times B_j$  such that both projections  $p_i$  and  $q_j$  are of infinite rank and yet the map  $(\lambda, \mu) \mapsto \lambda \cdot \mu$  is not constant on it. At least for one such a rectangle  $\sigma'_{ij} = \{\lambda \cdot \mu_j; \lambda \in A_i\}$  or

$\sigma''_{ij} = \{\lambda_i \cdot \mu; \mu \in B_j\}$  has to be infinite for otherwise a rectangle with the above property could not exist. Suppose  $\sigma'_{ij}$  is infinite. This set is the spectrum of the normal operator  $\mu_j \cdot \mathbf{a}|_{p_i \mathcal{H}}$  and we can find an infinite sequence of pairwise disjoint Borel subsets  $\delta_k \subset \sigma'_{ij}$  such that appropriate spectral projections  $p_{ij}(\delta_k) = \chi_{\delta_k}(\mu_j \cdot \mathbf{a})|_{p_i \mathcal{H}}$  for  $\mu_j \cdot \mathbf{a}|_{p_i \mathcal{H}}$  are all nontrivial. Now, take a partial isometry  $x \in \mathcal{B}(\mathcal{H})$ , mapping the subspace  $q_j \mathcal{H}$  onto the subspace  $p_i \mathcal{H}$  such that we have  $xx^* = p_i$ . Since both subspaces are of the same (infinite) dimension, such a partial isometry exists. Then we have

$$E_{\mathbf{a}, \mathbf{b}}(\delta_k)x = \chi_{\delta_k}(\mu_j \cdot \mathbf{a})p_i x.$$

Since

$$\begin{aligned} \|E_{\mathbf{a}, \mathbf{b}}(\delta_k)x\|^2 &= \|\chi_{\delta_k}(\mu_j \cdot \mathbf{a})p_i \chi_{\delta_k}(\mu_j \cdot \mathbf{a})\| \\ &= \|\chi_{\delta_k}(\mu_j \cdot \mathbf{a})p_i\| = \|p_{ij}(\delta_k)\| = 1, \end{aligned}$$

the map  $\delta \mapsto E_{\mathbf{a}, \mathbf{b}}(\delta)x$  is not countably additive in the norm topology and, hence,  $R_{\mathbf{a}, \mathbf{b}}$  is not spectral.

Let us now turn our attention to elementary operators of the second kind.

We first need the following simple result ( $A^\circ$  denotes the interior of the set  $A$ ).

**4.6. LEMMA.** *Let  $R_{\mathbf{a}, \mathbf{b}}$  be an elementary operator of the second kind. Then there exist two infinite sequences  $A_i$  and  $B_j$  of closed subsets in  $\mathbb{C}^n$  having the following properties*

- (i)  $A_i^\circ \cap \sigma(\mathbf{a}) \neq \emptyset$ ,  $B_j^\circ \cap \sigma(\mathbf{b}) \neq \emptyset$  for  $i, j = 1, 2, \dots$ .
- (ii)  $A_i \cap A_{i'} = \emptyset$ ,  $B_j \cap B_{j'} = \emptyset$  for  $i \neq i'$  and  $j \neq j'$ .
- (iii) If  $M_k = \{(i, j); 2^{k-1} \leq i, j \leq 2^k - 1\}$  for  $k = 1, 2, \dots$  and  $C_{i,j} = A_i \cdot B_j$  ( $= \{\lambda \cdot \mu; \lambda \in A_i, \mu \in B_j\}$ ) for  $i, j = 1, 2, \dots$ , then we have

$$C_{i,j} \cap C_{i',j'} = \emptyset$$

for  $(i, j), (i', j') \in \bigcup_{k=1}^{\infty} M_k$  and  $(i, j) \neq (i', j')$ .

**THE SKETCH OF THE PROOF.** Since the operator  $R_{\mathbf{a}, \mathbf{b}}$  is of the second kind, one can choose by definition infinitely many points  $\lambda_i, \lambda_2, \dots \in \sigma(\mathbf{a})$  and  $\mu_1, \mu_2, \dots \in \sigma(\mathbf{b})$  such that the points  $\lambda_i \cdot \mu_j$  ( $i, j = 1, 2, \dots$ ) are all different. Obviously, by taking an appropriate subsequence, this can be done in such a way that the points  $\lambda_i \cdot \mu_j$  are also isolated in the set  $\{\lambda_i \cdot \mu_j; i, j = 1, 2, \dots\}$ .

The desired sets  $A_i$  and  $B_j$  are defined inductively. Start with two closed polydiscs,  $A_1$  and  $B_1$ , centered in  $\lambda_1$  and  $\mu_1$ , respectively, such that  $\lambda_i \notin$

$A_1, \mu_j \notin B_1$  and  $\lambda_i \cdot \mu_j \notin \mathcal{E}_{1,1} = A_1 \cdot B_1$  for  $i, j > 1$ . Then, on the  $n$ th step, construct closed polydiscs  $A_i, B_j$  around the points  $\lambda_i, \mu_j$  with  $(i, j) \in M_n$  and small enough to ensure (separate) disjointness of the sets  $A_i$  for  $1 \leq i \leq n$  and  $\{\lambda_i; i > n\}$ , as well as of the sets  $B_j$  for  $1 \leq j \leq n$  and  $\{\mu_j; j > n\}$ , and also of the sets  $C_{i,j}$  for  $(i, j) \in \bigcup_{k=1}^n M_k$  and  $\{\lambda_i \cdot \mu_j; (i, j) \in \bigcup_{k>n} M_k\}$ . Proceeding in this way we get the required infinite sequences  $\{A_i\}$  and  $\{B_j\}$ .

4.7. We shall need also some elementary spectral properties of Hadamard matrices  $E_m$ , defined inductively in the following way

$$E_1 = [1], \quad E_{2m} = \begin{bmatrix} E_m & -E_m \\ E_m & E_m \end{bmatrix}$$

for  $m = 1, 2, 2^2, 2^3, \dots$ , and of related matrices  $E'_m$  constructed from  $E_m$  simply by replacing each  $-1$  with  $0$ .

It can be shown by a direct but rather cumbersome computation that for  $m = 1, 2, 2^2, 2^3, \dots$  the matrix  $|E_m| = (E_m^* E_m)^{1/2}$  has a single eigenvalue  $\lambda = m^{1/2}$  of multiplicity  $m$ , while for  $m = 2, 2^2, 2^3, \dots$  the matrix  $|E'_m| = (E'_m E_m^*)^{1/2}$  has an eigenvalue  $\lambda_0 = \frac{1}{2} m^{1/2}$  of multiplicity  $m - 2$  and two simple eigenvalues  $\lambda_1 = \lambda_0 u_m$  and  $\lambda_2 = \lambda_0 v_m$  where we have

$$u_m = \frac{1}{2}((m+8)^{1/2} + m^{1/2}) \quad \text{and} \quad v_m = \frac{1}{2}((m+8)^{1/2} - m^{1/2}).$$

In the following proposition  $E_{\mathbf{a}, \mathbf{b}}$  is the spectral resolution of identity for the operator  $R_{\mathbf{a}, \mathbf{b}}$  restricted to  $\mathcal{E}_2(\mathcal{H})$  (see Proposition 3.5).

4.8. PROPOSITION. Let  $R_{\mathbf{a}, \mathbf{b}}$  be the elementary operator defined by normal  $n$ -tuples  $\mathbf{a}$  and  $\mathbf{b}$  and let  $p > 2$ . If  $R_{\mathbf{a}, \mathbf{b}}$  is of the second kind, then there exist a Borel subset  $\delta \subset \mathbb{C}$  and a sequence of finite rank operators  $x_m$  such that  $\|x_m\|_p = 1$  for each  $m$  and  $\|E_{\mathbf{a}, \mathbf{b}}(\delta)x_m\|_p \rightarrow \infty$  as  $m \rightarrow \infty$ .

PROOF. If the operator  $R_{\mathbf{a}, \mathbf{b}}$  is of the second kind, one can construct closed sets  $A_i, B_j$  and  $C_{i,j} = A_i \cdot B_j$  as in Lemma 4.6. Let  $p$  and  $q$  be spectral resolutions of identity for  $n$ -tuples  $\mathbf{a}$  and  $\mathbf{b}$ , respectively, and denote  $p_i = p(A_i)$  and  $q_j = q(B_j)$  for  $i, j = 1, 2, \dots$ . Since  $A_i^\circ \cap \sigma(\mathbf{a}) \neq \emptyset$  and  $B_j^\circ \cap \sigma(\mathbf{b}) \neq \emptyset$  by 4.6(i), the projections  $p_i$  and  $q_j$  are nontrivial. Also, by 4.6(ii), the projections  $p_i$  are pairwise orthogonal and the same is true for  $q_j$ .

For each  $i$  and  $j$  choose  $\xi_i \in p_i \mathcal{H}$  and  $\eta_j \in q_j \mathcal{H}$  and put  $z_{ij} = \xi_i \otimes \eta_j$  (that is  $z_{ij} \zeta = \langle \zeta, \eta_j \rangle \xi_i$  for each  $\zeta \in \mathcal{H}$ ). Denote by  $\varepsilon_{ik}$  the elements of the direct sum of matrices  $E_m$  and by  $\varepsilon'_{ik}$  the elements of the direct sum of matrices  $E'_m$  ( $m = 1, 2, 2^2, 2^3, \dots$ ) from 4.7.

For  $m = 1, 2, 2^2, 2^3, \dots$  define

$$x_m = x^{-1/2-1/p} \sum_{r=m}^{2m-1} \sum_{s=m}^{2m-1} \varepsilon_{rs} z_{rs}$$

and

$$x'_m = m^{-1/2-1/p} \sum_{r=m}^{2m-1} \sum_{s=m}^{2m-1} \varepsilon'_{rs} z_{rs}.$$

Then  $x_m$  and  $x'_m$  are bounded operators on  $\mathcal{H}$ , of rank  $m$ , and the non-zero eigenvalues for  $|x_m|$  and  $|x'_m|$  are the same as the eigenvalues of matrices  $m^{-1/2-1/p}|E_m|$  and  $m^{-1/2-1/p}|E'_m|$ , respectively, together with their multiplicities.

So the norms of  $x_m$  and  $x'_m$  as elements of the class  $\mathcal{E}_p(\mathcal{H})$  can be computed as follows:

$$\|x_m\|_p = m^{-1/2-1/p} (m \cdot m^{p/2})^{1/p} = 1,$$

$$\|x'_m\|_p = m^{-1/2-1/p} ((m-2)\lambda_0^p + \lambda_1^p + \lambda_2^p)^{1/p} = \frac{1}{2} \left(1 - \frac{2}{m} + \frac{1}{m} u_m^p + \frac{1}{m} v_m^p\right)^{1/p}$$

with the notation from 4.7. It is easy to see that  $\frac{1}{m} u_m^p \rightarrow \infty$  and  $\frac{1}{m} v_m^p \rightarrow 0$  as  $m \rightarrow \infty$ . Hence,  $\|x'_m\|_p \rightarrow \infty$  as  $m \rightarrow \infty$ . This is true also for  $p = \infty$  because in this case we have  $\|x'_m\|_\infty = m^{-1/2} \lambda_1 = \frac{1}{2} u_m$ .

Now, let  $M = \{(i, j); \varepsilon_{ij} > 0\}$ . If  $M_k$  ( $k \geq 1$ ) is as in Lemma 4.6, then  $M \subset \bigcup_{k=1}^\infty M_k$  and by 4.6(iii)

$$\delta = \bigcup \{C_{i,j}; (i, j) \in M\}$$

is a countable union of pairwise disjoint closed sets  $C_{i,j}$  and hence a Borel subset in  $\mathbb{C}$ .

The rest of the proof consists in showing that  $E_{\mathbf{a}, \mathbf{b}}(\delta)x_m = x'_m$ . From Proposition 3.5 we have

$$\begin{aligned} E_{\mathbf{a}, \mathbf{b}}(\delta)x_m &= \iint_{\lambda, \mu \in \delta} dp_\lambda x_m dq_\mu \\ &= m^{-1/2-1/p} \sum_{r=m}^{2m-1} \sum_{s=m}^{2m-1} \varepsilon_{rs} \iint_{\lambda, \mu \in \delta} dq_\lambda z_{rs} dq_\mu \\ &= m^{-1/2-1/p} \sum_{r=m}^{2m-1} \sum_{s=m}^{2m-1} \varepsilon_{rs} \sum_{(i,j) \in M} \iint_{\lambda, \mu \in C_{i,j}} dp_\lambda z_{rs} dq_\mu. \end{aligned}$$

So, we have to compute the integrals

$$\iint_{\lambda, \mu \in C_{i,j}} dp_\lambda z_{rs} dq_\mu.$$



Since  $z_{rs}$  is of finite rank, the above integral is well defined (see 3.2). If  $\tau$  is a Borel subset in  $\mathbb{C}$  with  $\tau \cap A_r = \emptyset$  and  $\tau \cap B_s = \emptyset$ , then we have  $p(\tau)z_{rs} = z_{rs}q(\tau) = 0$ . This means the integration above spreads in fact only over  $A_r \times B_s$ . But, if  $\lambda \in A_r$  and  $\mu \in B_s$ , then we have  $\lambda \cdot \mu \in A_r \cdot B_s = C_{r,s}$ , and because of pairwise disjointness of these sets, the last integral is non-zero only when  $(i, j) = (r, s)$  (and hence  $(r, s) \in M$ ). In this case it is equal to

$$\iint_{\lambda \in A_r, \mu \in B_s} dp_\lambda z_{rs} dq_\mu = p(A_r)z_{rs}q(B_s) = p_r z_{rs} q_s = z_{rs}.$$

Thus, we have

$$\begin{aligned} E_{\mathbf{a}, \mathbf{b}}(\delta)x_m &= m^{-1/2-1/p} \sum_{\substack{r=m \\ (r,s)}}^{2m-1} \sum_{\substack{s=m \\ \in M}}^{2m-1} \varepsilon_{rs} z_{rs} \\ &= m^{-1/2-1/p} \sum_{r=m}^{2m-1} \sum_{s=m}^{2m-1} \varepsilon'_{rs} z_{rs} = x'_m. \end{aligned}$$

We can now formulate and prove our main result.

**4.9. THEOREM.** *Let  $R_{\mathbf{a}, \mathbf{b}}$  be an elementary operator with normal coefficients, defined on the space  $\mathcal{E}_p(\mathcal{H})$  with  $p \neq 2$ . Then  $R_{\mathbf{a}, \mathbf{b}}$  is a scalar-type prespectral operator of the class  $\Gamma \cong \mathcal{E}_q(\mathcal{H})$  if and only if it is of the first kind.*

**PROOF.** If  $R_{\mathbf{a}, \mathbf{b}}$  is of the first kind, its prespectrality follows from Proposition 4.3.

Now, let  $p > 2$ . If  $R_{\mathbf{a}, \mathbf{b}}$  is of the second kind, then by Proposition 4.8 we can find a Borel subset  $\delta \subset \mathbb{C}$  and a sequence of finite rank operators  $x_m$  with the property  $\|x_m\|_p = 1$  and  $\|E_{\mathbf{a}, \mathbf{b}}(\delta)x_m\|_p \rightarrow \infty$ . So, the spectral projection  $E_{\mathbf{a}, \mathbf{b}}(\delta)$  is not bounded in the norm  $\|\cdot\|_p$  and, by Proposition 4.1,  $R_{\mathbf{a}, \mathbf{b}}$  cannot be a prespectral operator on  $\mathcal{E}_p(\mathcal{H})$  of the class  $\Gamma \cong \mathcal{E}_q(\mathcal{H})$ .

In the case  $p > 2$  we use the duality between von Neumann-Schatten classes. If  $R_{\mathbf{a}, \mathbf{b}}$  is of the second kind on the space  $\mathcal{E}_p(\mathcal{H})$ , then its Banach adjoint  $R_{\mathbf{b}, \mathbf{a}}$  on the space  $\mathcal{E}_q(\mathcal{H})$  is also of the second kind. But now  $q > 2$  and by the first part of this proof  $R_{\mathbf{b}, \mathbf{a}}$  cannot be prespectral of the class  $\Gamma \cong \mathcal{E}_p(\mathcal{H})$ . Hence, neither can  $R_{\mathbf{a}, \mathbf{b}}$  be prespectral of any class (see Dowson [4, Theorem 5.22]).

**REMARK.** For  $1 \leq p < \infty$  the space  $\mathcal{E}_q(\mathcal{H})$  can be identified with the dual of  $\mathcal{E}_p(\mathcal{H})$  and Theorem 4.9 gives in fact a characterization of spectrality for the operator  $R_{\mathbf{a}, \mathbf{b}}$ . However, this is not true for  $p = \infty$ . In this case a characterization of spectrality has to be stated and proved separately.

**4.10. THEOREM.** *An elementary operator  $R_{\mathbf{a}, \mathbf{b}}$  with normal coefficients, acting of  $\mathcal{B}(\mathcal{H})$ , is a scalar-type spectral operator if and only if it is of the first kind and there exists a decomposition*

$$\sigma(\mathbf{a}) = \bigcup_{i=1}^r A_i, \quad \sigma(\mathbf{b}) = \bigcup_{j=1}^s B_j$$

*into closed disjoint subsets such that the map*

$$(\lambda, \mu) \mapsto \lambda \cdot \mu$$

*is constant on every rectangle  $A_i \times B_j$  with spectral projections  $p(A_i)$  and  $q(B_j)$  of infinite rank.*

**PROOF.** The theorem follows immediately from Theorems 4.5 and 4.9.

**CONCLUDING REMARKS.** (a) Since on the Hilbert space every spectral operator is similar to a normal operator modulo a commuting quasinilpotent operator (see Dowson [4]), there is a straightforward generalization of Theorem 4.9 to elementary operators with spectral coefficients (instead of only normal coefficients). The result is that such an operator is prespectral on  $\mathcal{E}_p(\mathcal{H})$  with  $p \neq 2$  of the class  $\Gamma \cong \mathcal{E}_q(\mathcal{H})$  if and only if it is of the first kind. Compare similar reduction to normal coefficients in [14, Section 1].

(b) In the special case of a generalized inner derivation  $D_{a,b}$  (or an elementary multiplication  $M_{a,b}$ ) with normal coefficients, acting on the space  $\mathcal{E}_p(\mathcal{H})$ ,  $p \neq 2$ , we find that it is a scalar-type prespectral operator of the class  $\Gamma \cong \mathcal{E}_q(\mathcal{H})$  if and only if at least one of the spectra,  $\sigma(\mathbf{a})$  or  $\sigma(\mathbf{b})$ , is finite. This follows immediately from the characterization in 4.9 and 2.5 (see also [13]). Acting on  $\mathcal{B}(\mathcal{H})$ ,  $D_{a,b}$  is a spectral operator if and only if both spectra,  $\sigma(\mathbf{a})$  and  $\sigma(\mathbf{b})$ , are finite (compare 4.10). The last result was given by J. Anderson and C. Foiaş in [1].

(c) For a generalized derivation  $D_{a,b}$  some kind of converse is true: if  $D_{a,b}$  is a prespectral operator on  $\mathcal{E}_p(\mathcal{H})$  of the class  $\Gamma \cong \mathcal{E}_q(\mathcal{H})$ , then the coefficients  $a$  and  $b$  also have to be spectral. This is proved in [14]. It is not known to the author whether the same holds for an elementary multiplication  $M_{a,b}$  instead of  $D_{a,b}$ .

(d) Finally, we can look on elementary operators as a source of interesting examples or counterexamples. For instance, it is well known that even on a reflexive Banach space the sum or the product of commuting spectral operators need not be spectral any more. Classical counterexamples were given by S. Kakutani and C. A. McCarthy many years ago (see, for example, Dowson [4, Chapter 9]). The generalized derivation  $D_{a,b}$  and the elementary multiplication  $M_{a,b}$  provide us with other very natural examples of the same

kind. Writing  $D_{a,b} = L_a - R_b$  and  $M_{a,b} = L_a R_b$ ; where  $L_a x = ax$  and  $R_b x = xb$  for every  $x \in \mathcal{B}(\mathcal{H})$ , we see from the main results in this paper that, for normal  $a$  and  $b$ , the operators  $L_a$  and  $R_b$  are spectral on  $\mathcal{E}_p(\mathcal{H})$ ,  $1 < p < \infty$ , while  $D_{a,b}$  and  $M_{a,b}$  are not spectral unless  $p = 2$  or at least one of the sets  $\sigma(a)$  and  $\sigma(b)$  is finite.

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### References

- [1] J. Anderson and C. Foiaş, 'Properties which normal operators share with normal derivations and related operators', *Pacific J. Math.* **61** (1975), 313–325.
- [2] J. B. Conway, *Subnormal operators*, (Research Notes in Mathematics 51, Pitman, Boston, London, Melbourne, 1981).
- [3] J. Diestel and J. J. Uhl, *Vector measures*, (Math. Surveys 15, Amer. Math. Soc., Providence, R. I., 1977).
- [4] H. R. Dowson, *Spectral theory of linear operators*, (Academic Press, London, New York, San Francisco, 1978).
- [5] N. Dunford and J. T. Schwartz, *Linear operators, part II: Spectral theory*, (Interscience, New York, London, 1963).
- [6] N. Dunford and J. T. Schwartz, *Linear operators, part III: Spectral operators*, (Wiley-Interscience, New York, London, Sydney, Toronto, 1971).
- [7] L. A. Fialkow, 'Spectral properties of elementary operators', *Acta Sci. Math. (Szeged)* **46** (1983), 269–282.
- [8] L. A. Fialkow, 'Spectral properties of elementary operators II', *Transl. Amer. Math. Soc.* **290** (1985), 415–429.
- [9] C. K. Fong and A. R. Sourour, 'On the operator identity  $\sum A_k X B_k = 0$ ', *Canad. J. Math.* **31** (1979), 845–857.
- [10] I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, (Transl. Math. Monographs 18, Amer. Math. Soc., Providence, R. I., 1960).
- [11] R. Harte, 'Spectral mapping theorems', *Proc. Roy. Irish Acad. Sect. A* **72** (1972), 89–107.
- [12] R. Harte, *Invertibility and singularity for bounded linear operators*, (Monographs and Textbooks in Pure and Applied Mathematics 109, Marcel Dekker, New York and Basel, 1988).
- [13] M. Hladnik, 'On prespectrality of generalized derivations', *Proc. Roy. Soc. Edinburgh Sect. A* **104** (1986), 93–106.
- [14] M. Hladnik, *When are generalized derivations spectral*, *Operator Theory: Advances and Appl.* **24**, pp. 215–226, Birkhäuser-Verlag, Basel, 1987).
- [15] M. Mathieu, 'Elementary operators on prime  $C^*$ -algebras I', *Math. Ann.* **284** (1989), 223–244.

- [16] W. Ricker, 'A commutativity criterion for prespectral operators', *Bull. Austral. Math. Soc.* **36** (1987), 113–119.
- [17] W. Rudin, *Functional analysis*, (McGraw-Hill, 1973).

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