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A CHARACTER-THEORY-FREE CHARACTERIZATION OF THE MATHIEU GROUP M_{12}

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Abstract

The known characterization of the Mathieu group M_{12} by the structure of the centralizer of a 2-central involution is based on the application of the theory of exceptional characters and uses in addition a block theoretic result which asserts that a simple group of order $|M_{12}|$ is isomorphic to M_{12}. The details of the proof of the latter result had never been published. We show here that M_{12} can be handled in a completely elementary and group theoretical way.

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The object of this paper is to present a character theory free proof of the following result.

THEOREM. *Let G be a finite, nonabelian simple group which possesses an involution such that its centralizer in G is isomorphic to the centralizer of a 2-central involution in M_{12}. Then G is isomorphic to M_{12}, the Mathieu group on* 12 *letters.*

The main point here is as in [7], that our proof will be completely free of applications of the theory of group characters. Predecessors of this theorem are [1, Theorem (6A)] and [12, Theorem]. The results of both papers had been easily combined in [4] to show that the above theorem holds. However, the proofs in [1] and in [12] rely upon a theorem of R. G. Stanton [8] which asserts that a simple group of order 95,040 or 244,823,040 is isomorphic to either M_{12} or M_{24}.

As a matter of fact, the Ph.D. thesis of R. G. Stanton, written under the supervision of Richard Brauer, had not been published, and there is only the short summary [8] available in which he describes his methods, which are heavily block theoretical and computational. Thus, it seems worthwhile to present an elementary and complete proof of the charcterization of M_{12} by a 2-central involution. We remark that in [6] we had shown that the characterization of M_{24} of [5] can be done without referring to the result of [8]; moreover, the second author has shown that for the characterizations of M_{22} and M_{23}, originally due to Z. Janko, one does not need recourse to the theory of exceptional characters.

1. The centralizer of a 2-central involution in M_{12}

Denote by H the centralizer of a 2-central involution z_1 of M_{12}. According to [4], H can be generated by elements $z_1, z_2, z_3, a, b, c, d$, where all elements except c are involutions and c is an element of order 3. The subgroup E generated by z_1, z_2, z_3 is elementary abelian of order 8 and normal in H, and $\langle a, b\rangle\langle c\rangle$ is isomorphic to A_4 such that $a^c = b$, $b^c = ab$. The involution d inverts c. The action on E for the elements a, b, c, d is described by the following matrices with entries from $GF(2)$ with respect to the basis $\{z_1, z_2, z_3\}$ of the "vector space" E over $GF(2)$:

$$a \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad b \to \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad ab \to \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$d \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad c \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We have $a^d = z_1 z_2 a$ and $b^d = z_1^\beta z_2 z_3 ab$, where $\beta \in \{0, 1\}$. Since d inverts c, we get from $a^{dcdc} = a$ that $\beta = 1$.

It is now a routine matter to calculate the conjugacy clases of H. The results are listed in Table I.

2. The fusion of involutions and the possible orders for G

In what follows, G denotes a finite simple group possessing an involution z such that $\mathbf{C}(z)$ is isomorphic to H. We put $\mathbf{C}(z) = H$ and use the notation developed for H so far. Thus, in particular, $z = z_1$, and since the center of a S_2-subgroup of H is cyclic, we see that z_1 is 2-central in G.

TABLE I

x	x^2	$o(x)$	$\mathbf{C}_H(x)$	$\lvert ccl_H(x)\rvert$
1		1	H	1
z_1		2	H	1
z_2		2	$E\langle a, d\rangle$	6
a		2	$\langle z_1, z_2, a, b\rangle$	12
z_3a	z_1	4	$\langle z_3a, z_2, b, z_3d\rangle$	6
z_2z_3a	z_1	4	$\langle z_2z_3a, z_2, z_3b, z_3d\rangle$	6
c		3	$\langle z_1, c\rangle$	32
d		2	$\langle z_1, z_2, d\rangle$	24
z_3d	z_2	4	$\langle z_1, z_3a, z_3d\rangle$	12
z_1z_3d	z_2	4	$\Vert$	12
bd	z_2z_3a	8	$\langle bd\rangle$	24
z_3bd	z_1z_3a	8	$\langle z_3bd\rangle$	24
z_1c		6	$\langle z_1, c\rangle$	32
				192

(2.1) LEMMA. *A S_2-subgroup of* $\mathbf{C}(a)$ *has order* 2^4.

PROOF. We have $\mathbf{C}_H(a) = \langle a\rangle \times \langle z_1, z_2, b\rangle$. Thus $\mathbf{C}_H(a)' = \langle z_1\rangle$. The assertion follows.

(2.2) LEMMA. *If $z_1 \sim z_2$ in G, then $O(\mathbf{C}(a)) = \langle 1\rangle$.*

PROOF. Put $K = O(\mathbf{C}(a))$ and act with the four-group $\langle z_1, z_2\rangle$ on K. If $x \in \langle z_1, z_2\rangle^\#$, then $\mathbf{C}(a) \cap \mathbf{C}(x)$ is a 2-group by the structure of H, and so, x operates fixed-point-freely on K. Application of [9, 5.1.9] yields that $[K, z_1z_2] = \langle 1\rangle$. We conclude that $K = \langle 1\rangle$.

(2.3) LEMMA. *If $z_1 \sim z_2$ in G, then $\mathbf{C}(a) = \langle a\rangle \times U$, where U possesses a subgroup of index* 2.

PROOF. A result of W. Gaschütz yields the existence of a subgroup U such that $\mathbf{C}(a) = \langle a\rangle \times U$. Clearly, $P = \langle a, z_1, z_2, b\rangle \cap U$ is dihedral of order 8 with center equal to z_1 as $\mathbf{C}_H(a)' = \langle z_1\rangle$. We have that a S_2-subgroup of U is selfnormalizing in U. Application of a result of O. Grün [3, 7.4.2] yields that $P \cap U'$ is contained in $\langle z_1, z_2\rangle$; note that the elements of $\langle z_1, z_2\rangle^\#$ are the only G-conjugates of z_1 in $\langle a, z_1, z_2, b\rangle$. Now, from [3, 7.3.1] it follows that U has a subgroup of index 2; use the fact that $P/P\cap U' \cong U/U^*$ for some normal subgroup U^* of U. The lemma is proved.

(2.4) LEMMA. *The normalizer of E in G is H.*

PROOF. By way of contradiction assume that $\mathbf{N}(E) \supset H$. Then $\mathbf{N}(E)/E \cong GL(3, 2)$; note that E is selfcentralizing. Thus, $z_1 \sim z_2$ and $a \sim d$ hold in $\mathbf{N}(E)$. It follows that G possesses precisely two classes of involutions. The representatives of these classes are z_1 and a.

From (2.3) we know that $\mathbf{C}(a) = \langle a \rangle \times U$, where U possesses a subgroup U_1 of index 2, and from (2.2) we get that $O(\mathbf{C}(a)) = \langle 1 \rangle$. If a S_2-subgroup of U_1 is cyclic then U is a 2-group by transfer results, and in this case $\mathbf{C}(a) = \mathbf{C}_H(a)$. Assume that a S_2-subgroup X of U_1 is a four-group. If $\mathbf{C}_H(a) \subset \mathbf{C}(a)$, then all involutions of U_1 are conjugate in U_1 as otherwise U_1 would possess a normal 2-complement which is $\langle 1 \rangle$ by (2.2). It follows that $(\mathbf{N}(X) \cap U_1)/(\mathbf{C}(X) \cap U_1)$ has order 3. Clearly, the commutator subgroup of $\mathbf{C}(a)$ lies in U_1. Since $\mathbf{C}_H(a)' = \langle z_1 \rangle$, we get $z_1 \in U_1$ and this implies that $\langle z_1, z_2 \rangle$ is a S_2-subgroup of U_1. Thus, $\mathbf{C}(X) \cap U_1 = X$ and $|\mathbf{N}(X) \cap U_1| = 2^2 \cdot 3$. Note that $\langle z_1, z_2 \rangle = \mathbf{Z}(E\langle a, d \rangle)$. Since $\mathbf{N}(E)/E \cong GL(3, 2)$, we see that $\mathbf{N}(\langle z_1, z_2 \rangle)$ lies in $\mathbf{N}(E)$ and has order $2^6 \cdot 3$. Thus $\mathbf{N}(\langle z_1, z_2 \rangle) \cap U_1 \subseteq \mathbf{N}(E)$. But then a would centralize an element of order 3 in $\mathbf{N}(E)$ which contradicts the structure of $GL(3, 2)$. We have shown that $\mathbf{C}(a) = \mathbf{C}_H(a)$.

We know that G has precisely two classes of involutions and that if x is an involution of $\mathbf{N}(E)$ then $\mathbf{C}(x) \subseteq \mathbf{N}(E)$. Application of [10, Lemma 5.35] yields $\mathbf{N}(E) = G$ which contradicts the simplicity of G. The assertion is proved.

(2.5) LEMMA. *The involution z_1 is conjugate in G to an element of $H \backslash \langle z_1 \rangle$.*

PROOF. Assume by way of contradiction that G is 2-normal. Put $T = E\langle a, b, d \rangle$. Since $\mathbf{Z}(T) = \langle z_1 \rangle$ and $T \in \mathrm{Syl}_2(G)$, we get from O. Grün's theorem [3, 7.5.2] that $T \cap G' = T \cap H'$. It would follow that G had a normal subgroup of index 2. Therefore G is not 2-normal. This implies the existence of an element g in G such that $z_1 \in T \cap T^g$, but $\langle z_1 \rangle \neq \mathbf{Z}(T^g)$. The center of T^g is $\langle z_1^g \rangle$, and so, $z_1 \neq z_1^g$. Since $z_1 \in T^g$, we have $[z_1, z_1^g] = 1$. It follows $z_1^g \in H \backslash \langle z_1 \rangle$.

(2.6) LEMMA. *In G we have $z_1 \sim z_2 \sim d$ and a is not conjugate to z_1.*

PROOF. We know that $\mathbf{N}(E) = H$ and that z_1 is not conjugate to a in G; remember that a S_2-subgroup of $\mathbf{C}(a)$ has order 2^4.

By way of contradiction we suppose that $d \sim a$ holds in G. From (2.5) we get $z_1 \sim z_2$. In $\langle a\rangle \times \langle z_1, z_2, b\rangle$ there are precisely two elementary abelian subgroups of order 8. Thus, as $d \sim a$, there are the following two possibilities:

(i) $$\langle a, z_1, z_2\rangle \sim \langle d, z_1, z_2\rangle;$$

(ii) $$\langle a, z_1, b\rangle \sim \langle d, z_1, z_2\rangle.$$

The possibility (ii) does not occur as in the group on the left there is only one G-conjugate of z_1 whereas in the group on the right there are three G-conjugates of z_1. Thus, we are in case (i). We get that the conjugation (i) is performed by an element of $\mathbf{N}(\langle z_1, z_2\rangle)$. Denote the latter group by N. Our assumptions imply that $|N| = 2^6 \cdot 3$, since $\langle z_1, z_2\rangle = \mathbf{Z}(E\langle a, d\rangle)$. Clearly, $O_2(N) = E\langle a, d\rangle = \mathbf{C}(\langle z_1, z_2\rangle)$. By a result of Baer-Suzuki [3, 3.8.2] there is an element k of order 3 in N which is inverted by b. We may assume that $k: z_1 \to z_2 \to z_1z_2$. The group $\langle k, b\rangle \cong \Sigma_3$ acts on $E\langle a, d\rangle/\langle z_1, z_2\rangle$. Clearly, $k \notin \mathbf{N}(E)$ as $\mathbf{N}(E) = H$. We have $o(z_3) = o(a) = o(d) = o(z_3ad) = 2$ and $o(z_3a) = o(z_3d) = o(ad) = 4$. Since k acts fixed-point-freely on $\langle z_1, z_2\rangle$, we see that k does not leave invariant any of the following cosets: $\langle z_1, z_2\rangle z_3a$, $\langle z_1, z_2\rangle z_3d$, $\langle z_1, z_2\rangle ad$. It follows that

$$k: \langle z_1, z_2\rangle z_3a \to \langle z_1, z_2\rangle z_3d \to \langle z_1, z_2\rangle ad.$$

Furthermore, $k: \langle z_1, z_2\rangle z_3 \to \langle z_1, z_2\rangle a$ and $k: \langle z_1, z_2\rangle z_3 \to \langle z_1, z_2\rangle d$ are both impossible as $z_1 \sim z_2 \sim z_3$, $a \sim d$, and a is not conjugate to z_1. But also $k: \langle z_1, z_2\rangle z_3 \to \langle z_1, z_2\rangle z_3ad$ cannot hold as $Ead \sim Ed$ in H and all involutions of Ed are conjugate. We have obtained a contradiction which shows that d cannot be conjugate to a in G.

Thompson's transfer theorem [10, 5.38] gives $d \sim z_2$ or $d \sim z_1$. Suppose that z_1 is not conjugate to z_2 in G. Application of (2.5) yields that $z_1 \sim d$ in G. Clearly, z_2 is not conjugate to a in G by (2.1). Thus, in G there are precisely three classes of involutions; representatives of these classes are z_1, z_2, a. There is an element x in $\mathbf{C}(d)$ normalizing $\langle z_1, z_2, d\rangle$ such that $x^2 \in \langle z_1, z_2, d\rangle$ and $x \notin \langle z_1, z_2, d\rangle$. Note that all elements of $\langle z_1, z_2\rangle d$ are conjugate, that $z_1 \sim d$, and that $z_2 \sim z_1z_2 \nsim z_1$. It follows that x centralizes z_1; this, however, is not possible. The contradiction shows that $z_1 \sim z_2 \sim d \nsim a$. The lemma is proved.

(2.7) Lemma. $\mathbf{C}(a) = \langle a\rangle \times L$, *where* $L \cong \Sigma_4$ *or* $L \cong \Sigma_5$. *The group* $\langle z_1, z_2\rangle\langle \gamma\rangle$ *is isomorphic to* A_4 *and lies in* L'; *here,* γ *is an element of order* 3 *of* $N = \mathbf{N}(\langle z_1, z_2\rangle)$.

PROOF. We have $\mathbf{C}_H(z_2) = E\langle a, d\rangle$ and $\mathbf{Z}(E(\langle a, d\rangle)) = \langle z_1, z_2\rangle$. As $z_1 \sim z_2 \sim z_1z_2$, we get that $N = \mathbf{N}(\langle z_1, z_2'\rangle)$ has order $2^6 \cdot 3$. Since $b \notin O_2(N)$, we get from the result of Baer-Suzuki that there is an element k of order 3 in N which is inverted by b. We know that $\mathbf{N}(E) = H$, and this implies that E is not normal in N. Hence, $[k, z_3] \notin \langle z_1, z_2\rangle$. Obviously, $\langle z_1, z_2\rangle z_3$ cannot be mapped onto $\langle z_1, z_2\rangle a$ as all involutions of E are conjugate and a is not conjugate to z_1. Via b, the cosets Ed and Ez_3ad are conjugate; so, all involutions of Ed and Ead are conjugate to z_1 in G. It follows that $\langle z_1, z_2\rangle a$ is kept fixed under the action of k in N. Therefore, k centralizes a conjugate of a in $\langle z_1, z_2\rangle a$. Clearly, $\langle z_1\rangle$ is a S_2-subgroup of $\mathbf{C}(c)$. If $\langle k\rangle$ was conjugate to $\langle c\rangle$ in G, then we would get $a \sim z_1$, which is impossible. We have shown that in G there are at least two classes of elements of order 3 and that 3 divides $|\mathbf{C}(a)|$; note that $c \sim c^{-1}$ in H.

Denote by γ an element of order 3 of N which centralizes a; there is such an element as all elements of $\langle z_1, z_2\rangle a$ are conjugate by the action of $\langle z_3, d\rangle$. Let $H_2 = \mathbf{C}(a)$. We have that $H_2 \supseteq \langle a, z_1, z_2, \gamma, b\rangle$ and that a S_2-subgroup of H_2 is of type $Z_2 \times D_8$. Since $\gamma \in N$, we get $\langle z_1, z_2\rangle\langle\gamma\rangle \cong A_4$. We know from (2.1) and (2.3) that $H_2 = \langle a\rangle \times L$ and that a S_2-subgroup of L is dihedral of order 8. Moreover, L possesses a subgroup L_1 of index 2. Obviously, $\langle z_1, z_2\rangle\langle\gamma\rangle \subseteq L_1$. Note that if $x \in \langle z_1, z_2\rangle^{\#}$ then $\mathbf{C}(x) \cap L_1 = \langle z_1, z_2\rangle$.

Let us assume that $\langle z_1, z_2\rangle\langle\gamma\rangle \subset L_1$. As $O(L_1)$ is characteristic in L_1 and hence normal in H_2, we get $O(L_1) = \langle 1\rangle$. Since $\mathbf{C}(z_1) \cap L_1 = \langle z_1, z_2\rangle$, there is no normal 2-subgroup of L_1 different from $\langle 1\rangle$. Denote by K a minimal normal subgroup of L_1. Then K has even order and is a simple group. It follows from [9, p. 129] that $K \cong A_5$. Thus, as K is normal in L_1, we get $L_1 = K$. It follows $L \cong \Sigma_5$. Clearly, if $L_1 \cong A_4$ then $L \cong \Sigma_4$. The lemma is proved.

In what follows we shall make use of J. G. Thompson's order formula [9, 5.1.7]. Thus, if x is an involution of G, we denote by $a(x)$ the number of pairs (u, v) such that $u \sim z_1$, $v \sim a$, and $x \in \langle uv\rangle$.

(2.8) LEMMA. *The integer* $a(z_1)$ *is equal to* 240.

PROOF. The roots of z_1 lie in the H-classes with the representatives z_1, z_3a,
z_2z_3a, bd, z_3bd, and z_1c.

Assume that $uv = z_1$. Then $u = z_1v$ and v is conjugate to a in H. But $z_1a \sim a$ in H and this shows that $uv = z_1$ is not possible.

Assume that $o(uv) = 4$. Then $uvuv = z_1$, and so $u^v = z_1 u$ which implies that $v \in \mathbf{N}_H(\langle z_1, u\rangle)\backslash\mathbf{C}(\langle z_1, u\rangle)$. First, we handle the case $u = z_2$. We have $\mathbf{C}(\langle z_1, z_2\rangle) = E\langle a, d\rangle$, and therefore, v is an involution of the coset $E\langle a, d\rangle b$. The relevant cosets with respect to E containing v are Eb, Eab. In $Ea \cup Eab$ there are 8 involutions conjugate to a. Thus, if $u = z_2$, there are 8 possibilities for v. But z_2 has precisely 6 conjugates in H, and therefore we get that there are $8 \cdot 6 = 48$ pairs (u, v) such that $o(uv) = 4$ and $u \sim z_2$ in H. Now assume that $u = d$. Thus $v \in \langle z_1, z_2, d\rangle z_3 a$. This coset contains precisely four involutions; these involutions form $\langle z_1, z_2\rangle d z_3 a$; but all these involutions are conjugate to z_1 in G and we cannot find such a v. Thus, if $o(uv) = 4$, the number of pairs (u, v) such that $z_1 \in \langle u, v\rangle$ is equal to 48.

Assume that $o(uv) = 6$. There is only one class of elements of order 6 in H; a representative for the class is $z_1 c$. We have $\mathbf{C}_H(z_1 c) = \langle z_1\rangle \times \langle c\rangle$. Thus $\mathbf{N}_H(\langle z_1 c\rangle) = \langle z_1\rangle \times \langle c, d\rangle$. Let $uv = z_1 c$. Then $v \sim a$ in H and v inverts c. This is not possible as all involutions of $\langle z_1, d\rangle$ are conjugate to z_1 in G. We have shown that $o(uv) = 6$ is not possible.

Finally, we handle the case $o(uv) = 8$. In H there are exactly two classes of elements of order 8; they are represented by bd and $z_3 bd$. The elements of $\langle bd\rangle$ are $1, bd, z_2 z_3 a, z_2 z_3 abd, z_1, z_1 bd, z_1 z_2 z_3 a, z_1 z_2 z_3 abd$. The involutions inverting bd lie all in $\langle bd\rangle d$ and are the following elements: $d, b, z_2 z_3 ad, z_2 z_3 ab, z_1 d, z_1 b, z_1 z_2 z_3 ad, z_1 z_2 z_3 ab$. Let $uv = bd$. Then $ubd = v$. We see that there are precisely four pairs (u, v) such that $uv = bd$. Thus there are precisely $4 \cdot 24 = 96$ pairs (u, v) such that uv lies in $ccl_H(bd)$. Let $uv = z_3 bd$. The elements of $\langle z_3 bd\rangle$ are $1, z_3 bd, z_1 z_3 a, abd, z_1, z_1 z_3 bd, z_3 a, z_1 abd$. The involutions inverting $z_3 bd$ are $d, z_3 b, z_1 z_3 ad, ab, z_1 d, z_1 z_3 b, z_3 ad, z_1 ab$. We have $uz_3 bd = v$, and it is easy to compute that there are precisely four pairs (u, v) such that $uv = z_3 bd$. Hence, there are precisely $4 \cdot 24 = 96$ pairs (u, v) such that $uv \in ccl_H(z_3 bd)$. It follows that $a(z_1) = 48 + 96 + 96 = 240$.

(2.9) Lemma. *If $L \cong \Sigma_4$ then $a(a) = 3$.*

Proof. From (2.7) we get $\mathbf{C}(a) = \langle a, z_1, z_2, \gamma, b\rangle$ and $\langle z_1, z_2\rangle\langle\gamma\rangle \cong A_4$. The conjugacy classes of $\mathbf{C}(a)$ are listed in Table II.

Put $H_2 = \mathbf{C}(a)$. Roots of the involution a are in the H_2-classes with the representatives a and $a\gamma$.

All involutions of H_2 conjugate to z_1 in G are conjugate to z_1 and H_2. It follows that there are precisely three pairs (u, v) such that $uv = a$.

Assume that $uv = a\gamma$. Clearly, u and v both invert γ, and $\mathbf{N}(\langle\gamma\rangle) \cap H_2 = (\langle a\rangle \times \langle\gamma\rangle)\langle b'\rangle$, where b' is an involution conjugate to a and G and inverting

TABLE II

x	$o(x)$	$\mathbf{C}_{\mathbf{C}(a)}(x)$	$\lvert ccl_{\mathbf{C}(a)}(x)\rvert$
1	1	$\langle a, z_1, z_2, \gamma, b\rangle$	1
a	2	$\Vert$	1
z_1	2	$\langle a, z_1, z_2, b\rangle$	3
az_1	2	$\Vert$	3
γ	3	$\langle a\rangle \times \langle \gamma\rangle$	8
$a\gamma$	6	$\Vert$	8
b	2	$\langle a, z_1, b\rangle$	6
ab	2	$\Vert$	6
bz_2	4	$\langle a, bz_2\rangle$	6
abz_2	4	$\Vert$	6
			48

γ. The involutions in $\langle z_1, z_2\rangle$ cannot invert γ. Thus, there is no pair (u, v) such that $uv = a\gamma$. we have proved that $a(a) = 3$.

(2.10) LEMMA. *If $L \cong \Sigma_4$, then $|G| = 2^6 \cdot 3^3 \cdot 7$.*

PROOF. We apply Thompson's order-formula and compute

$$\begin{aligned}|G| &= |\mathbf{C}(z_1)| \cdot a(a) + |\mathbf{C}(a)| \cdot a(z_1)\\ &= 192 \cdot 3 + 48 \cdot 240 = 12,096.\end{aligned}$$

(2.11) LEMMA. *If $L \cong \Sigma_5$ then $a(a) = 195$.*

PROOF. As we have remarked in (2.7), the group $\langle z_1, z_2\rangle\langle\gamma\rangle$ is isomorphic to A_4 and lies in L' which is isomorphic to A_5. We know that $\langle a, z_1, z_2, b\rangle \in \mathrm{Syl}_2(\mathbf{C}(a))$. The element γ of order 3 is centralized by an involution b' of $L\backslash L'$, and it is clear that $b' \sim a$ in G. Denote by w an element of order 5 of L. The conjugacy classes of $\mathbf{C}(a)$ are listed in Table III.

Put $H_2 = \mathbf{C}(a)$. The roots of a lie in the H_2-classes with the representatives $a, a\gamma, aw$.

Assume that $uv = a$. Then u runs through the 15 elements of $ccl_{\mathbf{C}(a)}(z_1)$. We have $z_1 a \sim a$ in H. Thus, there are precisely 15 pairs (u, v) such that $o(uv) = 2$ and $a \in \langle uv\rangle$.

Assume that $uv = a\gamma$. Clearly, u and v both invert γ. We have $\mathbf{C}(\gamma) \cap H_2 = \langle a, b', \gamma\rangle$, and there is an involution z conjugate to z_1 in L' which inverts γ and centralizes $\langle a, b'\rangle$. Thus, $\mathbf{N}(\langle\gamma\rangle) \cap H_2 = (\langle a, b'\rangle \times \langle\gamma\rangle)\langle z\rangle$.

TABLE III

x	$o(x)$	$\mathbf{C}_{\mathbf{C}(a)}(x)$	$\lvert ccl_{\mathbf{C}(a)}(x)\rvert$
1	1	$\langle a\rangle \times L$	1
a	2	$\lVert$	1
z_1	2	$\langle a, z_1, z_2, b\rangle$	15
γ	3	$\langle a, b', \gamma\rangle$	20
w	5	$\langle a, w\rangle$	24
$b'\gamma$	6	$\langle a, b', \gamma\rangle$	20
$b' \sim a$	2	$\langle a, z_1', b', \gamma\rangle$, $z_1' \sim z_1$	10
z_2b	4	$\langle a, z_2b\rangle$	30
az_1	2	$\langle a, z_1, z_2, b\rangle$	15
$a\gamma$	6	$\langle a, b', \gamma\rangle$	20
aw	10	$\langle a, w\rangle$	24
$ab'\gamma$	6	$\langle a, b', \gamma\rangle$	20
ab	2	$\langle a, z_1, b, \gamma'\rangle$, $\gamma' \sim \gamma$	10
az_2b	4	$\langle a, z_2b\rangle$	30
			240

There are precisely 12 involutions in $\mathbf{N}(\langle\gamma\rangle) \cap H_2$ which invert γ but only three of which are conjugate to z_1 in G. These are z, γz, $\gamma^{-1}z$. We have $z\cdot a\gamma^{-1}z = a\gamma$, $\gamma z\cdot az = a\gamma$, $\gamma^{-1}z\cdot a\gamma z = \gamma^{-1}a\gamma^{-1} = a\gamma$. It follows that there are precisely $3\cdot 20 = 60$ pairs (u, v) such that $o(uv) = 6$ and $a \in \langle uv\rangle$.

Assume finally that $uv = aw$; thus, $o(uv) = 10$. Clearly, u and v both invert w. We have $\mathbf{C}(w) \cap H_2 = \langle a, w\rangle$. Thus, $u, v \in \mathbf{C}^*(w) \cap H_2 = (\langle a\rangle \times \langle w\rangle)\langle z\rangle$, where $z \in L'$ and $zwz = w^{-1}$. As $z \in L'$, we have $z \sim z_1$ in L'. The involutions of $\mathbf{C}^*(w) \cap H_2$ conjugate to z_1 in G are precisely the five elements in $\langle w\rangle z$. Clearly, $w^iz\cdot x = aw$ has the solution $x = z^{-1}w^{-i}aw$ and $x \sim a$ in G. It follows that there are precisely $5\cdot 24 = 120$ pairs (u, v) such that $o(uv) = 10$. We conclude $a(a) = 15+60+120 = 195$. The lemma is proved.

(2.12) LEMMA. *If $L \cong \Sigma_5$ then $|G| = |M_{12}|$.*

PROOF. Compute

$$\begin{aligned}|\mathbf{C}(z_1)|\cdot a(a) + |\mathbf{C}(a)|\cdot a(z_1) &= 192\cdot 195 + 240\cdot 240\\ &= 95,040 = 2^6\cdot 3^3\cdot 5\cdot 11.\end{aligned}$$

3. The case $L \cong \Sigma_4$

We shall show that the group G with the title property does not exist.

Remember that G has at least two classes of elements of order 3, namely those represented by c and γ. Since $\langle z_1 \rangle$ is a S_2-subgroup of $\mathbf{C}(c)$, and $\langle a \rangle$ is one of $\mathbf{C}(\gamma)$, we see that both $\mathbf{C}(c)$ and $\mathbf{C}(\gamma)$ have normal 2-complements. Note that $\langle z_1, d \rangle$ is a S_2-subgroup of $\mathbf{N}(\langle c \rangle)$ and $\langle a, b' \rangle$ is a S_2-subgroup of $\mathbf{N}(\langle \gamma \rangle)$.

(3.1) LEMMA. *We have* $\mathbf{C}(c) = K_c \langle z_1 \rangle$, *where* K_c *is a normal subgroup of* $\mathbf{C}(c)$ *of order* 3^2 *or* 3^3, *and* $\mathbf{N}(\langle c \rangle) = K_c \langle z_1, d \rangle$. *Also,* $\mathbf{C}(\gamma) = K_\gamma \langle a \rangle$, *where* K_γ *is a normal subgroup of* $\mathbf{C}(\gamma)$ *of order* 3^2 *or* 3^3, *and* $\mathbf{N}(\langle \gamma \rangle) = K_\gamma \langle a, b' \rangle$.

PROOF. By a transfer result [3, 7.4.5] both $\mathbf{C}(c)$ and $\mathbf{C}(\gamma)$ have normal 2-complements. These normal 2-complements are normalized by four-groups. Consider for instance $\mathbf{C}(c)$. Put $K = K_c$. Then $K \trianglelefteq K\langle z_1, d \rangle$. The Frattini argument together with [9, 5.1.9] yields that K is a 3-group. As 3^3 divides $|G|$, we get that $|K| \in \{3^2, 3^3\}$.

(3.2) LEMMA. *Denote by* T *a* S_3*-subgroup and by* S *a* S_7*-subgroup of* G. *Then* $|G: \mathbf{N}(S)| \in \{64, 288\}$ *and* $|G: \mathbf{N}(T)| \in \{112, 448\}$. *If* $|G: \mathbf{N}(S)| = 288$, *then* $\mathbf{C}(S) = S$. *A* S_3*-subgroup of* G *is not cyclic.*

PROOF. From the order of G we get that G has no proper subgroup of index smaller than 9. Thus, from Sylow's theorem, we get $|G: \mathbf{N}(S)| \in \{36, 64, 288\}$ and $|G: \mathbf{N}(T)| \in \{16, 28, 64, 112, 488\}$. If $|G: \mathbf{N}(S)| = 36$, then $|\mathbf{N}(S)| = 2^4 \cdot 3 \cdot 7$; but then an involution would centralize an element of order 7 which is not the case.

Clearly, T is not cyclic, since $\langle c \rangle$ is not conjugate to $\langle \gamma \rangle$ in G. If we had $|G: \mathbf{N}(T)| \in \{16, 64\}$, then an element of order 7 in G would centralize a Sylow 3-subgroup which contradicts (3.1).

Suppose that $|G: \mathbf{N}(T)| = 28$. Then $|\mathbf{N}(T)| = 2^4 \cdot 3^3$. Assume that $T' \neq \langle 1 \rangle$. Then $|T'| = 3$ and an element of order 3 of G is centralized by a group of order 8 which is not possible. Thus $T' = \langle 1 \rangle$. If T was of type (3, 9), then T had a characteristic subgroup of order 3, and again we get a contradiction to the structures of centralizers of involutions in G. It follows that T is elementary abelian. From (3.1) we get $|\mathbf{C}(c)| = |\mathbf{C}(\gamma)| = 2 \cdot 3^3$. Therefore, in $\mathbf{N}(T)$ there are 8 conjugates of c and 8 conjugates of γ. By a lemma of Burnside [3, 7.1.1], there is an element x of order 3 in T which is not conjugate to c and not to γ in G. Thus, x is not centralized by an involution. It follows that x has 16 conjugates in $\mathbf{N}(T)$; but $8+8+16+1 > |T| = 27$. We have obtained a contradiction also in this case.

Assume finally that $|G: \mathbf{N}(S)| = 288$. Then $|\mathbf{N}(S)| = 2\cdot 3\cdot 7$. Let x be an element of order 3 in $\mathbf{C}(S)$. Then, the order of $\mathbf{C}(x)$ is either $3^2\cdot 7$ or $3^3\cdot 7$; note, that by (3.1), the element x cannot be centralized by an involution. This contradicts the order of $\mathbf{N}(S)$. The lemma is proved.

We shall now rule out all four cases of Lemma (3.2).

Case 1. Here we have $|\mathbf{N}(S)| = 3^3\cdot 7$ and $|\mathbf{N}(T)| = 2^2\cdot 3^3$. By assumption, an element of order 7 is centralized by a group of order 9. If $T' \neq \langle 1\rangle$, then T' is centralized by an involution and by an element of order 7 which is against (3.1). Thus T is abelian. Note that a S_2-subgroup of $\mathbf{N}(T)$ has order 4. Application of (3.1) yields that a S_2-subgroup of $\mathbf{N}(T)$ is conjugate in G to $\langle z_1, d\rangle$ and to $\langle a, b'\rangle$. But this contradicts the fact that $z_1 \sim d \sim z_1 d$ and a is not conjugate to z_1 in G. Case 1 is ruled out.

Case 2. Here we have $|\mathbf{N}(S)| = 3^3\cdot 7$ and $|\mathbf{N}(T)| = 3^3$. From a transfer result of Burnside we get $T' \neq \langle 1\rangle$. By assumption, an element of order 7 centralizes a subgroup of order 9. Thus, we may assume that $\mathbf{C}(S) = S\times R$, where R has order 9 and $R \subset T$. Evidently, $T' \subset R \subset T$. Put $S = \langle\sigma\rangle$ and $T' = \langle\xi\rangle$. As $\mathbf{N}(T) = T$, we see that ξ is not conjugate to its inverse; clearly, ξ is not centralized by an involution. Thus, $|\mathbf{C}(\xi)| = 3^3\cdot 7$. There are six G-conjugates of c and six G-conjugates of γ in T. Therefore, T is generated by elements of order 3. From [3, 5.3.9] it follows that T has exponent 3. Since $|\mathrm{Aut}_G(S)| = 3$, we see that σ is not conjugate to σ^{-1} in G. Let x, y be in $R^\#$ and $x \neq y$. Then, $x\sigma$ and $y\sigma$ have order 21 and are not conjugate in G, since such a conjugation would be performed in $\mathbf{C}(\sigma)$ which is abelian. Also, $x\sigma$, $x\sigma^{-1}$, and $y\sigma^{-1}$ lie in three pairwise different G-classes as $|\mathrm{Aut}_G(S)| = 3$. It follows that in $S\times R$ there are representatives for 16 G-classes of elements of order 21. If $x \in R^\#$, then $|\mathbf{C}(x\sigma)| = |\mathbf{C}(x\sigma^{-1})| = 3^2\cdot 7$. Our assumptions imply that $|\mathbf{C}(c)| = |\mathbf{C}(\gamma)| = 2\cdot 3^2$. As the centralizers of roots of involutions are known, we may write down the conjugacy classes of G discussed so far, and we see that G has at least 13,056 elements. Since $|G| = 12,096$, we have shown that Case 2 does not occur.

Case 3. Here we have $|\mathbf{N}(S)| = 2\cdot 3\cdot 7$ and $|\mathbf{N}(T)| = 2^2\cdot 3^3$. If T were abelian then we would get from (3.1) that $\langle z_1, d\rangle$ and $\langle a, b'\rangle$ are conjugate in G which, however, is not the case. Therefore, $T' \neq \langle 1\rangle$ and T' is centralized by an involution. Since $T' \sim \langle c\rangle$ or $T' \sim \langle\gamma\rangle$, there is a four-subgroup V in $\mathbf{N}(T)$. Acting with V on appropriate V-admissible sections of T/T', we see that $T = \Omega_1(T)$. Application of [3, 5.3.9] yields that $\exp(T) = 3$.

Put $\mathbf{N}(T) = T\langle\alpha, \beta\rangle$, where $\langle\alpha, \beta\rangle$ is a four-subgroup of G. Then, all involutions of $\langle\alpha, \beta\rangle$ are conjugate in G but fall into three $\mathbf{N}(T)$-classes. We know that $|\mathbf{C}_T(\alpha)| = |\mathbf{C}_T(\beta)| = |\mathbf{C}_T(\alpha\beta)| = 3$. Without loss of generality we may set $T' = \mathbf{C}_T(\alpha)$. The groups $\mathbf{C}_T(\beta)$ and $\mathbf{C}_T(\alpha\beta)$ are not conjugate in $\mathbf{N}(T)$, since $\beta \nsim \alpha\beta$ holds in $\mathbf{N}(T)$. It follows that a generator for $\mathbf{C}_T(\beta)$ has precisely six conjugates in $\mathbf{N}(T)$; the same is true for a generator for $\mathbf{C}_T(\alpha\beta)$. So far, we have got 14 elements of order 3 in T which are conjugate to an element of order 3 in $\mathbf{C}_G(\alpha)$. There is an element of order 3 in T which is centralized by an involution which is not conjugate to α. Such an element has precisely 12 conjugates in $\mathbf{N}(T)$. It follows that G has precisely two classes of elements of order 3. In particular, either c or γ is 3-central in G.

Writing down the complete table of the conjugacy classes of G, we get $|G| = 10, 752$. Therefore, Case 3 is ruled out.

Case 4. Here we have $|\mathbf{N}(S)| == 2 \cdot 3 \cdot 7$ and $\mathbf{N}(T) = T$. By a result of Burnside it is clear that $T' \neq \langle 1\rangle$. Hence $T' = \mathbf{Z}(T)$ has order 3. We have shown above that $|\mathbf{C}(S)| = 7$. Clearly, T' is not centralized by an involution and $|\mathbf{C}(T')| = 3^3$.

Consider $\mathbf{N}(\langle c\rangle)$. This group has order $2^2 \cdot 3^2$ and $\langle z_1, d\rangle$ as a S_2-subgroup with $z_1 \sim d \sim z_1 d$ in G and $dcd = c^{-1}$. From (3.1) and the order of H we get that the S_3-subgroup of $\mathbf{N}(\langle c\rangle)$ is elementary abelian. Let $\langle t, c\rangle$ be the subgroup of order 9 of $\mathbf{N}(\langle c\rangle)$. We may assume that t is 3-central in G. Thus $\langle t\rangle$ is not normalized by a 2-subgroup of G different from $\langle 1\rangle$. In particular, $t \nsim t^{-1}$ in G. It follows that t and t^{-1} each have four conjugates in $\mathbf{N}(\langle c\rangle)$. But c is not 3-central and has precisely two conjugates under the action of $\mathbf{N}(\langle c\rangle)$. Since $1 + 2 + 4 + 4 = 11 > 9$, we have obtained a contradiction which shows that Case 4 does not occur.

Summarizing we get

(3.3) LEMMA. *The case* $L \cong \Sigma_4$ *does not occur.*

4. The case $L \cong \Sigma_5$

From (3.3), (2.7), and (2.12) we conclude that $L \cong \Sigma_5$ and that $|G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$.

It is our aim to determine the structures of all Sylow normalizers and to write down the uniquelydetermined table of the conjugacy classes for G.

Further, we are interested in the normalizers of certain elementary subgroups of order 9 of G.

(4.1) LEMMA. *We have* $\mathbf{C}(c) = K_c\langle z_1\rangle$, *where* K_c *is a normal subgroup of* $\mathbf{C}(c)$ *of order* 3^2 *or* 3^3, *and* $\mathbf{N}(\langle c\rangle) = K_c\langle z_1, d\rangle$. *The four-group* $\langle a, b'\rangle$ *is a* S_2*-subgroup of* $\mathbf{C}(\gamma)$.

PROOF. Clearly $\langle z_1\rangle \in \mathrm{Syl}(\mathbf{C}(c))$. Thus $\mathbf{C}(c)$ has a normal 2-complement K_c. Now, $\langle z_1, d\rangle$ acts on K_c and all involutions of $\langle z_1, d\rangle$ are conjugate in G. Thus, the first assertion follows from [9, 5.1.9] and the Frattini argument. As for the second assertion, note that all involutions of $\langle a, b'\rangle$ are conjugate in G.

(4.2) LEMMA. *A Sylow 5-normalizer of* G *is contained in* $\mathbf{C}(a)$.

PROOF. Denote by w an element of order 5 of $\mathbf{C}(a)$. We have $\mathbf{C}(a) \cap \mathbf{C}(w) = \langle a\rangle \times \langle w\rangle$, and so $\langle a\rangle$ is a S_2-subgroup of $\mathbf{C}(w)$. Thus $\mathbf{C}(w)$ possesses a normal 2-complement K. There is an involution $z \in L'$ inverting w. Therefore, from the action of $\langle a, z\rangle$ on K and from [9, 5.1.9] together with the Frattini argument and (4.1) we get that $|K| \in \{5, 3\cdot 5, 3^2\cdot 5\}$. From the structure of $\mathbf{C}(a)$ follows that in G there is precisely one class of elements of order 5. Thus $\mathbf{N}(\langle w\rangle)/\mathbf{C}(w) \cong Z_4$. Since $|\mathbf{C}(w)| = 2\cdot|K|$, we get from Sylow's theorem that $\mathbf{C}(w) = \langle a\rangle \times \langle w\rangle$. The assertion follows.

(4.3) LEMMA. *A Sylow* 11*-normalizer of* G *is a Frobenius group of order* 55.

PROOF. Denote by e an element of order 11 in G. We know that $|\mathbf{C}(e)|$ is neither divisible by 2 nor by 3^3 as $|K_c| \in \{3^2, 3^3\}$. Also, there are no elements of order 55 in G. Thus $|\mathbf{C}(e)| \in \{11, 3\cdot 11, 3^2\cdot 11\}$. From a transfer result we get $|\mathrm{Aut}_G(\langle e\rangle)| \in \{2, 5, 10\}$. Application of Sylow's theorem yields $|\mathbf{N}(\langle e\rangle)| = 5\cdot 11$. The assertion follows.

(4.4) LEMMA. *A* S_3*-subgroup of* G *is nonabelian.*

PROOF. Assume by way of contradiction that a S_3-subgroup T of G is abelian. Since by (4.1) a four-group acts on T, we get that T is elementary abelian. From the above results we get $\mathbf{C}(T) = T$ and $\mathbf{N}(T) \supset T$. Since T cannot have automorphisms of order 5 or 11, we get that $\mathbf{N}(T)/T$ is a 2-group. From the order of $GL(3, 3)$ it follows that $|\mathbf{N}(T)/T| \le 2^5$. Sylow's theorem yields $|\mathbf{N}(T)| \in \{2^4\cdot 3^3, 2^2\cdot 3^3\}$. A lemma of Burnside implies that any two G-conjugate elements in T are conjugate under the action of $\mathbf{N}(T)$. Let $x, y \in T$ such that $x \sim c$, $y \sim \gamma$ in G. Then $\mathbf{C}(x)$ has order $2 \cdot 3^3$ and lies in $\mathbf{N}(T)$. We consider first the case that $|\mathbf{N}(T)| = 2^4 \cdot 3^3$. Then x has precisely 8 conjugates in T. We have $|\mathbf{C}(y)| = 2^2 \cdot 3^3$, and so, y has 4, 8, or 16 G-conjugates in T. Note that $1 + 8 + 16 = 25 < 27$. In $T^{\#}$ there must be an element t which is not centralized by an involution. Thus t has 16 conjugates in $\mathbf{N}(T)$. But $1 + 8 + 4 + 16 = 29 > 27$ gives a contradiction. Finally, consider the case $|\mathbf{N}(T)| = 2^2 \cdot 3^3$. From (4.1) we get that a S_2-subgroup of $\mathbf{N}(T)$ is conjugate to $\langle z_1, d\rangle$ in G. A result of W. Gaschütz implies that $\mathbf{C}(\gamma) = \langle\gamma\rangle \times X$, where $|X| = 2^2 \cdot 3^2$; remember that $\langle a, b'\rangle$ is a S_2-subgroup of X. Since z_1 is not conjugate to a in G, we see that a S_3-subgroup of $\mathbf{C}(\gamma)$ is not normal in $\mathbf{C}(\gamma)$. If a minimal normal subgroup of X is a 2-group or a group of order 3, then an involution of X is centralized by a subgroup of order 9 of $\mathbf{C}(\gamma)$. This contradiction proves the lemma.

(4.5) **Lemma.** *A S_3-subgroup T of G is nonabelian of exponent* 3 *and* $|\mathbf{N}(T)| = 2^2 \cdot 3^3$.

Proof. We know that $\mathbf{C}(T) = T' = \mathbf{Z}(T)$ from (4.1). For the order of $|\mathbf{N}(T)|$ we get the following possibilities from Sylow's theorem: 3^3, $2^2 \cdot 3^3$, $2^4 \cdot 3^3$. The case $|\mathbf{N}(T)| = 2^4 \cdot 3^3$ is not possible as an element of order 3 is not centralized by a group of order 8.

Let us assume $|\mathbf{N}(T)| = 3^3$. Put $T' = \langle\xi\rangle$. Then, by a lemma of Burnside, ξ is not inverted in G, and so, ξ is neither conjugate to c nor to γ in G. Thus, in G there are at least four classes of elements of order 3.

To obtain a contradiction we write down the table for the conjugacy classes of G obtained so far, and we get at least 105,600 elements in G. Thus, the case that $|\mathbf{N}(T)| = 3^3$ is ruled out.

We are left with the case $|\mathbf{N}(T)| = 2^2 \cdot 3^3$. Since $T' \neq \langle 1\rangle$, we get $T' \sim \langle c\rangle$ or $T' \sim \langle\gamma\rangle$. From the structure of $\mathbf{C}(a)$ it follows that a S_2-subgroup of $\mathbf{N}(\langle\gamma\rangle)$ is elementary of order 8. Thus, a S_2-subgroup of $\mathbf{N}(T)$ is elementary abelian of order 4. From the orders of the centralizers of involutions it follows that T is generated by elements of order 3. Since the nilpotency class of T is 2, application of [3, 5.3.9] yields that T has exponent 3. The lemma is proved.

(4.6) **Lemma.** *The element c is* 3-*central.*

Proof. Assume that there is a S_3-subgroup T of G such that $T' = \langle\gamma\rangle$. Then $\mathbf{C}(\gamma) = T\langle a, b'\rangle$. Consider the factor group $X = T\langle a, b'\rangle/\langle\gamma\rangle$.

From the orders of the centralizers of involutions and by the stabilizing-chain argument [3, 5.3.2], we get that no involution of X centralizes an element of order 3 of X. As this is not possible, we get that c is 3-central.

We are able to write down the uniquely determined table of the conjugacy classes of the simple group G of order 95,040.

TABLE IV

x	$o(x)$	$\lvert \mathbf{C}_G(x)\rvert$	$\lvert ccl_G(x)\rvert$
1	1	95,040	1
z_1	2	$2^6 \cdot 3$	495
a	2	$2^4 \cdot 3 \cdot 5$	396
z_3a	4	2^5	2970
z_2z_3a	4	2^5	2970
bd	8	2^3	11880
z_3bd	8	2^3	11880
c	3	$2 \cdot 3^3$	1760
z_1c	6	$2 \cdot 3$	15840
γ	3	$2^2 \cdot 3^2$	2640
$a\gamma$	6	$2^2 \cdot 3$	7920
w	5	$2 \cdot 5$	9504
aw	10	$2 \cdot 5$	9504
e_1	11	11	8640
e_2	11	11	8640
			95,040

We see that Table IV is identical with the table for the Mathieu group M_{12}.

(4.7) LEMMA. *Let $T \in \mathrm{Syl}_3(G)$ such that $T' = \langle c\rangle$. Then, $\mathbf{N}(T) = T\langle z_1, d\rangle$ and $\mathbf{N}(T)$ contains elementary abelian subgroups M_1, M_2 such that $\mathbf{N}(M_i)$ is a splitting extension of M_i by $GL(2, 3)$ for $i = 1, 2$. Further, M_1 is not conjugate to M_2 in G. There are two elementary abelian subgroups of order 9 in T which are conjugate in $\mathbf{N}(T)$ and have only two 3-central elements each.*

PROOF. Clearly $\mathbf{N}(T) = T\langle z_1, d\rangle$. Every involution of $\langle z_1, d\rangle$ is conjugate to z_1 in G and centralizes a subgroup of order 3 of T; clearly $T = \mathbf{C}_T(z_1) \cdot \mathbf{C}_T(d) \cdot \mathbf{C}_T(z_1 d)$. Put $\mathbf{C}_T(d) = \langle r\rangle$, $\mathbf{C}_T(z_1 d) = \langle s\rangle$. Then

$c \sim r \sim s$ holds in G. Put $M_1 = \langle c, r\rangle$ and $M_2 = \langle c, s\rangle$. Evidently, M_i is normal in $\mathbf{N}(T)$, and so, $M_i^{\#}$ consists only of 3-elements which are 3-central. In T there are 12 conjugates of γ, and this implies that $M_1 \cup M_2$ contains all the 14 3-central elements of T. Clearly, $\langle c, rs\rangle \sim \langle c, rs^{-1}\rangle$ via d and M_1 is not conjugate M_2 by a result of Burnside.

Since M_i possesses precisely four subgroups of order 3, we get that M_i is normalized by precisely four S_3-subgroups of G, one of which is T. Moreover, $\langle z_1, d\rangle$ normalizes M_i. Since $\mathbf{C}(M_i) = M_i$, we see that $\mathbf{N}(M_i)$ is a $(2, 3)$-group. As $|\mathbf{N}(M_i) : \mathbf{N}(M_i) \cap \mathbf{N}(T)| = 4$, we get $|\mathbf{N}(M_i)| = 2^4 \cdot 3^3$. It follows that $\mathbf{N}(M_i)$ is a splitting extension of M_i by $GL(2, 3)$ as T has exponent 3; $i = 1, 2$.

5. The identification of G with M_{12}

In what follows we shall change our notation completely, because we are going to find generators and relations for G as given in [11, p. 421]. So, we shall use from now on only the structural information obtained for G so far.

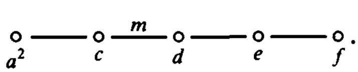
There is an elementary abelian subgroup M of order 9 of G, the normalizer of which is a splitting extension of M of $GL(2, 3)$. Since $\mathbf{C}(M) = M$, we get that $\mathbf{N}(M)$ is uniquely determined.

Studying Todd's presentation for M_{12}, we see that we may put $\mathbf{N}(M) = \langle a^2c, aca\rangle\langle a, b, e, f\rangle$ so that the relations between the generators of $\mathbf{N}(M)$ are those of Todd. Then, we have $M = \langle a^2c, aca\rangle$ and $\langle a, b, e, f\rangle \cong GL(2, 3)$.

Clearly, $\langle a, b, e, f\rangle$ lies in $\mathbf{C}(a^2) = H$ and a^2 is a 2-central involution. Since ef is an element of order 3 in H, it follows $f \in H \backslash O_2(H)$. We know that $\langle a, b\rangle$ is a normal quaternion subgroup of H. We may thus add a generator d for H so that Todd's relations hold between the generators of H. We get $H = \langle a, b, d, e, f\rangle$ and $\langle d, e, f\rangle \cong \Sigma_4$.

Thus, to prove $G \cong M_{12}$, it suffices to show that $(cd)^3 = 1$.

Consider the diagram



All relations represented by the diagram are known except $(cd)^m = 1$. It is easy to see that all involutions occurring in the diagram are 2-central in G.

We have $\langle a^2, c, d\rangle \subseteq \mathbf{C}(f) = H_f$. Since $o(a^2c) = 3$, we get that both a^2 and c are contained in $H_f \backslash O_{2,3}(H_f)$. The table of conjugacy classes of H shows that all involutions of $H \backslash O_2(H)$ are conjugate in H. It follows that

$\mathbf{C}(a^2) \cap H_f = \langle f, x_2, a^2 \rangle$ is elementary abelian of order 8, where $\langle f, x_2, x_3 \rangle$ is the normal elementary abelian subgroup of order 8 of H_f.

Now d lies in $\langle f, x_2, a^2 \rangle$. Assume first that $d \in \langle f, x_2 \rangle$. Then $cd \in \langle f, x_2, x_3 \rangle c$, and so the order of cd is 2 or 4.

Assume next that $d \in \langle f, x_2, a^2 \rangle \backslash \langle f, x_2 \rangle$. Then, $d = a^2 x$ with $x \in \langle f, x_2 \rangle$. It follows $cd \in ca^2 \langle f, x_2, x_3 \rangle$, and as $o(ca^2) = o(a^2 c) = 3$, we get that $o(cd) \in \{3, 6\}$. If $o(cd) = 6$, then the structure of H shows that $(cd)^3 = f$.

We have thus obtained the following possibilities for m: $m \in \{2, 4, 3, 6\}$.

First we shall eliminate the possibility $m = 2$. This is easy as the assumption $o(cd) = 2$ implies that in G the subgroup $\langle d, e, f \rangle$ which is isomorphic to Σ_4 is centralized by the element $a^2 c$ of order 3. But this contradicts the results of Table IV.

Next, we shall rule out the case $m = 6$. Assume that $o(cd) = 6$. Put $A = \langle c, d, e, f \rangle$. The generators of A respect the diagram

$$\underset{c}{\circ} \overset{6}{\text{———}} \underset{d}{\circ} \text{———} \underset{e}{\circ} \text{———} \underset{f}{\circ}$$

plus the additional relation $(cd)^3 f = 1$. Compute

$$(cd)^{edec} = c^{edec} d^{edec} = c^{dec} e = cedcdece = ecdcdc = efd.$$

It follows that the element efd of order 6 lies in the subgroup $\langle d, e, f \rangle$ of G which is isomorphic to Σ_4. This is a contradiction which shows that the case $m = 6$ is not possible.

Finally, we treat the case $o(cd) = 4$. Put $A = \langle a^2, c, d, e \rangle$ and $Y = \langle a^2, c, d \rangle$. The generators of A respect the diagram

$$\underset{a^2}{\circ} \text{———} \underset{c}{\circ} \overset{4}{\text{———}} \underset{d}{\circ} \text{———} \underset{e}{\circ}.$$

Therefore, A is an epimorphic image of the Coxeter group F_4 (see, for example, [2, Table 10]) and so, $|A|$ divides $2^6 \cdot 3^2$. We know that Y has order divisible by $2^3 \cdot 3$ and that Y is an epimorphic image of $Z_2 \times \Sigma_4$; see [2, Table 10]. Thus, Y is isomorphic to Σ_4 or to $Z_2 \times \Sigma_4$. The case $|A| = 24$ is not possible as then $Y = A$. But in Σ_4 there is no element of order 6. If $|A| = 2^4 \cdot 3$, then $A \cong Z_2 \times \Sigma_4$ which would imply $e \in \mathbf{Z}(A)$ as e centralizes the element $a^2 c$ of order 3 of A. Thus, if 3^2 does not divide $|A|$, then $|A| = 2^6 \cdot 3$ as $|\mathbf{N}_A(\langle a^2 c \rangle)| = 2^2 \cdot 3$; note that $a^2 c$ is 3-central in G. If $|A| = 2^6 \cdot 3$, then $|O_2(A)| = 2^5$, and the element $a^2 c$ centralizes the involution e which must lie in $O_2(A)$. This contradicts the fact that de has order 3.

Assume now that 3^2 divides $|A|$. Then $2^3 \cdot 3^2$ divides $|A|$ and since Σ_4 is present, we see that A is not 3-closed.

First suppose that $|A| = 2^3 \cdot 3^2$. A Sylow 2-subgroup of A is dihedral. If $O_3(A) = \langle 1 \rangle$, then $O_2(A)$ is a four-group which is centralized by a group of order 3, contradicting $O_3(A) = \langle 1 \rangle$. Thus, $O_3(A)$ has order 3. It follows that $A \cong (Z_3 \times A_4)Z_2$ with $A_4 Z_2 \cong \Sigma_4$. But then, a non-2-central involution of G would have a root of order 4 which is not the case.

We have proved that $2^4 \cdot 3^2$ divides $|A|$. Let X be a minimal normal subgroup of A. Then A is not a 3-group as A is not 3-closed and no element of order 3 of G is centralized by a subgroup of order 8. Hence, $|X| \in \{2, 2^2, 2^3\}$. From the structures of centralizers of involutions of G we get that X is a four-group. As X is centralized by an element of order 3 we get that all elements of $X^{\#}$ are non-2-central. Since a non-2-central involution of G has no roots of order 4, we see that all involultions of A centralize X. This implies that $\langle a^2, c \rangle$ centralizes X. But this is a contradiction, since $a^2 c$ is 3-central. The case $m = 4$ has been ruled out.

It remains $o(cd) = 3$. Thus G possesses elements a, b, c, d, e, f which satisfy the Todd relations for a presentation of the simple Mathieu group M_{12}. Since $|G| = |M_{12}|$, we get $G \cong M_{12}$, and we have reached our final goal.

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