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POSITIVE SEMIGROUPS OF OPERATORS ON BANACH SPACES

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Abstract

We prove a version of the Feller-Miyadera-Phillips theorem characterizing the infinitesimal generators of positive C_0-semigroups on ordered Banach spaces with normal cones. This is done in terms of $N(A)$ as well as the canonical half-norms of Arendt Chernoff and Kato defined by $N(a) = \inf\{\|b\| \,|b \geq a\}$, where $N(A) = \sup\{N(Aa)|N(a) \leq 1\}$ for operator A. A corresponding result on C_0^*-semigroups is also given.

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Let $(B, B_+, \|\ \|)$ be an ordered Banach space with proper closed convex cone B_+. The dual B^* is ordered by $B_+^* = \{f \in B^* | f(b) \geq 0$ for all $b \in B_+\}$. As in [1], [3], [6] and [7], the canonical half-norm N by $N(a) = \inf\{\|b\| \,|a \leq b\}$ for $a \in B$. For a linear operator A from B into itself, we define $N(A) = \sup\{N(Ax)|N(x) \leq 1\}$. We extend some recent results of Robinson [8], [9] by proving the following analog of the Feller-Miyadera-Phillips theorem (see [2] and [4]).

THEOREM 1. *Suppose B_+ is normal. Let H be a closed linear operator with domain $D(H)$, a dense subspace of B. Then, for constants M, ω, the following statements are equivalent.*

(i) *H generates a C_0-semigroup $\{S_t\}$ (so $S_t = e^{-tH}$) with $S_t \geq 0$ (that is $S_t(B_+) \subseteq B_+$) and $N(S_t) \leq Me^{\omega t}$, $t \geq 0$.*

(ii) *For all small* $\alpha > 0$, $(I + \alpha H)^{-1}$ *exists and is a positive linear operator on* B *such that*

$$N((I + \alpha H)^{-n}x) \leq M(1 - \alpha\omega)^{-n}N(x)$$

for all $x \in B$, $n \geq 1$.

(iii) *The range* $R(I + \alpha H) = B$ *and*

$$N((I + \alpha H)^n a) \geq (1 - \alpha\omega)^n N(a)/M$$

for all $a \in D(H^n)$, $n \geq 1$, *and for all small* $\alpha > 0$.

The equivalence of (ii) and (iii) follows easily from the closed graph theorem and the fact that $N(a) = 0$ if and only if $a \leq 0$. For (iii) $\Rightarrow$ (i), we use a suggestion in [1, Remark 4.2]: let $\|a\|_N = N(a) + N(-a)$. Then $\| \ \|_N$ is a norm on B equivalent to the given norm $\| \ \|$, because B_+ is assumed to be normal. The N-dissipative condition in (iii) implies the $\| \ \|_N$-dissipative condition:

$$\|(1 + \alpha H)^n a\|_N \geq (1 - \alpha\omega)^n \|a\|_N / M.$$

By the Feller-Miyadera-Phillips Theorem, H is the infinitesimal generator of a C_0-semigroup $\{S_t\}$, and $S_t x = \lim_{n\to\infty}(I + (t/n)H)^{-n}x$ for all $x \in B$. Since each $(I + (t/n)H)^{-n} \geq 0$ by (ii), it follows that $S_t \geq 0$. Also, by continuity of N, it follows from the N-dissipativity in (ii) that

$$N(S_t x) \leq \lim_{n\to\infty}\left[M\left(1 - \frac{t}{n}\omega\right)^{-n} N(x)\right] = Me^{t\omega}N(x)$$

for all $x \in B$. This shows that $N(S_t) \leq Me^{t\omega}$. Conversely, if (i) holds then, by the standard theory, $(I + \alpha H)^{-1}$ exists and is a continuous linear operator on B such that

$$(I + \alpha H)^{-n}x = \int_0^\infty (S_{\alpha t}x)\frac{t^{n-1}}{(n-1)!}e^{-t}\,dt.$$

Since $S_{\alpha t} \geq 0$ it follows that $(1 + \alpha H)^{-n} \geq 0$. Also, since N is convex and positively homogeneous, one has, by the following lemma and (i), that

$$\begin{aligned}
N((1 + \alpha H)^{-n}x) &\leq \int_0^\infty N(S_{\alpha t}x)\frac{t^{n-1}}{(n-1)!}e^{-t}\,dt \\
&\leq \int_0^\infty N(S_{\alpha t})N(x)\frac{t^{n-1}}{(n-1)!}e^{-t}\,dt \\
&\leq \int_0^\infty Me^{\alpha\omega t}N(x)\frac{t^{n-1}}{(n-1)!}e^{-t}\,dt \\
&= MN(x)(1 - \alpha\omega)^{-n},
\end{aligned}$$

proving (i) $\Rightarrow$ (ii)

LEMMA 1. *Let A be a linear operator on B and $\gamma \in \mathbf{R}$, $\gamma > 0$. The following statements are equivalent*:

(i) $N(A) \leq \gamma$;
(ii) $N(Ax) \leq \gamma N(x)$ *for all* $x \in B$.

We omit the proof of this easy lemma.

REMARK. If $N(A) < +\infty$ then $A \geq 0$.

LEMMA 2. *Suppose* $\| \;\|$ *is monotone on* B *and on the dual* B^*, *and let* A *be a positive linear operator on* B. *Then*

$$N(A) = \operatorname{Sup}\{N(Aa) | a \geq 0, N(a) \leq 1\} = \|A\|_+ \tag{1}$$

where $\|A\|_+$ *is the Robinson norm of* A *and is defined in* [9] *by*

$$\|A\|_+ = \sup\{\|Aa\| | a \geq 0, \|a\| \leq 1\}.$$

PROOF. Since $\| \;\|$ is monotone on B, $N(a) = \|a\|$ for $a \in B_+$. Since $\| \;\|$ is monotone on B^*, $N(a) = \inf\{\|b\| \,|b \geq a, 0\}$ for all $a \in B$ (see [7, Theorem 2.4], and also [5, Proposition 6]). Hence the second equality in (1) is clear. Moreover, for $a \in B$ with $N(a) \leq 1$,

$$\begin{aligned} N(Aa) &= \inf\{\|c\| \,|c \geq Aa, 0\} \\ &\leq \inf\{\|Ab\| \,|b \geq a, 0\} \leq \inf\{\|A\|_+\|b\| \,|b \geq a, 0\} \\ &= \|A\|_+ N(a) \leq \|A\|_+ \end{aligned}$$

which shows that $N(A) \leq \|A\|_+$. That $N(A) \geq \|A\|_+$ holds trivially in view of the second equality in (1). This completes our proof.

NOTE. In view of this lemma, Theorem 1, in the special case when $\| \;\|$ is monotone on B and B^*, is exactly the same as the theorem of Robinson [9, Theorem 1.1] which in turn generalizes [8, Theorem 3.5], and results in [1], [3] (extensions in line of Theorem 1 were also anticipated in [2, page 264] with less specific bounds). Likewise, our Theorem 2 below was given by Robinson [9], [8] for the special case stated. The following duality result will be important for our discussion of C_0^*-version of Theorem 1.

LEMMA 3. *Suppose* $(B, B_+, \| \;\|)$ *is the dual of an ordered Banach space* $(B_*, B_{*+}, \| \;\|)$ *with closed convex cone* B_{*+}. *Let* $A \in \mathscr{L}(B)$ *be the dual of an operator* $A_* \in \mathscr{L}(B_*)$. *Then* (i) $A \geq 0$ *if and only if* $A_* \geq 0$,
(ii) $N(A_*) = \|A\|_+$, *if* $A \geq 0$.

PROOF. As (i) is well known and easy to verify, we only prove (ii). General elements of B_* and B will usually be denoted by x and f respectively. By

[7, Theorem 2.1],

$$\begin{aligned} N(A_*x) &= \sup\{f(A_*x) | f \geq 0, \|f\| \leq 1\} \\ &= \sup\{(Af)(x) | f \geq 0, \|f\| \leq 1\} \\ &\leq \sup\{g(x) | g \in B, g \geq 0, \|g\| \leq \|A\|_+\} \\ &= \|A\|_+ N(x), \end{aligned}$$

which shows that $N(A_*) \leq \|A\|_+$. Here we have used the fact that if $g = Af$ with $f \geq 0$ and $\|f\| \leq 1$ then $g \geq 0$ and $\|g\| \leq \|A\|_+ \|f\| \leq \|A\|_+$.

On the other hand, for $f \geq 0$, $\|f\| \leq 1$, one has

$$\begin{aligned} \|Af\| &= \sup\{(Af)(x) | \|x\| \leq 1\} \\ &= \sup\{f(A_*x) | \|x\| \leq 1\} \\ &\leq \sup\{N(A_*x) | \|x\| \leq 1\} \\ &\leq \sup\{N(A_*)N(x) | \|x\| \leq 1\} \\ &\leq N(A_*), \end{aligned}$$

which shows that $\|A\|_+ \leq N(A_*)$. Here [7, Theorem 2.1] has been used again.

THEOREM 2. *Let $(B, B_+, \| \|)$ and $(B_*, B_{*+}, \| \|)$ be as in Lemma 3. Suppose $B = B_+ - B_+$. Let H be a w^*-closed linear operator with domain $D(H)$ a w^*-dense subspace of B. The following conditions are equivalent.*

(i) *H generates a C_0^*-semigroup $\{S_t\}$ with $S_t \geq 0$ and $\|S_t\|_+ \leq Me^{\omega t}$, $t \geq 0$.*

(ii) *For all small $\alpha > 0$, $(I + \alpha H)^{-1}$ exists such that*

$$\|(I + \alpha H)^{-n} f\| \leq M(1 - \alpha\omega)^{-n} \|f\| \tag{2}$$

for all $f \in B_+$, $n \geq 1$.

PROOF. We note first that since $B = B_+ - B_+$, the cone B_{*+} is normal in B_*. Since (2) is equivalent to

$$\|(I + \alpha H)^{-n}\|_+ \leq M(1 - \alpha\omega)^{-n} \tag{2'}$$

the proof of (i) $\Rightarrow$ (ii) is the same as that given in [8, Theorem 3.4] and [9, Theorem 1.2]. Conversely, if (ii) holds then, by Lemma 3, $(I + \alpha H)_*^{-n} = (I + \alpha H_*)^{-n}$ is a positive continuous linear operator on B_* such that

$$N((I + \alpha H_*)^{-n}) \leq M(1 - \alpha\omega)^{-n} \tag{3}$$

for all n and all small $\alpha > 0$, where H_* is norm-densely defined, normed-closed adjoint of H on B_*. By Theorem 1 applied to H_* and B_*, we conclude that H_* generates a C_0-semigroup $\{S_t^*\}$ on B_* with $S_t^* \geq 0$ and $N(S_t^*) \leq Me^{\omega t}$, $t \geq 0$. Then H generates the dual semigroup $\{S_t\}$ of $\{S_t^*\}$. Furthermore, by Lemma 3, $S_t \geq 0$ and $\|S_t\|_+ = N(S_t^*) \leq Me^{\omega t}$ for all $t \geq 0$.

Remark. In the special case $M = 1$ and $\omega = 0$, Theorem 1 corresponds to the Hille-Yosida theorem, that is, S is N-contractive (in the sense that $N(S_t) \leq 1$ for all t). The dissipative condition in (iii) then reduces to the single condition $N((I + \alpha H)a) \geq N(a)$ because the higher order conditions follow by iteration. Similarly, for $M = 1$ and $\omega = 0$, Theorem 2 simply states that H generates a C_0^*-semigroup of positive $\| \|_+$-contractions if and only if $(I + \alpha H)^{-1}$ is a positive w^*-continuous $\| \|_+$-contraction for all small $\alpha > 0$.

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