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ON THE ERROR ESTIMATES FOR THE RAYLEIGH-SCHRÖDINGER SERIES AND THE KATO-RELLICH PERTURBATION SERIES

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Abstract

Let λ be a simple eigenvalue of a bounded linear operator T on a Banach space X, and let (T_n) be a resolvent operator approximation of T. For large n, let S_n denote the reduced resolvent associated with T_n and λ_n, the simple eigenvalue of T_n near λ. It is shown that

$$\sup_{k=1,2,\dots} \frac{\|(T-T_n)S_n^k(T-T_n)S_n\|}{\|S_n\|^{k-1}} \to 0, \quad \text{as } n \to \infty,$$

under the assumption that all the spectral points of T which are nearest to λ belong to the discrete spectrum of T. This is used to find error estimates for the Rayleigh-Schrödinger series for λ and φ with initial terms λ_n and φ_n, where φ (respectively, φ_n) is an eigenvector of T (respectively, T_n) corresponding to λ (respectively, λ_n), and also for the Kato-Rellich perturbation series for PP_n, where P (respectively, P_n) is the spectral projection for T (respectively, T_n) associated with λ (respectively, λ_n).

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1. Introduction and preliminaries

Let X be a complex Banach space, and let T belong to the space $BL(X)$ of all bounded linear operators on X. Let λ be an isolated simple eigenvalue of T. We assume that $T \neq \lambda I$. Let Γ denote a circle with centre λ and radius $a < \operatorname{dist}(\lambda, \sigma(T)\setminus\{\lambda\})$. Then $\Gamma \subset \rho(T)$ and Γ isolates λ from the rest of the spectrum of T.

For z in $\rho(T)$, let $R(z) = (T - zI)^{-1}$ be the resolvent operator of T. Then the spectral projection P associated with T and λ is given by

$$P = -\frac{1}{2\pi i}\int_\Gamma R(z)\,dz. \tag{1}$$

We choose s such that $0 < s < a/2$ and define

$$f(z) = \begin{cases} 0, & \text{if } |z-\lambda| < s, \\ \dfrac{1}{z-\lambda}, & \text{if } |z-\lambda| > 2s. \end{cases}$$

Then f is locally analytic on a neighbourhood of $\sigma(T)$ and at ∞, if we define $f(\infty) = 0$.

The reduced resolvent S associated with T and λ can then be defined as

$$S = f(T) = \frac{1}{2\pi i}\int_\Gamma f(z)R(z)\,dz = \frac{1}{2\pi i}\int_\Gamma \frac{R(z)}{z-\lambda}\,dz. \tag{2}$$

Then we have

$$S(T-\lambda I) = (T-\lambda I)S = I - P, \qquad SP = PS = 0. \tag{3}$$

The spectrum $\sigma(S)$ and the resolvent set $\rho(S)$ of S are given by

$$\sigma(S) = \left\{\frac{1}{\tilde\lambda - \lambda} : \tilde\lambda \in \sigma(T)\backslash\{\lambda\}\right\} \cup \{0\}, \tag{4}$$

$$\rho(S) = \left\{\frac{1}{z-\lambda} : z \in \rho(T)\right\}. \tag{5}$$

(See Taylor and Lay [9, Theorem 9.5].) It follows that the spectral radius $r_\sigma(S)$ of S is given by

$$r_\sigma(S) = \frac{1}{\operatorname{dist}(\lambda, \sigma(T)\backslash\{\lambda\})}.$$

Also, $\tilde\lambda$ is an isolated point of the spectrum of T if and only if $1/(\tilde\lambda - \lambda)$ is an isolated point of the spectrum of S, and in that case, the spectral projection associated with T and $\tilde\lambda$ coincides with the spectral projection associated with $f(T) = S$ and $f(\tilde\lambda) = 1/(\tilde\lambda - \lambda)$. (See Taylor and Lay [9, Theorem 9.8].)

Let (T_n) be a *resolvent operator approximation* of T on $\rho(T)$ ($T_n \xrightarrow{ro} T$), that is,

$$\begin{gathered} T_n x \to Tx \text{ for every } x \text{ in } X, \text{ and} \\ \|(T_n - T)R(z)(T_n - T)\| \to 0 \text{ for every } z \text{ in } \rho(T). \end{gathered} \tag{6}$$

(In Chatelin and Lemordant [3], and in Kulkarni and Limaye [5], the resolvent operator approximation was considered under the name 'strong convergence'. It can be proved that if the spectrum of T is simply connected, then

$T_n \xrightarrow{ro} T$ if and only if $\|(T-T_n)T^k(T-T_n)\| \to 0$ for each $k = 1, 2, \ldots$. This is certainly the case when T is a compact operator.

If either (T_n) converges to T *in the norm* ($T_n \xrightarrow{\|\;\|} T$), or if (T_n) converges to T in a *collectively compact fashion* ($T_n \xrightarrow{cc} T$), that is, if $T_n x \to Tx$ for every x in X, and $\bigcup_{n=1}^{\infty}\{(T_n - T)x : \|x\| \le 1\}$ is a relatively compact subset of X, then $T_n \xrightarrow{ro} T$.

Since Γ is a compact subset of $\rho(T)$, we have

$$\max_{z\in\Gamma} \|(T_n - T)R(z)(T_n - T)\| \to 0. \tag{7}$$

Then for all n large enough, $\Gamma \subset \rho(T_n)$ and it can be seen that

$$\max_{z\in\Gamma} \|(T_n - T)R_n(z)(T_n - T)\| \to 0, \tag{8}$$

where $R_n(z) = (T_n - zI)^{-1}$ for $z \in \rho(T_n)$.

For all n large enough, the spectrum of T_n inside Γ consists of a simple eigenvalue λ_n. (See Chatelin and Lemordant [3, Lemma 4].) Let

$$P_n = -\frac{1}{2\pi i}\int_\Gamma R_n(z)\,dz \tag{9}$$

and

$$S_n = \frac{1}{2\pi i}\int_\Gamma \frac{R_n(z)}{z - \lambda_n}\,dz \tag{10}$$

be the spectral projection and the reduced resolvent associated with T_n and λ_n, respectively.

Since for every $x \in X$, $R_n(z)x \to R(z)x$, uniformly for z in Γ, and since $\lambda_n \to \lambda$, it can be easily verified that

$$P_n x \to Px \quad \text{and} \quad S_n x \to Sx \quad \text{for all } x \text{ in } X.$$

As $\dim P_n X = \dim PX = 1$, we have, in fact, $P_n \xrightarrow{cc} P$ (Chatelin [2, Proposition 3.13]), and hence

$$\|(T_n - T)P_n\| \to 0 \tag{11}$$

(Anselone [1, Corollary 1.9]). By the uniform boundedness principle, we see that for all large n,

$$\|P_n\|, \|S_n\|, \|(T - T_n)S_n\| \le C < \infty. \tag{12}$$

Fix n sufficiently large. Following Chatelin [2], we consider the Rayleigh-Schrödinger series

$$\lambda = \sum_{k=0}^{\infty} \lambda_n^{(k)} \quad \text{and} \quad \varphi = \sum_{k=0}^{\infty} \varphi_n^{(k)},$$

where $\lambda_n^{(0)} = \lambda_n$, and $\varphi_n^{(0)} = \varphi_n$ is an eigenvector of T_n corresponding to λ_n, and for $k \geq 1$,

$$\lambda_n^{(k)} = \langle (T - T_n)\varphi_n^{(k-1)}, \varphi_n^* \rangle,$$

where φ_n^* is the eigenvector of T_n^* corresponding to $\overline{\lambda}_n$ satisfying $\langle \varphi_n, \varphi_n^* \rangle = 1$, and

$$\varphi_n^{(k)} = S_n \left(-(T - T_n)\varphi_n^{(k-1)} + \sum_{i=1}^{k} \lambda_n^{(i)} \varphi_n^{(k-i)} \right). \tag{13}$$

In case $T_n \xrightarrow{cc} T$, Redont [8] gave error bounds for $|\lambda - \sum_{i=0}^{k} \lambda_n^{(i)}|$ and $\|\varphi - \sum_{i=0}^{k} \varphi_n^{(i)}\|$ in terms of $\|(T_n - T)P_n\|$ and a quantity α_n defined by

$$B = \{x \in X: \|x\| \leq 1\}, \quad K_n = \bigcup_{k \geq 0} \left(\frac{S_n}{\|S_n\|} \right)^k (T_n - T)S_n B,$$
$$\alpha_n = \text{diameter}((T_n - T)S_n K_n).$$

He claimed that $\alpha_n \to 0$ as $n \to \infty$. However, his proof does not seem to be justified, as shown by us in [4] by citing a counter example. Instead, we introduced in [4] a parameter $r \geq 1$ and proved that if $r > 1$, then $\alpha_n(r) \to 0$ as $n \to \infty$, where

$$\alpha_n(r) = \text{diameter}((T - T_n)S_n K_n(r)),$$

with

$$K_n(r) = \bigcup_{k \geq 0} \left(\frac{S_n}{r\|S_n\|} \right)^k (T - T_n)S_n B.$$

Note that $\alpha_n(1) = \alpha_n$.

In his thesis [7], Nair introduced another quantity

$$\tilde{\alpha}_n(r) = \sup_{k \geq 1} \frac{\|(T - T_n)S_n^k (T - T_n)S_n\|}{(r\|S_n\|)^{k-1}} \tag{14}$$

for $r \geq 1$, and gave error bounds for the convergence of the Rayleigh-Schrödinger series in terms of $\tilde{\alpha}_n(r)$. He proved that if $r > 1$, then $\tilde{\alpha}_n(r) \to 0$. Note that $\tilde{\alpha}_n(r) \leq \alpha_n(r)$. Thus, the original question regarding the case $r = 1$ remained unanswered. (See also [6].)

In the present paper we prove under the assumption of resolvent operator approximation (which is weaker than collectively compact approximation), that

$$\tilde{\alpha}_n(1) \to 0 \quad \text{as } n \to \infty;$$

the only restriction we impose is that all the spectral values of T nearest to λ are eigenvalues of T of finite algebraic multiplicities. If T is a compact operator and $\lambda \neq 0$, then this merely says that 0 is not the nearest spectral value of T from λ. Our proof is motivated by Redont's considerations.

Using the fact that $\tilde{\alpha}_n(1) \to 0$ as $n \to \infty$, we can also improve some results for the approximation of the spectral projection P given in [5, Theorem 4.2 and Theorem 4.3(b)]. These results use the Kato-Rellich perturbation series. We are able to give better error bounds for the approximations of PP_n.

The discrete spectrum $\sigma_d(T)$ of T is defined as follows:

$$\sigma_d(T) = \{\mu \in \sigma(T) \colon \mu \text{ is an eigenvalue of finite algebraic multiplicity}\}.$$

We first prove that if one of the spectral values of T nearest to λ is in the discrete spectrum of T, then $r_\sigma(S_n)$ tends to $r_\sigma(S)$. Recall that λ is a simple eigenvalue of T, separated by a circle Γ of radius a from the rest of $\sigma(T)$. If (T_n) is a resolvent operator approximation of T, then T_n has a simple eigenvalue λ_n inside Γ and it is the only spectral value of T_n inside Γ. We begin with the following elementary lemma.

LEMMA 1.1. *Let $T_n \xrightarrow{ro} T$. If (μ_n) is a sequence of spectral values of T_n, and if (μ_n) converges to μ, then μ is a spectral value of T.*

PROOF. Let, if possible, $\mu \in \rho(T)$. Consider a simple closed curve Γ in $\rho(T)$ enclosing μ and such that the interior of Γ is contained in $\rho(T)$. Then

$$P = -\frac{1}{2\pi i}\int_\Gamma R(z)\,dz = 0.$$

Since $T_n \xrightarrow{ro} T$, $\Gamma \subset \rho(T_n)$ for all n large enough. As $\mu_n \to \mu$, μ_n lies in the interior of Γ for all large n and hence

$$P_n = -\frac{1}{2\pi i}\int_\Gamma R_n(z)\,dz \neq 0.$$

This is a contradiction, since $\dim P_n X = \dim PX$ for all large n. (See Chatelin and Lemordant [3, Lemma 4].) Hence $\mu \in \sigma(T)$.

PROPOSITION 1.2. *Let $T_n \xrightarrow{ro} T$ and assume that there exists $\tilde{\lambda}$ in the discrete spectrum of T such that $|\tilde{\lambda} - \lambda| = \operatorname{dist}(\lambda, \sigma(T)\backslash\{\lambda\})$. Then*

$$\operatorname{dist}(\lambda_n, \sigma(T_n)\backslash\{\lambda_n\}) \to \operatorname{dist}(\lambda, \sigma(T)\backslash\{\lambda\}) \tag{15}$$

and hence

$$r_\sigma(S_n) \to r_\sigma(S). \tag{16}$$

PROOF. Let $\lambda_n' \in \sigma(T_n)$ be such that $|\lambda_n' - \lambda_n| = \operatorname{dist}(\lambda_n, \sigma(T_n)\backslash\{\lambda_n\})$. Then $|\lambda_n' - \lambda_n| \geq \delta$ for some $\delta > 0$ and for all large n. Since $\tilde{\lambda}$ belongs to the discrete spectrum of T, there exists $\tilde{\lambda}_n$ in $\sigma(T_n)$ such that $\tilde{\lambda}_n \to \tilde{\lambda}$. (See Chatelin and Lemordant [3, Lemma 4].) Now, $|\lambda'_n - \lambda_n| \le |\tilde{\lambda}_n - \lambda_n| \to |\tilde{\lambda} - \lambda|$, so

$$\overline{\lim_{n\to\infty}} |\lambda'_n - \lambda_n| \le |\tilde{\lambda} - \lambda|. \tag{17}$$

In order to prove (15), it is enough to show that

$$|\tilde{\lambda} - \lambda| \le \underline{\lim}_{n\to\infty} |\lambda'_n - \lambda_n|. \tag{18}$$

Suppose that this is not the case. Then there exist subsequences (λ'_{n_k}) and (λ_{n_k}) such that $|\lambda'_{n_k} - \lambda_{n_k}| \to \varepsilon < |\tilde{\lambda} - \lambda|$. By passing to a subsequence, if necessary, we can assume that $\lambda'_{n_k} \to \lambda'$ for some $\lambda' \in \mathbb{C}$. Since $T_n \xrightarrow{ro} T$, it follows by Lemma 2.1 that $\lambda' \in \sigma(T)$ and $|\lambda' - \lambda| < |\tilde{\lambda} - \lambda|$, a contradiction to the fact that $|\tilde{\lambda} - \lambda| = \operatorname{dist}(\lambda, \sigma(T)\backslash\{\lambda\})$. Thus, (15) follows from (17) and (18). Finally, (16) follows by

$$r_\sigma(S_n) = \frac{1}{\operatorname{dist}(\lambda_n, \sigma(T_n)\backslash\{\lambda_n\})}$$

and

$$r_\sigma(S) = \frac{1}{\operatorname{dist}(\lambda, \sigma(T)\backslash\{\lambda\})}.$$

2. Main results

Consider the following inclusion, which we call by the name 'Assumption $(*)$'.

$$\{\tilde{\lambda} \in \sigma(T) : |\tilde{\lambda} - \lambda| = \operatorname{dist}(\lambda, \sigma(T)\backslash\{\lambda\})\} \subset \sigma_d(T). \tag{$*$}$$

In this case, the spectral points of T nearest to λ are finite in number and each such point belongs to the discrete part of the spectrum of T.

Note that this assumption is stronger than the one made in Proposition 1.2.

In case T is compact and $\lambda \ne 0$, Assumption $(*)$ is satisfied if

$$|\lambda| \ne \operatorname{dist}(\lambda, \sigma(T)\backslash\{\lambda\}),$$

that is, if 0 is not one of the nearest spectral points from λ. We write

$$\beta_{n,k} = \frac{\|(T - T_n)S_n^k(T - T_n)S_n\|}{\|S_n\|^{k-1}}, \qquad k \ge 1.$$

Then for large n,

$$\tilde{\alpha}_n(1) = \sup\{\beta_{n,k} : k = 1, 2, \dots\}.$$

Theorem 2.1. *Let $T_n \xrightarrow{ro} T$ and let Assumption $(*)$ be satisfied. Then*

$$\tilde{\alpha}_n(1) \to 0 \quad \text{as } n \to \infty. \tag{19}$$

Proof. We denote the eigenvalues of T nearest to λ, that is, the elements of the set

$$E = \{\tilde{\lambda} \in \sigma(T) : |\tilde{\lambda} - \lambda| = \operatorname{dist}(\lambda, \sigma(T)\backslash\{\lambda\})\}$$

by $\lambda^j = 1/\mu^j + \lambda$, $j = 1, \ldots, q$. Then each μ^j is an eigenvalue of S. Note that

$$\{\mu \in \sigma(S) : |\mu| = r_\sigma(S)\} = \{\mu^1, \mu^2, \ldots, \mu^q\}.$$

Let m_j denote the algebraic multiplicity of λ^j, $j = 1, \ldots, q$. For $j = 1, \ldots, q$, let Γ_j denote a curve in $\rho(T)$ isolating λ^j from the rest of the spectrum, and let P_{λ^j} be the associated spectral projection. Then P_{λ^j} is also the spectral projection associated with S and μ^j. If we write

$$\tilde{P} = P_{\lambda^1} + \cdots + P_{\lambda^q},$$

then

$$r_\sigma(S(I - \tilde{P})) < r_\sigma(S). \tag{20}$$

Since $T_n \xrightarrow{ro} T$, $\Gamma_j \subset \rho(T_n)$ for all n large enough and $j = 1, \ldots, q$. Let $P_{n,j}$ denote the spectral projection associated with T_n and $\sigma(T_n) \cap \operatorname{Int}\Gamma_j$, where $\operatorname{Int}\Gamma_j$ denotes the interior of Γ_j. Then the spectral projection $\tilde{P}_n$ associated with T_n and $\bigcup_{j=1}^q (\sigma(T_n) \cap \operatorname{Int}\Gamma_j)$ is given by $\tilde{P}_n = P_{n,1} + \cdots + P_{n,q}$. By Assumption $(*)$, $\operatorname{rank}\tilde{P} = m_1 + \cdots + m_q < \infty$. Hence $\tilde{P}_n \xrightarrow{cc} \tilde{P}$ and

$$\|(T_n - T)\tilde{P}_n\| \to 0 \quad \text{as } n \to \infty. \tag{21}$$

(See Anselone [1, Corollary 1.9].) Also, $\|(T_n - T)S_n\|$ and $\|(T_n - T)(I - \tilde{P}_n)\|$ are uniformly bounded. Now, we write

$$S_n = S_n\tilde{P}_n + S_n(I - \tilde{P}_n).$$

Since S_n and $\tilde{P}_n$ commute,

$$\|(T - T_n)S_n^k\tilde{P}_n\| \le \|S_n\|^k \|(T - T_n)\tilde{P}_n\|. \tag{22}$$

Also,

$$\begin{aligned}\|(T - T_n)S_n^k(T - T_n)S_n\| \le{}& \|(T - T_n)S_n^k\tilde{P}_n(T - T_n)S_n\| \\ &+ \|(T - T_n)S_n^k(I - \tilde{P}_n)(T - T_n)S_n\tilde{P}_n\| \\ &+ \|(T - T_n)S_n^k(I - \tilde{P}_n)(T - T_n)S_n(I - \tilde{P}_n)\|.\end{aligned}$$

Using (21) and (22), we see that in order to prove (19) it is enough to prove that

$$\sup_{k \ge 1} \|(T - T_n)S_n^k(I - \tilde{P}_n)(T - T_n)S_n(I - \tilde{P}_n)\| / \|S_n\|^{k-1}$$

tends to zero as $n \to \infty$.

We recall from (2) and (10) that

$$S = \frac{1}{2\pi i}\int_\Gamma \frac{R(z)}{z-\lambda}\,dz \quad \text{and} \quad S_n = \frac{1}{2\pi i}\int_\Gamma \frac{R_n(z)}{z-\lambda_n}\,dz,$$

where Γ represents a circle with centre λ and radius a with

$$a < \operatorname{dist}(\lambda, \sigma(T)\backslash\{\lambda\}) = \frac{1}{r_\sigma(S)}.$$

Hence

$$S(I-\tilde{P}) = \frac{1}{2\pi i}\int_\Gamma \frac{R(z)(I-\tilde{P})}{z-\lambda}\,dz \tag{23}$$

and

$$S_n(I-\tilde{P}_n) = \frac{1}{2\pi i}\int_\Gamma \frac{R_n(z)(I-\tilde{P}_n)}{z-\lambda_n}\,dz. \tag{24}$$

We note that $R(z)(I-\tilde{P})$ has a removable singularity at λ^j, $j = 1, \ldots, q$. Hence, we can choose a circle Γ' with centre λ and radius a' satisfying $a < a' < 1/r_\sigma(S(I-\tilde{P}))$. Then (23) remains valid with Γ replaced by Γ'.

Now we wish to show that even in (24) we can replace Γ by Γ'.

Consider

$$P_{\Gamma'}(T) = -\frac{1}{2\pi i}\int_{\Gamma'} R(z)\,dz$$

and for all n large,

$$P_{\Gamma'}(T_n) = -\frac{1}{2\pi i}\int_{\Gamma'} R_n(z)\,dz.$$

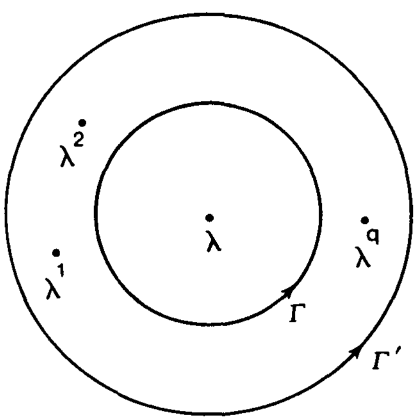

Figure 1

Then $P_{\Gamma'}(T) = P + P_{\lambda^1} + \cdots + P_{\lambda^q}$ and $P_{\Gamma'}(T_n) = P_n + P_{n,1} + \cdots + P_{n,q}$. Now

$$\operatorname{rank} P_{\Gamma'}(T) = 1 + m_1 + \cdots + m_q = \operatorname{rank} P_{\Gamma'}(T_n).$$

Since

$$\operatorname{rank} \tilde{P}_n = \operatorname{rank}(P_{n,1} + \cdots + P_{n,q}) = m_1 + \cdots + m_q,$$

the only singularity of $R_n(z)(I - \tilde{P}_n)$ inside Γ' is at λ_n. Thus, we can replace Γ by Γ' in (24) and write

(25) $$S_n(I - \tilde{P}_n) = \frac{1}{2\pi i} \int_{\Gamma'} \frac{R_n(z)(I - \tilde{P}_n)}{z - \lambda_n}\, dz.$$

In Proposition 1.2 we have proved that $r_\sigma(S_n) \to r_\sigma(S)$ as $n \to \infty$. Hence we can choose $0 < \eta' < \eta < 1$ such that

(26) $$r_\sigma(S(I - \tilde{P})) < \eta' r_\sigma(S) \le \eta r_\sigma(S_n).$$

If we let $a' = 1/(\eta' r_\sigma(S))$, then

(27) $$\frac{1}{\eta \|S_n\|} \le \frac{1}{\eta r_\sigma(S_n)} \le a' < \frac{1}{r_\sigma(S(I - \tilde{P}))}.$$

Since $\lambda_n \to \lambda$, λ_n is inside the circle with centre λ and radius $(1 - \eta)a'$ for all large n. Then for z in Γ',

(28) $$|z - \lambda_n| \ge |z - \lambda| - |\lambda - \lambda_n| \ge a' - (1 - \eta)a' = \eta a'.$$

Now, for $k = 1, 2, \ldots,$

$$\begin{aligned}
&\|(T - T_n)S_n^k(I - \tilde{P}_n)(T - T_n)S_n(I - \tilde{P}_n)\| \\
&\quad \le \left\| \frac{1}{4\pi^2} \int_{\Gamma'} \int_{\Gamma'} \frac{(T - T_n)R_n(z)(I - \tilde{P}_n)(T - T_n)R_n(w)(I - \tilde{P}_n)\, dz\, dw}{(z - \lambda_n)^k (w - \lambda_n)} \right\| \\
&\quad \le \max_{z,w \in \Gamma'} (a')^2 \frac{\|(T - T_n)R_n(z)(I - \tilde{P}_n)(T - T_n)R_n(w)(I - \tilde{P}_n)\|}{|z - \lambda_n|^k |w - \lambda_n|} \\
&\quad \le \max_{z,w \in \Gamma'} \|(T - T_n)R_n(z)(I - \tilde{P}_n)(T - T_n)R_n(w)(I - \tilde{P}_n)\| / (\eta a')^{k-1} \eta^2.
\end{aligned}$$

Hence by (27),

$$\begin{aligned}
&\sup_{k \ge 1} \|(T - T_n)S_n^k(I - \tilde{P}_n)(T - T_n)S_n(I - \tilde{P}_n)\| / \|S_n\|^{k-1} \\
&\qquad \le \left(\max_{z,w \in \Gamma'} \|(T - T_n)R_n(z)(I - \tilde{P}_n)(T - T_n)R_n(w)(I - \tilde{P}_n)\| \right) / \eta^2.
\end{aligned}$$

Since $T_n \xrightarrow{ro} T$ on $\rho(T)$ and Γ' is compact, we have

$$\begin{aligned}
&\max_{z,w \in \Gamma'} \|(T - T_n)R_n(z)(T - T_n)R_n(w)\| \\
&\qquad \le \max_{w \in \Gamma'} \|R_n(w)\| \max_{z \in \Gamma'} \|(T - T_n)R_n(z)(T - T_n)\|,
\end{aligned}$$

which tends to 0 as $n \to \infty$. Also by (21) we have

$$\|(T - T_n)\tilde{P}_n\| \to 0.$$

Since $R_n(z)$ commutes with $\tilde{P}_n$, we obtain

$$\begin{aligned}
&\max_{z,w\in\Gamma'} \|(T - T_n)R_n(z)(I - \tilde{P}_n)(T - T_n)R_n(w)(I - \tilde{P}_n)\| \\
&\quad\le \max_{z,w\in\Gamma'} \|(T - T_n)R_n(z)(T - T_n)R_n(w)\| \|I - \tilde{P}_n\| \\
&\qquad + \left(\max_{z\in\Gamma'} \|R_n(z)\|\right)\left(\max_{w\in\Gamma'} \|(T - T_n)R_n(w)\|\right) \|(T - T_n)\tilde{P}_n\| \|I - \tilde{P}_n\|,
\end{aligned}$$

which tends to 0 as $n \to \infty$. This completes the proof of $\tilde{\alpha}_n(1) \to 0$ as $n \to \infty$.

Let

$$\begin{aligned}
\eta_n &= \|(T - T_n)\varphi_n\|, \\
\mu_n &= \max\{\|(T - T_n)S_n\|, \|(T - T_n)\varphi_n\| \|P_n\| \|S_n\|\}, \\
\varepsilon_n &= \max\{\alpha_n, \|(T - T_n)\varphi_n\| \|P_n\| \|S_n\| \mu_n\}, \\
\tilde{\varepsilon}_n &= \max\{\tilde{\alpha}_n, \|(T - T_n)\varphi_n\| \|P_n\| \|S_n\| \mu_n\}
\end{aligned}$$

and

$$a_0 = 1, \qquad a_k = \sum_{i=1}^{k} a_{i-1}a_{k-i}, \qquad k = 1, 2, \ldots.$$

The following error bounds for the Rayleigh-Schrödinger iterates have been obtained by Redont. (See [8, Remark 3.3].) For $k = 0, 1, 2, \ldots,$

$$|\lambda_n^{(2k+1)}| \le a_{2k}\eta_n\|P_n\|(\sqrt{\varepsilon_n})^{2k}, \qquad |\lambda_n^{(2k+2)}| \le a_{2k+1}\eta_n\|P_n\|\mu_n(\sqrt{\varepsilon_n})^{2k}$$

and

$$\|\varphi_n^{(2k+1)}\| \le a_{2k+1}\eta_n\|S_n\|(\sqrt{\varepsilon_n})^{2k}, \qquad \|\varphi_n^{(2k+2)}\| \le a_{2k+2}\eta_n\|S_n\|\mu_n(\sqrt{\varepsilon_n})^{2k}.$$

The error bounds obtained in [6] and [7] are similar to the above bounds with ε_n replaced by $\tilde{\varepsilon}_n$. We have proved that if $T_n \xrightarrow{ro} T$, then $\tilde{\varepsilon}_n \to 0$ as $n \to \infty$. Hence we have the following theorem.

THEOREM 2.3. *Let $T_n \xrightarrow{ro} T$ and let Assumption $(*)$ be satisfied. Then for $k = 0, 1, 2, \ldots,$*

$$\left|\lambda - \sum_{i=0}^{2k} \lambda_n^{(i)}\right| = O(\eta_n\|P_n\|\tilde{\varepsilon}_n^k), \qquad \left|\lambda - \sum_{i=0}^{2k+1} \lambda_n^{(i)}\right| = O(\eta_n\|P_n\|\mu_n\tilde{\varepsilon}_n^k)$$

and

$$\left\|\varphi - \sum_{i=0}^{2k} \varphi_n^{(i)}\right\| = O(\eta_n\|S_n\|\tilde{\varepsilon}_n^k), \qquad \left\|\varphi - \sum_{i=0}^{2k+1} \varphi_n^{(i)}\right\| = O(\eta_n\|S_n\|\mu_n\tilde{\varepsilon}_n^k).$$

Now we consider the Kato-Rellich perturbation series for the spectral projection P. We choose n large enough so that $\max_{z\in\Gamma} r_\sigma((T-T_n)R_n(z))<1$, where Γ is a circle with centre λ and radius $a<\operatorname{dist}(\lambda,\sigma(T)\backslash\{\lambda\})$. The Kato-Rellich perturbation series for P is given by

$$P = P_n - \sum_{k=1}^{\infty}\sum_{(*)_{k+1}} S_n^{p_1}(T_n-T)S_n^{p_2}\dots(T_n-T)S_n^{p_{k+1}}, \tag{29}$$

where $(*)_{k+1}$ denotes the conditions

$$p_1+\cdots+p_{k+1}=k \quad \text{and} \quad p_j\geq 0, \qquad j=1,\dots,k+1.$$

We adopt the notation $S_n^0=-P_n$. The number n_{k+1} of the ordered $(k+1)$-tuples $(p_1,\dots,p_{k+1})$ satisfying $(*)_{k+1}$ is the coefficient of x^k in the binomial expansion of $(1-x)^{-(k+1)}$. Thus,

$$n_{k+1}=\frac{(2k)!}{k!k!}.$$

We define

$$h(x)=\sum_{k=1}^{\infty} n_{k+1}x^k \text{ for } |x|<\tfrac{1}{4}. \tag{30}$$

Let

$$h_1(x)=\frac{h(x)+h(-x)}{2}, \qquad h_2(x)=\frac{h(x)-h(-x)}{2}.$$

We have

$$PP_n = P_n+\sum_{k=1}^{\infty} P_n^{(k)}, \tag{31}$$

where for $k\geq 1$

$$P_n^{(k)} = -\sum_{\substack{(*)_{k+1}\\ p_{k+1}=0}} S_n^{p_1}(T_n-T)S_n^{p_2}\cdots(T_n-T)S_n^{p_{k+1}}. \tag{32}$$

We set

$$P_n^0=P_n^{(0)}=P_n, \qquad P_n^m=\sum_{k=0}^{m} P_n^{(k)}.$$

Recalling that

$$\beta_{n,k}=\frac{\|(T_n-T)S_n^k(T_n-T)S_n\|}{\|S_n\|^{k-1}},$$

we write

$$\gamma_n=\max\{\|S_n\|\|(T_n-T)P_n\|,\|(T_n-T)S_n\|\},$$

$$\delta_{n,k}=\max\left\{\|S_n\|\|(T_n-T)P_n\|\|(T_n-T)S_n\|,\ \max_{1\leq i\leq k}\beta_{n,i}\right\},$$

$$\delta_n=\max\left\{\|S_n\|\|(T_n-T)P_n\|\gamma_n,\sup_{1\leq i}\beta_{n,i}\right\}.$$

By (11) and (12) we know that $\|(T_n - T)P_n\| \to 0$ as $n \to \infty$ and that $\|(T_n - T)S_n\| \le C < \infty$ for all large n. In Theorem 2.1, we have proved that $\tilde{\alpha}_n(1) = \sup_{k\ge1} \beta_{n,k} \to 0$, as $n \to \infty$ under Assumption $(*)$, so that $\delta_n \to 0$ as $n \to \infty$.

THEOREM 2.4. *Let $T_n \xrightarrow{ro} T$, and Assumption $(*)$ be satisfied. The series $PP_n - P_n = \sum_{k=1}^{\infty} P_n^{(k)}$ is dominated term by term by the following series*

$$\|P_n\|\|S_n\|\varepsilon_n \left[h_1(\sqrt{\delta_n}) + \frac{\gamma_n}{\sqrt{\delta_n}} h_2(\sqrt{\delta_n})\right]. \tag{33}$$

Hence for $k \ge 0$

$$\|PP_n - P_n^{2k}\| = O(\|P_n\|\|S_n\|\|(T_n - T)P_n\|(\delta_n)^k) \tag{34}$$

and

$$\|PP_n - P_n^{2k+1}\| = O(\|P_n\|\|S_n\|\|(T_n - T)P_n\|\gamma_n(\delta_n)^k). \tag{35}$$

PROOF. It is easy to see that for $p, q \ge 0$

$$\|(T_n - T)S_n^p(T_n - T)S_n^q\| \le \|S_n\|^{p+q-2}\delta_n. \tag{36}$$

Let $p_1 + \cdots + p_{k+1} = k$, $p_j \ge 0$, $j = 1, \dots, k$ and $p_{k+1} = 0$. Then

$$\|S_n^{p_1}(T_n - T)S_n^{p_2} \cdots (T_n - T)S_n^{p_{k+1}}\|$$
$$\le \begin{cases} \|S_n\|^{p_1}\|(T_n - T)S_n^{p_2}(T_n - T)S_n^{p_3}\| \cdots \\ \quad \|(T_n - T)S_n^{p_{k-1}}(T_n - T)S_n^{p_k}\|\|(T_n - T)P_n\|, \\ \hfill \text{if } k \text{ is odd}, \\ \|S_n\|^{p_1}\|(T_n - T)S_n^{p_2}(T_n - T)S_n^{p_3}\| \cdots \\ \quad \|(T_n - T)S_n^{p_{k-2}}(T_n - T)S_n^{p_{k-1}}\|\|(T_n - T)S_n^{p_k}\|\|(T_n - T)P_n\|, \\ \hfill \text{if } k \text{ is even}, \end{cases}$$
$$\le \begin{cases} \|P_n\|\|S_n\|\|(T_n - T)P_n\|(\delta_n)^{(k-1)/2}, & \text{if } k \text{ is odd}, \\ \|P_n\|\|S_n\|\|(T_n - T)P_n\|\gamma_n(\delta_n)^{(k-2)/2}, & \text{if } k \text{ is even}. \end{cases}$$

Hence

$$\|P_n^{(k)}\| \le \begin{cases} n_{k+1}\|P_n\|\|S_n\|\|(T_n - T)P_n\|(\delta_n)^{(k-1)/2}, & \text{if } k \text{ is odd}, \\ n_{k+1}\|P_n\|\|S_n\|\|(T_n - T)P_n\|\gamma_n(\delta_n)^{(k-2)/2}, & \text{if } k \text{ is even}. \end{cases}$$

Thus, the result follows.

REMARK 2.5. The above theorem should be compared with the following result [3, Theorem 4.2].

Let T_n converge to T in a collectively compact fashion. Let $p \ge 1$ be a fixed integer. Then there exists n_0 such that for every fixed $n \ge n_0$ and for $k = 0, \dots, p-1$, we have

$$\begin{aligned} \|PP_n - P_n^{2k}\| &= O(\|P_n\|\|S_n\|\|(T_n - T)P_n\|\nu_n^k), \\ \|PP_n - P_n^{2k+1}\| &= O(\|P_n\|\|S_n\|\|(T_n - T)P_n\|\gamma_n\nu_n^k), \end{aligned} \tag{37}$$

where $\nu_n = \max\{\|S_n\|\|(T_n - T)P_n\|\gamma_n, \delta_{n,k+1}\}$.

We see from the above result that P_n^j approximates PP_n in a semi-geometric fashion for $j = 0, \dots, 2k-1$.

Since $\|S_n\| \|(T_n - T)P_n\| \gamma_n \leq \delta_n$ and $\delta_{n,k+1} \leq \delta_n$ for all k, the bounds given in (37) are sharper than those in (34) and (35), but they have the disadvantage that they depend upon k. Also, the proof of the above result given in [5] is much more complicated.

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