J. Austral. Math. Soc. (*Series A*) **46** (1989), 251–261

QF-3 RINGS AND TORSION THEORIES

JOSÉ L. GÓMEZ PARDO and NIEVES RODRÍGUEZ GONZÁLEZ

(Received 3 February 1987)

Communicated by B. J. Gardner

Abstract

In this paper, the rings which have a torsion theory τ with associated torsion radical $\mathbf{t}$ such that $R/\mathbf{t}(R)$ has a minimal τ-torsionfree cogenerator are studied. When τ is the trivial torsion theory these are precisely the left QF-3 rings. For $\tau = \tau_L$, the Lambek torsion theory, this class of rings is wider but, with an additional hypothesis on τ_L it is shown that if R has this property with respect to the Lambek torsion theory on both sides, then R is a (left and right) QF-3 ring. The results obtained are applied to get new characterizations of QF-3 rings with the ascending chain condition on left annihilators.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 16 A 36, 16 A 63.

A ring R is called left QF-3 if it has a minimal faithful left R-module and left QF-3′ if the injective envelope $E({}_RR)$ is a torsionless module. These rings have been the object of extensive study (for example in [3, 9, 11, 12, 13, 16]) and, recently, Baccella has obtained in [2] structure results for the class of nonsingular, finite-dimensional QF-3 rings. In the present paper we consider a (hereditary) torsion theory τ in R-mod with associated torsion radical $\mathbf{t}$ and the property that $R/\mathbf{t}(R)$ has a minimal τ-torsionfree cogenerator X, in the sense that X is a τ-torsionfree module which cogenerates $R/\mathbf{t}(R)$ and is a direct summand of every τ-torsionfree cogenerator of $R/\mathbf{t}(R)$. When τ is the trivial torsion theory in which all R-modules are τ-torsionfree, this property defines left QF-3 rings. The class of rings which have this property with respect to the Lambek torsion theory τ_L is wider than the class of left QF-3 rings but when τ_L is strongly semiprime and R has this property for the Lambek torsion theory on both sides,

This work was partially supported by the CAICYT (0784-84).
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then we show that R is a (left and right) QF-3 ring (Theorem 4). This allows us to get characterizations of the class of QF-3 rings with the ascending chain condition (ACC) on left annihilators, which is shown to be equal to the class of QF-3′ rings with ACC on left annihilators.

Our property behaves nicely when R is, furthermore, supposed to be τ-artinian and this case is studied in Theorem 7. For trivial τ the conditions of this theorem reduce to known characterizations of left artinian left QF-3 rings (in particular, we get as a consequence the main result of [12]) but, taking τ equal to the Lambek torsion theory we obtain new characterizations of left QF-3′ rings with the descending chain condition on left annihilators, QF-3 rings with ACC on left annihilators and, in the nonsingular case, of finite-dimensional left QF-3′ rings.

Throughout this paper, R denotes an associative ring with identity and R-mod (mod-R) the category of left (right) R-modules. A module N is said to be a cogenerator of M if there is a monomorphism from M to a direct product N^I of copies of N; in this case we also say that M is N-torsionless and, in particular, the ${}_RR$-torsionless modules are called simply torsionless modules. We use the notation $E(M)$ to stand for an injective envelope of the module M. When R is left and right QF-3, we will say simply that R is a QF-3 ring and the analogous convention holds for other classes of rings (for example QF-3′ rings).

If τ is a torsion theory of R-mod, a submodule N of M is said to be τ-dense in M if M/N is a τ-torsion module; M is τ-finitely generated when it has a finitely generated τ-dense submodule; and M is called τ-noetherian (τ-artinian) if the lattice $\mathbf{C}_\tau(M)$ of τ-closed submodules of M (that is, of those submodules X of M such that M/X is τ-torsionfree) satisfies the ascending chain condition (ACC for short), respectively the descending chain condition (DCC). The Teply-Miller theorem [10, Theorem 1.4] asserts that if R is a τ-artinian ring (that is, ${}_RR$ is τ-artinian), then every τ-artinian module is τ-noetherian. Also, it is well known that if M is τ-noetherian, then M is τ-finitely generated. The Lambek torsion theory of R-mod τ_L is the torsion theory cogenerated by $E({}_RR)$ [15], that is, the τ_L-torsionfree modules are precisely the $E({}_RR)$-torsionless modules. The τ_L-finitely generated modules are also called almost finitely generated modules and the elements of $C_{\tau_L}(R)$ are called rationally closed left ideals.

The reader is referred to [1], [7] and [15] for all ring-theoretic and torsion-theoretic notions used in the text.

Recall that a torsion theory τ is called jansian if the corresponding torsion class is closed under direct products or, equivalently, R has a smallest τ-dense left ideal. Then we have

PROPOSITION 1. *Let τ be a torsion theory of R-mod. If $R/\mathbf{t}(R)$ is a left* QF-3 *ring, then $R/\mathbf{t}(R)$ has a minimal τ-torsionfree cogenerator. If, furthermore, τ is jansian, then the converse holds.*

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PROOF. Assume first that $R/\mathbf{t}(R)$ is left QF-3 and let X be a minimal faithful left $R/\mathbf{t}(R)$-module. Then X is isomorphic to a left ideal of $R/\mathbf{t}(R)$ and hence it is τ-torsion free as left R-module. If Y is a τ-torsion free R-module which cogenerates $R/\mathbf{t}(R)$, then Y is canonically a faithful left $R/\mathbf{t}(R)$-module and hence X is a direct summand of Y (as $R/\mathbf{t}(R)$-module and also as R-module), which shows that it is a minimal τ-torsionfree cogenerator.

Conversely, assume that τ is jansian and let X be a minimal τ-torsionfree cogenerator of $R/\mathbf{t}(R)$. Then X is a faithful $R/\mathbf{t}(R)$-module and if Y is another faithful $R/\mathbf{t}(R)$-module, it follows from the facts that $R/\mathbf{t}(R)$ is τ-torsionfree and that $\mathbf{t}$ preserves products that $Y/\mathbf{t}(Y)$ is also a cogenerator of $R/\mathbf{t}(R)$. Then our hypothesis implies that X is a direct summand of $Y/\mathbf{t}(Y)$. On the other hand, X is a direct summand of $R/\mathbf{t}(R)$ and hence it is $R/\mathbf{t}(R)$-projective, from which it follows easily that X is also a direct summand of Y, which completes the proof.

The ring considered in the remarks following Theorem 4, with the Lambek torsion theory τ_L, shows that the converse of Proposition 1 is not true in general. On the other hand, if R is a right perfect ring, then every torsion theory of R-mod is jansian [15, Corollary VIII.6.3]. In particular τ_L is jansian and so R is left QF-3 if and only if it has a minimal τ_L-torsionfree faithful module.

Let τ be a torsion theory and denote by M_τ the module of quotients corresponding to M (so that R_τ is the ring of quotients of R with respect to τ). Then we recall that τ is called perfect when the localization functor R-mod $\to$ R_τ-mod which assigns to each M its module of quotients M_τ is naturally equivalent to the functor $R_\tau \otimes_R -$: R-mod $\to$ R_τ-mod. This is equivalent to every R_τ-module being τ-torsionfree as R-module [7, Proposition 45.1]. We have

PROPOSITION 2. *Let τ be a perfect torsion theory. If $R/\mathbf{t}(R)$ has a minimal τ-torsionfree cogenerator, then R_τ is a left* QF-3 *ring.*

PROOF. Let X be a minimal τ-torsionfree cogenerator of $R/\mathbf{t}(R)$ and E an injective cogenerator of τ (so that the τ-torsionfree modules are precisely the E-torsionless modules). Then X is a direct summand of E and hence it is injective and τ-torsionfree. Thus X is an injective R_τ-module in a canonical way [15, Proposition IX.2.7]. Since X is an injective R-module, it is clear that X cogenerates $E(R/\mathbf{t}(R))$ and since $R/\mathbf{t}(R)$ is an essential left R-submodule of R_τ and $E(R/\mathbf{t}(R)) = E({}_{R_\tau}R_\tau)$, we see that X cogenerates R_τ. Now, let Y be a faithful R_τ-module. As τ is perfect, Y is a τ-torsionfree R-module which cogenerates $R/\mathbf{t}(R)$ and hence X is, by hypothesis, a direct summand of Y, which shows that X is a minimal faithful R_τ-module.

The converse of Proposition 2 does not hold. For instance, if $R = \mathbf{Z}$ and τ is the usual torsion theory, then $\mathbf{Z}$ has not a minimal torsionfree cogenerator.

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We recall that a module is called finitely cogenerated if it has a finitely generated essential socle and, more generally, for a torsion theory τ, M is τ-finitely cogenerated [4] when $M/\mathbf{t}(M)$ contains an essential finite direct sum of τ-cocritical modules (a module C is τ-cocritical when it is τ-torsionfree and each proper quotient of C is a torsion module [7]). In [3] it is shown that R is left QF-3 if and only if it has a finitely cogenerated injective faithful left ideal. In general we cannot get a similar result in our situation, for a torsion theory τ needs not have any τ-cocritical module. If τ is cogenerated by the injective envelopes of the τ-cocritical modules, then τ is called strongly semiprime [7] and in this case we have

PROPOSITION 3. *Let τ be a strongly semiprime torsion theory. Then $R/\mathbf{t}(R)$ has a minimal τ-torsionfree cogenerator if and only if it has an injective cogenerator submodule which is τ-finitely cogenerated.*

PROOF. Necessity. Let X be a minimal τ-torsionfree cogenerator of $R/\mathbf{t}(R)$ and $E = \bigoplus_I E_i$, where $\{E_i\}$ is a representative set of the isomorphism classes of injective envelopes $E_i = E(C_i)$ of τ-cocritical modules C_i. Since τ is strongly semiprime, E cogenerates $R/\mathbf{t}(R)$ and hence X is a direct summand of E. But X is also a direct summand of $R/\mathbf{t}(R)$ and hence it is a cyclic left R-module, so that X is, in fact, a direct summand of a finite direct sum $\bigoplus_1^n E_i = E(\bigoplus_1^n C_i)$. Thus X is τ-finitely cogenerated by [4, Proposition 1.6].

Sufficiency. Let Q be an injective cogenerator submodule of $R/\mathbf{t}(R)$ such that Q is τ-finitely cogenerated. Then $Q = \bigoplus_1^n E_i$ where $E_i = E(C_i)$ is an injective envelope of a τ-cocritical module C_i. Let $\{E_j : j = 1, \ldots, r\}$ be a set of representatives of the isomorphism classes of these injective envelopes and set $X = \bigoplus_1^r E_j$. Let Y be a τ-torsionfree cogenerator of $R/\mathbf{t}(R)$. Then Y cogenerates X and hence it cogenerates each E_j. Therefore, for $j = 1, \ldots, r$, there exists a nonzero homomorphism $f_j : E_j \to Y$. The restriction of f_j to C_j is a monomorphism (for C_j is τ-cocritical and Y τ-torsionfree) and since C_j is essential in E_j we see that f_j must also be a monomorphism. On the other hand, if $\operatorname{Im} f_j \cap \operatorname{Im} f_k \neq 0$ for $j \neq k$, then we would have that, since E_j and E_k are uniform, $E_j \simeq E(\operatorname{Im} f_j \cap \operatorname{Im} f_k) \simeq E_k$, contradiction. Therefore Y contains a submodule isomorphic to X and the proof is complete.

Let now τ_L be the Lambek torsion theory. Then R is τ_L-torsionfree and the τ_L-torsionfree modules are the $E({}_RR)$-torsionless modules, so R is left QF-3 relative to τ_L if and only if it has a minimal $E({}_RR)$-torsionless faithful R-module. For these rings, we have

PROPOSITION 4. *Let R be a ring such that τ_L is strongly semiprime and R has a minimal $E({}_RR)$-torsionless faithful left R-module and a minimal $E(R_R)$-torsionless faithful right R-module. Then R is a* QF-3 *ring.*

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PROOF. Let X be a minimal $E({}_RR)$-torsionless faithful left R-module. Then X is, by Proposition 3, a τ_L-finitely cogenerated injective left ideal of R. By [6, Corollaries 1.6 and 1.9] ${}_RX$ is a dominant module in the sense of Kato (see [14]) so that by [14, Theorem 1.4] the trace of X in R is a minimal dense (that is, Lambek-dense) right ideal. Thus the Lambek torsion theory of mod-R is jansian and hence R is right QF-3 by Proposition 1. The same argument as before shows that the Lambek torsion theory on the left, τ_L, is also jansian and, using again Proposition 1 we get that R is left QF-3.

REMARKS. The one-sided version of Theorem 4 fails; that is, there is a ring such tht τ_L is perfect (and so is the Lambek torsion theory of mod-R) and R has a minimal $E({}_RR)$-torsionless faithful left R-module but R is neither a left nor a right QF-3 ring. An easy example is given by the triangular matrix ring $R = \left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Q} \\ 0 & \mathbf{Q} \end{smallmatrix}\right)$. If $e = e_{22}$, then Re is a faithful injective left ideal. Moreover, Re is minimal among the $E({}_RR)$-torsionless faithful left R-modules for if Y is another module with this property, then there is a nonzero homomorphism $f\colon \mathrm{Re} \to Y$ and since Y is $E({}_RR)$-torsionless, there is a homomorphism $g\colon Y \to E({}_RR)$ such that $g \circ f \neq 0$. But it is easily seen that $g \circ f$ must be a monomorphism and hence f is also a monomorphism, so that Re is a direct summand of Y. Note also that since the maximal (left and right) ring of quotients of R is the full matrix ring $M_2(\mathbf{Q})$, both Lambek torsion theories of R are perfect (and hence strongly semiprime).

The triangular matrix ring considered in the above remark shows that the class of left QF-3′ rings with ACC on left annihilators is larger than the class of left QF-3 rings with ACC on left annihilators. This cannot happen in the two-sided case, for we have

COROLLARY 5. *Let R be a ring with* ACC (*or* DCC) *on left annihilators. Then the following conditions are equivalent:*

(i) *R is a* QF-3 *ring;*

(ii) *R is a* QF-3′ *ring;*

(iii) *R has a minimal $E({}_RR)$-torsionless faithful left module and a minimal $E(R_R)$-torsionless faithful right module.*

PROOF. Note that, in all cases, every element of $E({}_RR)$ ($E(R_R)$) is also an element of a torsionless module and hence the left (right) annihilator ideals of R coincide with the annihilators of subsets of $E({}_RR)$ (respectively $E(R_R)$). Thus the ACC (DCC) on left annihilators is equivalent to R being τ_L-noetherian (τ_L-artinian) and similarly on the right. In particular, τ_L is strongly semiprime (see [7, page 324]) and so the equivalence between (i) and (iii) follows from Theorem 4. Since (i) implies (ii) obviously, to complete the proof it is enough to show that (ii) implies (iii). Since R has ACC on left annihilators, it has DCC on right

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annihilators and hence, by the Teply-Miller theorem [10, Theorem 1.4], it has ACC on right annihilators. Thus R has also DCC on left annihilators, that is, R is τ_L-artinian and by [10, Corollary 3.3] $E({}_RR)$ is a finite direct sum of injective envelopes of τ_L-cocritical modules. Taking the direct sum of representatives of each isomorphism class of these injective envelopes we have a module X which is clearly faithful and, since R is left QF-3′, X is in fact isomorphic to a left ideal of R. Now, it follows from Proposition 3 that R has a minimal τ_L-torsionfree cogenerator, that is, a minimal $E({}_RR)$-torsionless faithful left module. Since R has also DCC on right annihilators and is right QF-3′, the analogous result holds on the right and the proof is complete.

We will make use of the following lemma.

LEMMA 6. *Let R be a τ-artinian ring and C a τ-cocritical module such that every τ-finitely generated submodule of $E(C)$ is cogenerated by $R/\mathbf{t}(R)$. Then $E(C)$ is a τ-noetherian module.*

PROOF. See the proof that (i) implies (ii) in [8, Theorem 3.1].

Left artinian left QF-3 rings can be characterized by the property that every finitely generated submodule of $E({}_RR)$ is torsionless (this was proved by Rutter in [12]), and also as the left noetherian rings such that $E({}_RR)$ is projective. We are going to give a torsion-theoretic generalization of these facts in such a way that, corresponding to the trivial torsion theory we recover Rutter's result and for the Lambek torsion theory we get characterizations of the much larger class of left QF-3′ rings with ACC on left annihilators.

THEOREM 7. *Let τ be a torsion theory of R-*mod*. The following conditions are equivalent:*

(i) *R is τ-artinian and $R/\mathbf{t}(R)$ has a minimal τ-torsionfree cogenerator;*

(ii) *R is τ-artinian and every τ-finitely generated submodule of $E(R/\mathbf{t}(R))$ is $R/\mathbf{t}(R)$-torsionless;*

(iii) *R is τ-noetherian and each product of copies of $E(R/\mathbf{t}(R))$ is a projective $R/\mathbf{t}(R)$-module;*

(iv) *R is τ-noetherian and $E(R/\mathbf{t}(R))$ is projective as $R/\mathbf{t}(R)$-module.*

PROOF. (i)⇒(ii). If $R/\mathbf{t}(R)$ has a minimal τ-torsionfree cogenerator, then it is clear that $E(R/\mathbf{t}(R))$ is $R/\mathbf{t}(R)$-torsionless from which (ii) follows.

(ii)⇒(iii). Since R is τ-artinian, R is τ-noetherian by the Teply-Miller theorem. Let $E = E(R/\mathbf{t}(R))$. By [10, Corollary 3.3] we have that each product E^I is a direct sum of injective envelopes of τ-cocritical modules. Thus it is enough to prove that if C is a τ-cocritical submodule of E^I, then $E(C)$ is isomorphic to a submodule of $R/\mathbf{t}(R)$. If $C \leq E^I$, then there is a monomorphism $j\colon E(C) \to E^I$

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and for some index i of I we have that $(p_i \circ j)|_C \neq 0$ (where $p_i: E^I \to E$ is the ith projection). Since C is τ-cocritical we see that $(p_i \circ j)|_C$ is a monomorphism and, since C is essential in $E(C)$, this implies that $p_i \circ j$ is a monomorphism, so that $E(C)$ is isomorphic to a submodule of E. Thus (ii) implies that every τ-finitely generated submodule of $E(C)$ is $R/\mathbf{t}(R)$-torsionless and from Lemma 6 it follows that $E(C)$ is τ-noetherian and hence τ-finitely generated (see [7, Proposition 20.1]). Therefore, our hypothesis implies that $E(C)$ is $R/\mathbf{t}(R)$-torsionless and as before, we see that, in fact, $E(C)$ embeds in $R/\mathbf{t}(R)$.

(iii)⇒(iv). This is obvious.

(iv)⇒(i). Note that the proof of [9, Theorem 1] works to show that in our hypothesis R is τ-artinian. Moreover, τ is in this case strongly semiprime [7, Proposition 34.14] and hence, from Proposition 3, to show that $R/\mathbf{t}(R)$ has a minimal torsionfree cogenerator it will be enough to prove that $R/\mathbf{t}(R)$ has an injective cogenerator submodule which is τ-finitely cogenerated. Since R is τ-artinian, ${}_RR$ is τ-finitely cogenerated [4, Proposition 1.5] and hence $E(R/\mathbf{t}(R))$ is isomorphic to a finite direct sum of injective envelopes of τ-cocritical modules. Let $\{E_1, \ldots, E_n\}$ be a representative set of the isomorphism classes of these injective envelopes. Since $E(R/\mathbf{t}(R))$ is $R/\mathbf{t}(R)$-torsionless by hypothesis, we see that each E_i, $i = 1, \ldots, n$, is isomorphic to a submodule of $R/\mathbf{t}(R)$ and, as in the proof of Proposition 3, we have that $R/\mathbf{t}(R)$ contains a submodule which is isomorphic to $\bigoplus_1^n E_i$. It is clear that this submodule satisfies the required conditions and the proof is complete.

REMARK. If τ is the trivial torsion theory, then condition (i) of Theorem 7 is clearly equivalent to R being a left artinian left QF-3 ring, so that the equivalence of (i) and (ii) gives [12, Corollary 3].

We may apply Theorem 7 to τ_L in order to characterize the rings which have a minimal $E({}_RR)$-torsionless faithful left module and the DCC on left annihilators. Recall that a ring R is called left **1**-Gorenstein if $E({}_RR)$ is projective [2] and that a module X is Π-projective when each product of copies of X is projective. We get

COROLLARY 8. *The following conditions are equivalent for a ring R.*

(i) *R has a minimal $E({}_RR)$-torsionless faithful left module and satisfies the* DCC *on left annihilators.*

(ii) *R is a left* QF-3′ *ring with* DCC *on left annihilators.*

(iii) *R is a left* **1**-*Gorenstein ring with* ACC *on left annihilators.*

(iv) *Every almost finitely generated submodule of $E({}_RR)$ is torsionless and R satisfies the* DCC *on left annihilators.*

(v) *Every almost finitely generated $E({}_RR)$-torsionless module embeds in a free module and $E({}_RR)$ is a finite direct sum of injective envelopes of minimal rationally closed submodules.*

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(vi) $E({}_RR)$ *is* Π*-projective.*

(vii) *Every* $E({}_RR)$*-torsionless module embeds in a free module.*

(viii) R *is a left* QF-3′ *ring and every torsionless module embeds in a free module.*

(ix) R *is a left* QF-3′ *ring with semiprimary* QF-3 *maximal two-sided ring of quotients.*

PROOF. Since in all cases any element of $E({}_RR)$ is containd in a torsionless module, the ACC and the DCC on left annihilators are equivalent, respectively, to R being τ_L-noetherian and τ_L-artinian. Now, the equivalence between (i), (ii), (iii) and (iv) follows in a strightforward way from Theroem 7. From the equivalence between (iii) and (iv) of Theorem 7 it also follows that these conditions imply condition (vi).

(v)⇔(vi). This follows from [8, Theorem 3.5] applied to the Lambek torsion theory.

That (vi)⇒(vii) and (vii)⇒(viii) are clear.

(viii)⇒(iii). This follows from [10, Proposition 3.1].

That (iv)⇒(ix) and (ix)⇒(ii) are consequences of [9, Theorem 2].

REMARKS. Note that the equivalence between (iii) and (vii) above gives a proof of [9, Corollary 1] and shows that the converse of that corollary is also true. The class of rings which satisfy the conditions of Corollary 8 contains properly the class of left QF-3 rings with DCC on left annihilators (for example, $\left(\begin{smallmatrix} \mathbf{Z} & \mathbf{Q} \\ 0 & \mathbf{Q} \end{smallmatrix}\right)$ is left **1**-Gorenstein and has ACC on left annihilators but is not left QF-3) and is contained properly in the class of rings which have a semiprimary QF-3 maximal two-sided ring of quotients (this class is characterized in [9, Theorem 2]). For instance, any commutative domain R belongs to the latter class but $E({}_RR)$ is not projective in the nontrivial cases (this also shows that in (iv) and (v) we cannot replace almost finitely generated modules by finitely generated ones). On the other hand, we also remark that it is not possible to replace DCC by ACC in conditions (i), (ii) and (iv) of Corollary 8. The example of Ringel and Tachikawa [11, page 60] of a left noetherian left QF-3 ring which is not right noetherian is not left **1**-Gorenstein and hence does not satisfy the equivalent conditions of the above corollary.

We have obtained in Corollary 5 characterizations of the class of QF-3 rings with ACC on left annihilators. Other equivalent conditions are given in [13, Theorem] and in [9, page 385]. A different characterization is the following (recall that a ring R is said to be left FC if ${}_RR$ is finitely cogenerated).

COROLLARY 9. *The following conditions are equivalent for a ring* R.

(i) R *is a* QF-3 *ring with* ACC *on left annihilators.*

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(ii) *R satisfies the* ACC *on left annihilators and every almost finitely generated submodule of $E({}_RR)$ and of $E(R_R)$ is torsionless.*

(iii) *R is a left* FC *ring such that every almost finitely generated $E({}_RR)$-torsionless left module embeds in a free module and every almost finitely generated $E(R_R)$-torsionless right module is torsionless.*

(iv) *R is a left* QF-3′ *ring and $E(R_R)$ is* Π*-projective.*

PROOF. The equivalences between (i) and (ii) and (i) and (iv) follow from Corollary 5 and Corollary 8.

(i)⇒(iii). By Corollary 8, ${}_RR$ has an essential submodule of the form $\bigoplus_1^n C_i$, where the C_i are minimal rationally closed left ideals (that is, τ_L-saturated and τ_L-cocritical left ideals). Since R is right QF-3, τ_L is jansian and if D is a minimal dense left ideal of R, each DC_i is a simple left ideal of R (see the proof of [7, Proposition 14.8]) which is essential in the uniform left ideal C_i and hence ${}_RR$ is finitely cogenerated. The rest of condition (iii) follows from Corollary 8.

(iii)⇒(ii). Clearly (iii) implies that R satisfies condition (v) of Corollary 8, so that R has ACC on left annihilators and hence we get that (ii) holds.

For left nonsingular rings, the τ_L-dense left ideals are precisely the essential left ideals of R [15]. Denoting by $Q = R_{\tau_L}$ the maximal left ring of quotients of R, we get

PROPOSITION 10. *The following conditions are equivalent for a ring R.*

(i) *R is a let nonsingular left* QF-3′ *ring with* ACC (*or* DCC) *on left annihilators.*

(ii) *R is a left nonsingular left finite-dimensional left* QF-3′ *ring.*

(iii) *Q is semisimple and ${}_RQ$ is projective.*

(iv) *Q is semisimple and every nonsingular left R-module embeds in a free module.*

(v) *R is a left* QF-3′ *ring with semisimple maximal two-sided ring of quotients.*

(vi) *R is a right* FC *ring with projective right socle and every finitely generated nonsingular right R-module embeds in a free module.*

PROOF. (i)⇔(ii). We observed in the proof of Corollary 5 that if R is left QF-3′, then ACC (or DCC) on left annihilators is equivalent to R being τ_L-noetherian. Thus the equivalence between (i) and (ii) follows from [15, Proposition XIII.3.2 and Proposition XIII.3.3].

(ii)⇔(iii)⇔(iv). These equivalences follows readily from [2, Proposition 2.1 and Theorem 2.5], since Q is semisimple if and only if R is left nonsingular and left finite-dimensional.

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(iv)⇒(v). Since, by (iv), R is left nonsingular and finite-dimensional it follows from [15, Corollary XII.7.3], that Q is also the maximal right ring of quotients of R.

(v)⇒(vi). By (v), R is right nonsingular and right finite-dimensional so that, by [15, Corollary XII.7.3] every finitely generated nonsingular right R-module embeds in a free module. Furthermore, from Corollary 8 it follows that ${}_RQ$ is projective and so R is a right FC ring with projective right socle by [14, Corollary 2.5].

(vi)⇒(iii). If Q' denotes the maximal right ring of quotients of R, then Q' is semisimple and ${}_RQ'$ is projective by [14, Corollary 2.5]. From [15, Corollary XII.7.3] it follows that $Q = Q'$, which completes the proof.

REMARKS. The left nonsingular left QF-3′ rings which contain no infinite sets of orthogonal idempotents are characterized in [2, Theorem 2.5]. The class of rings defined by Proposition 10 is smaller than the class of rings such that Q is semisimple and every finitely generated nonsingular left R-module embeds in a free module, for among the rings satisfying the last conditions are all the two-sided orders in semisimple rings [15, Proposition II.4.6]. Also, there are rings R with maximal right ring of quotients Q' semisimple and ${}_RQ'$ projective (these rings are studied in [14, Corollary 2.5]) which do not satisfy the conditions of Corollary 11. An example is given by the ring considered in [15, Exercise XII.5]. On the other hand, $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{smallmatrix}\right)$ satisfies the conditions of Proposition 10 but not their left-right symmetric, for it is not a right QF-3′ ring. On the contrary, if we replace left QF-3′ by left QF-3 in (i), (ii) or (v), we get a class of rings which is left-right symmetric [2, Theorem 2.8].

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Departamento de Matemáticas
Universidad de Murcia
30001 Murcia,
Spain

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