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SCALAR OPERATORS AND INTEGRATION

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Abstract

The notion of a scalar operator on a Banach space, in the sense of N. Dunford, is widened so as to cover those operators which can be approximated in the operator norm by linear combinations of disjoint values of an additive and multiplicative operator valued set function, P, on an algebra of sets in a space Ω such that $P(\Omega) = I$, subject to some conditions guaranteeing that this definition is unambiguous. An operator T turns out to be scalar in this sense, if and only if, there exists a (not necessarily bounded) Boolean algebra of bounded projections such that the Banach algebra of operators it generates is semisimple and contains T.

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Let E be a complex Banach space. Let $B(E)$ be the algebra of all bounded linear operators on E. Then $B(E)$ is a Banach algebra with respect to the operator (uniform) norm defined by $\|T\| = \sup\{|Tx|\colon |x| \leq 1, x \in E\}$, for every $T \in B(E)$. The identity operator is denoted by I.

A spectral measure is a multiplicative and σ-additive (in the strong operator topology) map $P\colon \mathscr{Q} \to B(E)$, whose domain, $\mathscr{Q}$, is a σ-algebra of sets in a space Ω, such that $P(\Omega) = I$. An operator $T \in B(E)$ is said to be of scalar type if there exists a spectral measure P and a P-integrable function f such that

$$T = \int_\Omega f\,dP. \tag{1}$$

This notion, due to N. Dunford, extends to arbitrary Banach spaces the idea of an operator with diagonalizable matrix on a finite-dimensional space. It proved to be very fruitful as is shown by the exposition in monograph [4]. Many powerful techniques in which scalar operators play a role are based on the requirements that $\mathscr{Q}$ be a σ-algebra and that P be σ-additive. But precisely these requirements are responsible for excluding many operators of prime interest from the class of scalar-type operators. Suggestions for extending this class lead to new interesting theories.

So, C. Foiaş introduced the notion of a generalized scalar operator, which was extended further by I. Colojoară and C. Foiaş. They take Ω to be the complex plane and replace the algebra of all bounded measurable functions by a suitable but, possibly, much poorer algebra of functions on Ω and the integration map by a certain kind of homomorphism of such an algebra into $B(E)$. The resulting theory is systematically presented in [1].

The theory of well-bounded operators has its origin in the work of D. R. Smart, [12], and J. R. Ringrose, [11]. It is discussed in Section XV.16 of [4] and, more completely, in the monograph [3]. The underlying idea of this theory is that even if the set function P is not σ-additive and is not defined on a σ-algebra, it may still be possible to introduce the integral with respect to P, based on strong operator convergence, for sufficiently many functions.

The theory of extended spectral operators, due to W. Ricker, [2], is not yet available in a monograph form. Its point of departure is the observation that the failure of an operator T to be of scalar type may be, so to say, not the fault of the operator T itself but, rather, of the space E. Indeed, there often exist a space F, continuously and densely containing E, and an extension, S, of the operator T, by continuity, onto the whole of F such that S is a scalar-type operator.

The purpose of this note is to propose still another generalization of the notion of a scalar-type operator. It is suggested by the fact that the integral (1) exists if and only if there exist $\mathscr{Q}$-simple functions f_j, $j = 1, 2, \dots$, such that

$$\sum_{j=1}^{\infty} \left\| \int_{\Omega} f_j \, dP \right\| < \infty \tag{2}$$

and the equality

$$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega) \tag{3}$$

holds for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty. \tag{4}$$

In that case,

$$\int_{\Omega} f \, dP = \sum_{j=1}^{\infty} \int_{\Omega} f_j \, dP. \tag{5}$$

So, the integral with respect to P can be characterized purely in terms of the operator-norm convergence. Moreover, to use this characterization as a definition of the integral with respect to P, it is not necessary to assume that the set function P be bounded, let alone σ-additive, nor that $\mathscr{Q}$ be a σ-algebra. These assumptions can be replaced by less stringent ones which nevertheless guarantee that the integral is defined unambiguously, that any operator expressible in the form (1) can be approximated by linear combinations of disjoint values of P, that the spectrum of T is equal to the essential range of the function f and the family of all operators so expressed, with fixed P but varying f, is a semisimple commutative Banach algebra.

Thus, as scalar operators in a wider sense, we propose operators which can be expressed in the form (1), assuming that P is an additive and multiplicative $B(E)$-valued function on an algebra of sets in a space Ω, with $P(\Omega) = I$, such that the integral with respect to P can be defined in the indicated manner and has the mentioned properties. Such operators can also be characterized intrinsically, that is, without the reference to any particular definition of integral. Namely, an operator $T \in B(E)$ turns out to be scalar in this sense if and only if there exists a (not necessarily bounded) Boolean algebra of projections belonging to $B(E)$ such that the Banach algebra of operators it generates is semisimple and contains T. However, in contrast with the classical theory, the Gelfand representation of this Banach algebra is not necessarily the algebra of all continuous functions on the structure space but only a dense subalgebra.

So, we have two alternative approaches to the notion of a scalar operator: one using integration and the other operator algebras. The fact that these two approaches tally can be taken as a good indication that the introduced concept would lead to a rich and fruitful theory. Indeed, the successes which P. G. Dodds and W. Ricker obtained in [2] and elsewhere, by injecting techniques related to integration into an area pertaining seemingly exclusively to operator algebras, show convincingly the advantage of complementing and harmonizing the two approaches.

To demonstrate the viability of the introduced concepts and methods, we use them to obtain new information about some multiplier operators in L^p spaces. We show, in particular, that for any $p \in (1, \infty)$ translations are scalar operators in the indicated wider sense. This is particularly significant if $p > 2$, because, as proved in [5], in this case translations are not extended spectral operators in the sense of W. Ricker [10].

The presented theory is open to the extension along the lines suggested by W. Ricker in [10]. Indeed, such an extension becomes necessary if we want to produce a theory comprehensive enough to cover also certain operators which commute with too few projections but nevertheless are natural candidats for being of scalar type in some sense. However, the natural context for such an extension is that of general locally convex spaces rather than normed spaces. Therefore, for the sake of simplicity, no attempt in this direction will be made here.

For this and other reasons, the influence of my happy association with Werner Ricker may not be apparent in what follows. Therefore I would like to acknowledge this influence explicity. I also recollect with a pleasure that the proposed approach to scalar operators and its application to multipliers occured to me during and in consequence of the discussions I had with Earl Berkson.

1

Let Ω be a non-empty set to be called the space. To save subscripts and circumlocution, subsets of Ω will be identified with their characteristic functions.

A family, $\mathscr{Q}$, of subsets of Ω is called a quasialgebra of sets in the space Ω if the family of the unions of all finite collections of pair-wise disjoint sets from $\mathscr{Q}$ form an algebra of sets. By $\operatorname{sim}(\mathscr{Q})$ is denoted the set of all $\mathscr{Q}$-simple functions, that is the vector space spanned by $\mathscr{Q}$. If $\mathscr{Q}$ is a quasialgebra of sets then $\operatorname{sim}(\mathscr{Q})$ is an algebra of functions on Ω under pointwise operations. If $\mathscr{R}$ is the algebra of sets generated by the quasialgebra $\mathscr{Q}$, then $\operatorname{sim}(\mathscr{R}) = \operatorname{sim}(\mathscr{Q})$. Consequently, the introduction of quasialgebras instead of algebras does not bring with it greater generality; it is dicated simply by convenience in considering the families of sets which classically occur in integration and spectral theories but are merely quasialgebras and not algebras.

If $\mathscr{Q}$ is a quasialgebra of sets then a map $P: \mathscr{Q} \to B(E)$ is additive if and only if it has a linear extension onto the whole of $\operatorname{sim}(\mathscr{Q})$. Because the linear extension is unique, we shall not distinguish in the notation between an additive map on $\mathscr{Q}$ and its linear extension on $\operatorname{sim}(\mathscr{Q})$. We shall also write

$$\int_\Omega f\, dP = P(f) \tag{6}$$

for every $f \in \operatorname{sim}(\mathscr{Q})$.

Let $\mathscr{Q}$ be a quasialgebra of sets in the space Ω. An additive map $P: \mathscr{Q} \to B(E)$ is said to be multiplicative if $P(fg) = P(f)P(g)$ for every $f \in \operatorname{sim}(\mathscr{Q})$ and $g \in \operatorname{sim}(\mathscr{Q})$. For an additive map, P, to be multiplicative if suffices that $P(X \cap Y) = P(X)P(Y)$ for every $X \in \mathscr{Q}$ and $Y \in \mathscr{Q}$.

An additive and multiplicative map $P: \mathscr{Q} \to B(E)$ such that $P(\Omega) = I$ will be called a $B(E)$-valued spectral set function on $\mathscr{Q}$.

In virtue of the Stone representation theorem, a set $W \subset B(E)$ is a Boolean algebra of projection operators if and only if there exist an algebra of sets, $\mathscr{R}$, in a space Ω and a spectral set function, $P: \mathscr{R} \to B(E)$, such that $W = \{P(X): X \in \mathscr{R}\}$. Accordingly, a set of operators $W \subset B(E)$ is called a Boolean quasialgebra of projection operators if it is the range of a $B(E)$-valued spectral set function, that is, if there exist a quasialgebra of sets, $\mathscr{Q}$, in a space Ω and a spectral set function, $P: \mathscr{Q} \to B(E)$, such that $W = \{P(X): X \in \mathscr{Q}\}$.

If $W \subset B(E)$, then by $A(W)$ is denoted the least uniformly closed algebra of operators which contains W. If $W = \{P(X): X \in \mathscr{Q}\}$ is the range of a spectral set function $P: \mathscr{Q} \to B(E)$, we write $A(W) = A(P)$. Clearly, $A(P)$ is then the closure of the family of operators $\{P(f): f \in \operatorname{sim}(\mathscr{Q})\}$ in the space $B(E)$.

Recall that, if A is a commutative Banach algebra with unit, then the structure space, Δ, of A is the set of all homomorphisms of A onto the field of complex numbers. For an element T of A, by $\hat{T}$ is denoted the Gelfand transform of T; it is the function on Δ defined by $\hat{T}(h) = h(T)$, for every $h \in \Delta$. It is well known (see, for example, [9], 23B) that $\sup\{|\hat{T}(h)|: h \in \Delta\} \leq \|T\|$ and that the coarsest topology on Δ which makes all the functions, $\hat{T}$, $T \in A$, continuous turns Δ into a compact Hausdorff space. Hence the Gelfand transform is a norm-decreasing homomorphism of the algebra A into the algebra, $C(\Delta)$, of all complex continuous functions on Δ. If the Gelfand transform is injective, then the algebra A is called semisimple.

Recall that an operator $T \in B(E)$ is called nonsingular if it is invertible in $B(E)$, that is, if there exists an operator $S \in B(E)$ such that $ST = TS = I$. Then of course $S = T^{-1}$ is the inverse of T. A full algebra of operators is uniformly closed algebra of operators which contains the inverse of each of its nonsingular elements (see [4], Definition XVII.1.1).

LEMMA 1. *Let $\mathscr{Q}$ be a quasialgebra of sets in a space Ω and let $P: \mathscr{Q} \to B(E)$ be a spectral set function.*

(i) *If $f \in \operatorname{sim}(\mathscr{Q})$, then the operator $P(f)$ is nonsingular if and only if the function f can be represented in the form*

$$f = \sum_{j=1}^{n} c_j X_j, \tag{7}$$

where n is a natural number, c_j are non-zero complex numbers and X_j are pairwise disjoint sets from $\mathscr{Q}$, $j = 1, 2, \ldots, n$, such that $\sum_{j=1}^{n} P(X_j) = I$. In that case, $(P(f))^{-1} = P(g)$, where $g = \sum_{j=1}^{n} c_j^{-1} X_j$.

(ii) *Let $f \in \operatorname{sim}(\mathscr{Q})$ be a function expressed in the form (7) where $X_j \in \mathscr{Q}$ are pairwise disjoint sets such that $P(X_j) \neq 0$, for every $j = 1, 2, \ldots, n$, and let $c = \sup\{|c_j|: j = 1, 2, \ldots, n\}$ and*

$$d = \sup\left\{\left\|\sum_{j\in J} P(X_j)\right\| : J \subset \{1, 2, \ldots, n\}\right\}.$$

Then $c \leq \|P(f)\| \leq 4cd$.

(iii) *$A(P)$ is a full algebra of operators.*

PROOF. Let $n \geq 1$ be an integer. Let $X_j \in \mathcal{Q}$ be pairwise disjoint sets, such that $P(X_j) \neq 0$, for every $j = 1, 2, \ldots, n$ and the sum of the operators $P(X_j)$, $j = 1, 2, \ldots, n$, is equal to I. Then the family of operators $\sum_{j=1}^{n} c_j P(X_j)$, with arbitrary complex $c_j = 1, 2, \ldots, n$, is a closed algebra of operators generated by the Boolean algebra of projections $\sum_{j \in J} P(X_j)$, where J varies over all subsets of $\{1, 2, \ldots, n\}$. Then (i) holds by Lemma XVII.2.1 and (ii) by Lemma XVII.2.2 in [4].

To show that $A(P)$ is a full algebra of operators let T be a non-singular element of $A(P)$. Let $f_n \in \operatorname{sim}(\mathcal{Q})$, $n = 1, 2, \ldots$, be functions such that $\|T - P(f_n)\| \to 0$, as $n \to \infty$. Then for all sufficiently large n, the operator $P(f_n)$ is nonsingular and $\|T^{-1} - (P(f_n))^{-1}\| \to 0$. But, by (i), for each such n, there exists a function $g_n \in \operatorname{sim}(\mathcal{Q})$ such that $(P(f_n))^{-1} = P(g_n)$. Therefore, $T^{-1} \in A(P)$.

2

Let $\mathcal{Q}$ be a quasialgebra of sets in a space Ω and let $P\colon Q \to B(E)$ be a spectral set function.

Let us call P-null any set $Y \subset \Omega$ for which there exist sets $X_j \in \mathcal{Q}$ such that $P(X_j) = 0$, for every $j = 1, 2, \ldots$, and

$$Y \subset \bigcup_{j=1}^{\infty} X_j. \tag{8}$$

PROPOSITION 2. *A set $Y \subset \Omega$ is P-null if and only if there exist functions $f_j \in \operatorname{sim}(\mathcal{Q})$, $j = 1, 2, \ldots$, satisfying condition* (2), *such that*

$$\sum_{j=1}^{\infty} |f_j(\omega)| = \infty \tag{9}$$

for every $\omega \in Y$.

PROOF. Let the set Y be P-null. Let $X_j \in \mathcal{Q}$ be sets such that $P(X_j) = 0$, for every $j = 1, 2, \ldots$, and (8) holds. Let us repeat each set countably many times, arrange the resulting family of sets into a single sequence and call their characteristic functions f_j, $j = 1, 2, \ldots$. Then $f_j \in \operatorname{sim}(\mathcal{Q})$, $j = 1, 2, \ldots$, the inequality (2) holds and the equality (9) holds for every $\omega \in Y$.

Conversely, assume that $f_j \in \operatorname{sim}\mathcal{Q})$, $j = 1, 2, \ldots$, are functions, satisfying condition (2), such that the equality (9) holds for every $\omega \in Y$. Let $f_j = \sum_{j=1}^{n_j} c_{jk} X_{jk}$, with some integer $n_j \geq 1$, numbers c_{jk} and pair-wise disjoint sets $X_{jk} \in \mathcal{Q}$, $k = 1, 2, \ldots n_j$, for every $j = 1, 2, \ldots$. By Lemma 1, $\|P(f_j)\| \geq |c_{jk}|$, whenever $P(X_{jk}) \neq 0$. Therefore if we modify each function f_j by omiting those sets X_{jk}, together with the corresponding numbers c_{jk} for which $P(X_{jk}) \neq 0$, then (9) will remain satisfied for every $\omega \in Y$. But then, Y is covered by the remaining sets X_{jk}, $k = 1, 2, \ldots, n_j$, $j = 1, 2, \ldots$.

For a function f on Ω, let

$$\|f\|_\infty = \inf\{\sup\{|f(\omega): \omega \in \Omega \setminus Y\}: Y \in \mathscr{N}\},$$

where $\mathscr{N}$ is the family of all P-null sets. Then $0 \leq \|f\|_\infty \leq \infty$. The function f is said to be P-essentially bounded if $\|f\|_\infty < \infty$. In that case, the infimum is actually a minimum because any subset of the union of countably many P-null sets is P-null. That is to say, for any P-essentially bounded function f, there exists a P-null set Y such that

$$\|f\|_\infty = \sup\{|f(\omega)|: \omega \in \Omega \setminus Y\}.$$

Following the custom, we shall call P-null any function f on Ω such that $\|f\|_\infty = 0$. The P-equivalence class of a function f will be denoted by $[f]$. To be sure, $[f]$ is the set of all functions g on Ω such that $\|f - g\|_\infty = 0$.

Let $\mathscr{L}^\infty(P)$ be the family of all functions f on Ω such that, for every $\varepsilon > 0$, there exists a function $g \in \mathrm{sim}(\mathscr{Q})$ for which $\|f - g\|_\infty < \varepsilon$. Then $\mathscr{L}^\infty(P)$ is an algebra under the point-wise operations.

Let $L^\infty(P) = \{[f]: f \in \mathscr{L}^\infty(P)\}$. Then $L^\infty(P)$ is a Banach algebra with respect to the operations induced by the operators in the algebra $\mathscr{L}^\infty(P)$ and the norm, $\|\cdot\|_\infty$, induced by the seminorm $f \mapsto \|f\|_\infty$, $f \in \mathscr{L}^\infty(P)$.

The Banach algebra $L^\infty(P)$ is semisimple (see, for example, [9], Theorem 24C). Actually, if Δ is the structure space of $L^\infty(P)$, then the Gelfand transform is an isometric isomorphism of $L^\infty(P)$ onto the whole of $C(\Delta)$. Moreover, for any function $f \in \mathscr{L}^\infty(P)$, the equality

$$\{[f]\hat{}(h): h \in \Delta\} = \bigcap_{Y \in \mathscr{N}} \{f(\omega): \omega \in \Omega \setminus Y\}^- \tag{10}$$

holds, where $\mathscr{N}$ is the family of all P-null sets and the bar indicates the closure in the complex plane. The set (10) is called the P-essential range of the function f.

3

Let $\mathscr{Q}$ be a quasialgebra of sets in a space Ω and let $P: \mathscr{Q} \to B(E)$ be a spectral set function.

The spectral set function P will be called closable if

$$\sum_{j=1}^{\infty} \int_\Omega f_j \, dP = 0 \tag{11}$$

for any functions $f_j \in \operatorname{sim}(\mathscr{Q})$, $j = 1, 2, \ldots$, satisfying condition (2), such that

$$\sum_{j=1}^{\infty} f_j(\omega) = 0 \tag{12}$$

for every $\omega \in \Omega$ satisfying the inequality (4).

The reason for using this term will become apparent after Proposition 6 has been stated.

Now, assume that the spectral set function P is closable.

A function f on Ω will be called P-integrable if there exist functions $f_j \in \operatorname{sim}(\mathscr{Q})$, $j = 1, 2, \ldots$, satisfying condition (2), such that the equality (3) holds for every $\omega \in \Omega$ for which the inequality (4) does. The integral with respect to P of such a function f is then defined by the formula (5). Because the set function P is closable, it is determined unambiguously by the function f. For convenience, we shall use also the simple notation (6) for every P-integrable function f.

The family of all P-integrable functions is denoted by $\mathscr{L}(P)$. It is straightforward that $\mathscr{L}(P)$ is a vector space of functions and the integration map $P\colon \mathscr{L}(P) \to B(E)$ is linear. Therefore, the functional $f \mapsto \|P(f)\|$, $f \in \mathscr{L}(P)$, is a seminorm on $\mathscr{L}(P)$. The normed space, obtained by the identification of any two elements of $\mathscr{L}(P)$ such that this seminorm vanishes on their difference, will be denoted by $L(P)$. The integration map then induces a linear map of $L(P)$ into $B(E)$ still to be called the integration map and denoted by P.

LEMMA 3. *If $f \in \mathscr{L}(P)$ then*

$$\|P(f)\| = \inf \sum_{j=1}^{\infty} \|P(f_j)\|,$$

where the infimum is taken over all choices of functions $f_j \in \operatorname{sim}(\mathscr{Q})$, $j = 1, 2, \ldots$, satisfying condition (2), *such that the equality* (3) *holds for every $\omega \in \Omega$ for which the inequality* (4) *does.*

PROOF. Clearly, $\|P(f)\| \le \sum_{j=1}^{\infty} \|P(f_j)\|$ for any such choice of the $\mathscr{Q}$-simple functions f_j, $j = 1, 2, \ldots$. On the other hand, one can choose them so that the numbers $\sum_{j=2}^{\infty} \|P(f_j)\|$ and $|\|P(f)\| - \|P(f_1)\||$ are both arbitrarily small. This can always be achieved: it suffices to take first any such functions and then, if necessary, to replace the first one by the sum of the first n of them, for sufficiently large n, and renumber the rest.

The following proposition is a formulation of the Beppo Levi theorem in the present setting.

PROPOSITION 4. *Let $f_j \in \mathscr{L}(p)$, $j = 1, 2, \dots$, be function satisfying condition* (2) *and let f be a function on Ω such that the equality* (3) *holds for every $\omega \in \Omega$ for which the inequality* (4) *does. Then $f \in \mathscr{L}(P)$ and the equality* (5) *holds.*

PROOF. Let, for every $j = 1, 2, \dots, f_{jk} \in \operatorname{sim}(\mathscr{Q})$ be functions, $k = 1, 2, \dots$, such that $\sum_{k=1}^{\infty} \|P(f_{jk})\| < \|P(f_j)\| + 2^{-j}$ and $f_j(\omega) = \sum_{k=1}^{\infty} f_{jk}(\omega)$ for every $\omega \in \Omega$ for which $\sum_{k=1}^{\infty} |f_{jk}(\omega)| < \infty$. Then, for every $n = 0, 1, 2, \dots$,

$$\sum_{j=n+1}^{\infty} \sum_{k=1}^{\infty} |P(f_{jk})\| < \sum_{j=n+1}^{\infty} \|P(f_j)\| + 2^{-n} < \infty$$

and

$$f(\omega) - \sum_{j=1}^{n} f_j(\omega) = \sum_{j=n+1}^{\infty} \sum_{k=1}^{\infty} f_{jk}(\omega)$$

for every $\omega \in \Omega$ for which $\sum_{j=n+1}^{\infty} \sum_{k=1}^{\infty} |f_{jk}(\omega)| < \infty$. Therefore, for every $n = 0, 1, 2, \dots$, the function $f - \sum_{j=1}^{n} f_j$ belongs to $\mathscr{L}(P)$ and

$$\left\| P\left(f - \sum_{j=1}^{n} f_j \right) \right\| < \sum_{j=n+1}^{\infty} \|P(f_j)\| + 2^{-n}.$$

PROPOSITION 5. *The equality $\|f\|_\infty = 0$ holds for a function f on Ω if and only if $f \in \mathscr{L}(P)$ and $P(f) = 0$. Furthermore, $\mathscr{L}(P) \subset \mathscr{L}^\infty(P)$.*

If $f \in \mathscr{L}(P)$ and $g \in \mathscr{L}(P)$, then $fg \in \mathscr{L}(P)$ and $P(fg) = P(f)P(g)$. So, $\mathscr{L}(P)$ is an algebra of functions.

The range of the integration map $P\colon \mathscr{L}(P) \to B(E)$ is equal to $A(P)$. The Banach algebra $A(P)$ is semisimple. The integration map $P\colon L(P) \to A(P)$ is an isomorphism of the algebra $L(P)$ onto the algebra $A(P)$.

If $f \in \mathscr{L}(P)$, then the spectrum of the operator $T = P(f)$ is equal to the P-essential range of the function f.

PROOF. If f is a function on Ω such that $\|f\|_\infty = 0$, then, by the definitions of the P-null sets, P-null functions and integral, $f \in \mathscr{L}(P)$ and $P(f) = 0$.

Let $f \in \mathscr{L}(P)$. Let $f_j \in \operatorname{sim}(\mathscr{Q})$, $j = 1, 2, \dots$, be functions, satisfying condition (2), such that (3) holds for every $\omega \in \Omega$ for which (4) does. Then, by Lemma 1,

$$\sum_{j=1}^{\infty} \|f_j\|_\infty < \infty. \tag{13}$$

By the completeness of the space $L^\infty(P)$, there exists a function $g \in \mathscr{L}^\infty(P)$ such that

$$(14) \qquad [g] = \sum_{j=1}^{\infty} [f_j]$$

in $L^\infty(P)$. Since, by Proposition 2, the set of points $\omega \in \Omega$ for which the equality (3) does not hold is P-null, we have $\|f-g\|_\infty = 0$, and so, $f \in \mathscr{L}^\infty(P)$. Moreover, by Lemma 1,

$$\left\| \sum_{j=1}^{n} f_j \right\|_\infty \leq \left\| P\left(\sum_{j=1}^{n} f_j \right) \right\|,$$

for every $n = 1, 2, \ldots$. Therefore, by Proposition 4 and the continuity of norms, $\|f\|_\infty \leq \|P(f)\|$.

If, moreover, $g \in \mathrm{sim}(\mathscr{Q})$, then, by Lemma 1, $P(f_j g) = P(g_j)P(g)$, for every $j = 1, 2, \ldots$, and, by (2),

$$(15) \qquad \sum_{j=1}^{\infty} \|P(f_j g)\| \leq \sum_{j=1}^{\infty} \|P(f_j)\| \, \|P(g)\| < \infty.$$

Hence, $fg \in \mathscr{L}(P)$ and $P(fg) = P(f)P(g)$. But then, we can write (15) for any function $g \in \mathscr{L}(P)$. Consequently, by Proposition 4, $fg \in \mathscr{L}(P)$ and $P(fg) = P(f)P(g)$ for any $f \in \mathscr{L}(P)$ and $g \in \mathscr{L}(P)$.

It is clear, from the definition of the integral, that for any $f \in \mathscr{L}(P)$, the operator $P(f)$ belongs to $A(P)$, the closure of the set $\{P(h) : h \in \mathrm{sim}(\mathscr{Q})\}$. Hence, to show that $\{P(h) : h \in \mathscr{L}(P)\} = A(P)$, it suffices to show that the set $\{P(h) : h \in \mathscr{L}(P)\}$ is closed in $B(E)$. So, let the operator T be in the closure of this set. Let $h_j \in \mathscr{L}(P)$ be functions such that $\|T - P(h_j)\| < 2^{-j}$ for every $j = 1, 2, \ldots$. Let $f_1 = h_1$ and $f_j = h_j - h_{j-1}$, for every $j = 2, 3, \ldots$. Then the condition (2) is satisfies, and, so by Proposition 4, if f is a function such that (3) holds for every $\omega \in \Omega$ for which (4) does, then $f \in \mathscr{L}(P)$ and $T = P(f)$.

If is now obvious that the integration map $P: L(P) \to A(P)$ is an isomorphism of the algebras $L(P)$ and $A(P)$. Because the algebra $L(P)$ is semisimple, being a dense subalgebra of $L^\infty(P)$, the algebra $A(P)$ too is semisimple.

By Lemma 1, the algebra $A(P)$ is full. Therefore, the spectrum of an operator T belonging to $A(P)$ coincides with its spectrum as an element of this algebra. Because of the isomorphism of $A(P)$ and $L(P)$, this spectrum coincides with the spectrum of the element $[f]$ of the algebra $L(P)$ such that $T = P(f)$ which is equal to the essential range of the function f.

4

Let $\mathscr{Q}$ be a quasialgebra of sets in a space Ω.

If $P: \mathscr{Q} \to B(E)$ is a closable spectral set function, then by Proposition 5, $L(P) \subset L^\infty(P)$. Clearly, if P is not bounded on the algebra generated by $\mathscr{Q}$, then the integration map is not continuous in the norm of the space $L^\infty(P)$ and its domain, $L(P)$, is not equal to the whole of $L^\infty(P)$. This domain is of course dense in $L^\infty(P)$ and the following proposition implies that the integration map is closed.

PROPOSITION 6. *A spectral set function $P: \mathscr{Q} \to B(E)$ is closable if and only if there exists an injective map $\Phi: A(P) \to L^\infty(P)$ such that $\|\Phi(T)\|_\infty \le \|T\|$, for every $T \in A(P)$, and $\Phi(P(f)) = [f]$, for every $f \in \mathrm{sim}(\mathscr{Q})$.*

If the spectral set function $P: \mathscr{Q} \to B(E)$ is indeed closable, then such a map Φ is unique, its range is equal to $L(P)$ and the map Φ is equal to the inverse of the integration map.

PROOF. If such a map $\Phi: A(P) \to L^\infty(P)$ exists, then it is unique and linear because $\{P(f): f \in \mathrm{sim}(\mathscr{Q})\}$ is a dense subspace of $A(P)$. Let then $f_j \in \mathrm{sim}(\mathscr{Q})$, $j = 1, 2, \ldots$, be functions satisfying condition (2) and let the equality (12) hold for every $\omega \in \Omega$ for which the inequality (4) does. Let $T \in B(E)$ be the operator such that $\lim_{n\to\infty} \|T - \sum_{j=1}^n P(f_j)\| = 0$. Then of course $T \in A(P)$. Because the map Φ is norm-decreasing, condition (2) implies that (13) holds and, if $[g] = \Phi(T)$, then (14) does. Now, by Proposition 2, the set of points $\omega \in \Omega$ for which (9) holds is P-null, and so, $[g] = 0$. Consequently, $T = 0$ because the map Φ is injective. That is, (11) holds and the set function P is closable.

If the set function P is closable, then, by Proposition 5, such a map $\Phi: A(P) \to L^\infty(P)$ exists: it is the inverse of the integration map.

Let us now mention a sufficient condition for a spectral set function to be closable which implies in particular that spectral measures are closable spectral set function. But first a definition.

A spectral set function $P: \mathscr{Q} \to B(E)$ is said to be stable if $P(Y) = 0$ for every P-null set Y which belongs to $\mathscr{Q}$.

PROPOSITION 7. *If $\mathscr{Q}$ is an algebra of sets and $P: \mathscr{Q} \to B(E)$ a bounded and stable spectral set function, then P is closable.*

PROOF. Let $[\mathrm{sim}(\mathscr{Q})] = \{[f]: f \in \mathrm{sim}(\mathscr{Q})\}$. Because P is stable, there is a map $\tilde{P}: [\mathrm{sim}(\mathscr{Q})] \to B(E)$, unambiguously defined by $\tilde{P}([f]) = P(f)$, for every $f \in \mathrm{sim}(\mathscr{Q})$. Because P is bounded and $\mathscr{Q}$ is an algebra, by Lemma 1, the map $\tilde{P}$ is bounded. Then $\tilde{P}$ has a unique continuous extension onto the whole of

$L^\infty(P)$. By Lemma 1, $\tilde{P}$ and its extension are norm-increasing. Therefore, $\tilde{P}$ so extended has an inverse, Φ, which is norm-decreasing. Because both maps, $\tilde{P}$ and Φ, are bounded, the domain of Φ is closed and, hence, equal to $A(P)$. So, by Proposition 6, the set function P is closable.

COROLLARY 8. *Let $P: \mathscr{Q} \to B(E)$ be a spectral set function such that, for every $x \in E$ and $x' \in E'$, the set function $X \mapsto x'P(X)x$, $X \in \mathscr{Q}$, generates a σ-additive measure of finite variation. Then the set function P is closable.*

PROOF. The assumption implies that the additive extension of P onto the algebra of sets generated by $\mathscr{Q}$ is bounded and stable.

In particular, if $\mathscr{Q}$ is a σ-algebra and $P: \mathscr{Q} \to B(E)$ a spectral measure, then P is a closable spectral set function.

5

Let us call a Boolean quasialgebra of projections $W \subset B(E)$ semisimple if the Banach algebra, $A(W)$, it generates is semisimple.

PROPOSITION 9. *A Boolean quasialgebra of projection operators, $W \subset B(E)$, is semisimple if and only if there exists a quasialgebra of sets, $\mathscr{Q}$, in a space Ω, and a closable spectral set function, $P: \mathscr{Q} \to B(E)$, such that $A(W) = A(P)$.*

PROOF. Let W be semisimple. Let Ω be the structure space of the Banach algebra $A(W)$. Let us denote by Φ the Gelfand transform and put $\mathscr{Q} = \{\Phi(S): S \in W\}$. Because we identify sets with their characteristic functions, $\mathscr{Q}$ is a quasialgebra of sets in the space Ω. Let $P(\Phi(S)) = S$, for every $S \in W$. This defines a spectral set function $P: \mathscr{Q} \to B(E)$ such that the empty set is the only P-null set. Therefore, $L^\infty(P) = C(\Omega)$ and the Gelfand transform is clearly a norm-decreasing injective map from $A(P) = A(W)$ into $L^\infty(P)$ such that $\Phi(P(f)) = [f]$ for every $f \in \operatorname{sim}(\mathscr{Q})$. So, by Proposition 6, the spectral set function P is closable.

Conversely, if a closable spectral set function P such that $A(W) = A(P)$ exists, then, by Proposition 5, the Banach algebra $A(W)$ is semisimple.

COROLLARY 10. *Any bounded Boolean algebra of projections is semisimple.*

PROOF. By the Stone representation theorem, for any Boolean algebra of operators, W, there exists and algebra of sets $\mathscr{Q}$ and a spectral set function

$P: \mathscr{Q} \to B(E)$ such that $\varnothing$ is the only P-null set and $\{P(X): X \in \mathscr{Q}\} = W$. By Proposition 7, the set function P is closable.

Let us call an operator $T \in B(E)$ scalar in wider sense if there exists a semisimple Boolean quasialgebra of operators $W \subset B(E)$ such that $T \in A(W)$. By Proposition 9 and Proposition 5, an operator T is scalar in wider sense if and only if there exist a quasialgebra of sets, $\mathscr{Q}$, in a space Ω, a closable spectral set function $P: \mathscr{Q} \to B(E)$ and a P-integrable function f such that $T = P(f)$.

Clearly, operators which are scalar in the sense of Dunford are scalar in wider sense. Moreover, these operators can be characterized in terms introduced here.

By a Boolean σ-algebra of projection operators is understood a Boolean algebra of projection operators which contains the strong limit of every monotonic sequence of its elements.

PROPOSITION 11. *An operator $T \in B(E)$ is scalar in the sense of Dunford if and only if there exists a Boolean σ-algebra of projection operators, $W \subset B(E)$, such that $T \in A(W)$.*

PROOF. If the operator $T \in B(E)$ is scalar in the sense of Dunford, then there exist a σ-algebra of sets, $\mathscr{Q}$, in a space Ω, a spectral measure $P: \mathscr{Q} \to B(E)$ and a function $f \in \mathscr{L}(P)$ such that $T = P(f)$. The range, $W = \{P(X): X \in \mathscr{Q}\}$, of the spectral measure P is then a Boolean σ-algebra of projections such that $T \in A(W)$.

Conversely, let $W \subset B(E)$ be a Boolean σ-algebra of projections such that $T \in A(W)$. By the Stone representation theorem there exist a compact space Ω, an algebra $\mathscr{R}$ consisting to its compact and open subsets and a spectral set function $P: \mathscr{R} \to B(E)$ such that $W = \{P(X): X \in \mathscr{R}\}$. Let $\mathscr{Q}$ be the σ-algebra of sets generated by $\mathscr{R}$. Because P is in fact σ-additive and W is a σ-algebra of operators, the set function P has a strongly σ-additive extension onto $\mathscr{Q}$, still denoted by P, whose range remains equal to W; see, for example, [8]. Then $P: \mathscr{Q} \to B(E)$ is a spectral measure such that, by Proposition 5, $T = P(f)$, for some function $f \in \mathscr{L}(P)$.

In view of this proposition, an alternative terminology suggests itself: operators scalar in wider sense could be simply called scalar and operators scalar in the sense of Dunford could be called σ-scalar.

6

Let G be a locally compact Abelian group and Γ its dual group. The value of a character $\xi \in \Gamma$ on an element $x \in G$ is denoted by $\langle x, \xi \rangle$.

Let $1 < p < \infty$ and let $E = L^p(G)$, with respect to a fixed Haar measure on G.

Let $\mathscr{M}^p(\Gamma)$ be the family of all individual functions on Γ which determine multiplier operators on E. That is, $f \in \mathscr{M}^p(\Gamma)$ if and only if there exists an operator $T_f \in B(E)$ such that $(T_f\varphi)\hat{} = f\hat{\varphi}$, for every $\varphi \in L^2 \cap L^p(G)$. Here, of course, $\hat{\varphi}$ denotes the Fourier-Plancherel transform of an element φ of $L^2(G)$.

Functions belonging to $\mathscr{M}^p(\Gamma)$ are essentially bounded. In fact, $\|f\|_\infty \leq \|T_f\|$, for every $f \in \mathscr{M}^p(\Gamma)$, where $\|f\|_\infty$ is the essential supremum norm of f with respect to the Haar measure. The operator T_f depends only on the equivalence class of a function f. That is, if $f \in \mathscr{M}^p(\Gamma)$ and if g is a function on Γ such that $g(\xi) = f(\xi)$ for almost every $\xi \in \Gamma$, relative to the Haar measure, then $g \in \mathscr{M}^p(\Gamma)$ and $T_g = T_f$.

It is well known that an operator $T \in B(E)$ commutes with all translations of G if and only if there exists a function $f \in \mathscr{M}^p(\Gamma)$ such that $T = T_f$. So, $\{T_f : f \in \mathscr{M}^p(\Gamma)\}$ is a commutative algebra of operators, containing the identity operator, which is closed in $B(E)$. Clearly, $\mathscr{M}^p(\Gamma)$ is an algebra of functions and the map $f \mapsto T_f$, $f \in \mathscr{M}^p(\Gamma)$, is multiplicative and linear.

Let $\mathscr{R}^p(\Gamma)$ be the family of all sets $X \subset \Gamma$ such that $X \in \mathscr{M}^p(\Gamma)$. Let $P^p_\Gamma(X) = T_X$, for every $X \in \mathscr{R}^p(\Gamma)$.

PROPOSITION 12. *The family $\mathscr{R}^p(\Gamma)$ is an algebra of sets in Γ and P^p_Γ: $\mathscr{R}^p(\Gamma) \to B(L^p(G))$ is a closable spectral set function.*

PROOF. It follows from the mentioned properties of the map $f \mapsto T_f$, $f \in \mathscr{M}^p(\Gamma)$, that $\mathscr{R}^p(\Gamma)$ is an algebra of sets and the set function $P = P^p_\Gamma$ is spectral. Furthermore, a set $Y \subset \Gamma$ is P-null if and only if it is null with respect to the Haar measure on Γ. Consequently, the Haar measure equivalence classes of functions on Γ are the same as the P-equivalence classes and so are their ∞-norms. Therefore, $L^\infty(P)$ is a Banach subspace of $L^\infty(\Gamma)$. Now, $A(P)$ is a closed subalgebra of the Banach algebra $\{T_f : f \in \mathscr{M}^p(\Gamma)\}$. For every $T \in A(P)$, let $\Phi(T) = [f]$, where $f \in \mathscr{M}^p(\Gamma)$ is a function such that $T = T_f$. Then Φ is an unambiguously defined norm-decreasing map from $A(P)$ into $L^\infty(P)$ such that $\Phi(P(f)) = [f]$, for every $f \in \mathrm{sim}(\mathscr{R}^p(\Gamma))$. Therefore, by Proposition 6, the set function P is closable.

The usefulness of this proposition depends of course on how rich is the algebra of sets $\mathscr{R}^p(\Gamma)$. A result of T. A. Gillespie implies that it is rich enough to permit complete spectral analysis of translation operators. Let us introduce the necessary relevant notation.

Let $\mathbf{T}$ be the circle group, $\{z \in \mathbf{C} : |z| = 1\}$, with its usual topology as a subset of the complex plane. Connected subsets of $\mathbf{T}$ will be called arcs. For an element x of the group G and an arc $Z \subset \mathbf{T}$, let

$$X_{Z,x} = \{\xi \in \Gamma : \langle x, \xi \rangle \in Z\}.$$

Let $\mathscr{K}_1(\Gamma)$ be the family of all sets $X_{Z,x}$ corresponding to arcs $Z \subset \mathbf{T}$ and elements of $x \in G$. The classes of sets $\mathscr{K}_n(\Gamma)$, $n = 2, 3, \ldots$, are then defined recursively by requiring that $\mathscr{K}_n(\Gamma)$ consist of all sets $X \cap Y$ such that $X \in \mathscr{K}_{n-1}(\Gamma)$ and $Y \in \mathscr{K}_1(\Gamma)$.

LEMMA 13. *The inclusion $\mathscr{K}_n(\Gamma) \subset \mathscr{R}^p(\Gamma)$ is valid for every $p \in (1, \infty)$ and every $n = 1, 2 \ldots$. Moreover, for every $p \in (1, \infty)$, there exists a constant $C_p \geq 1$ such that $\|P_\Gamma^p(X)\| \leq C_p^n$, for every $X \in \mathscr{K}_n(\Gamma)$, every $n = 1, 2, \ldots$. and every locally compact Abelian group Γ.*

PROOF. For $n = 1$, this is a simple re-formulation of [6, Lemma 6]. (See also [3, Lemma 20.15].) By induction, the result follows for every $n = 2, 3, \ldots$.

Let $\mathscr{T}_1$ be the family of all subsets of $\mathbf{R}$ which contains all members of $\mathscr{K}_1(\mathbf{R})$ and all intervals in $\mathbf{R}$ and no other sets. The families $\mathscr{T}_n$, $n = 2, 3, \ldots$, are then defined recursively by requiring that $\mathscr{T}_n$ consist of all sets $X \cap Y$ such that $X \in \mathscr{T}_{n-1}$ and $Y \in \mathscr{T}_1$.

If we combine Lemma 13 with a classical theorem of M. Riesz (interpreted to the effect that intervals belong to $\mathscr{M}^p(\mathbf{R})$ and determine a bounded family of multiplier operators, see, for example, [1, Theorem 6.3.3]) we obtain the following

COROLLARY 14. *The inclusion $\mathscr{T}_n \subset \mathscr{R}^p(\mathbf{R})$ is valid for every $p \in (1, \infty)$ and every $n = 1, 2, \ldots$. Moreover, for every $p \in (1, \infty)$, there exists a constant $D_p \geq 1$ such that $\|P_\mathbf{R}^p(X)\| \leq D_p^n$, for every $X \in \mathscr{T}_n$ and $n = 1, 2, \ldots$.*

The (total) variation of a function f of bounded variation on $\mathbf{R}$ or on $\mathbf{T}$ will be denoted by $\operatorname{var}(f)$. Recall that every function, f, of bounded variation has a decomposition, $f = f_1 + f_2 + f_3$, such that the function f_1 is absolutely continuous, f_2 is continuous and singular (its derivative vanishes almost everywhere) and f_3 is a jump-function. If the function f vanishes at a point (or at $-\infty$) then there is only one such decomposition with all the three components, f_1, f_2 and f_3, vanishing at that point. If the continuous singular component, f_2, is identically equal to zero, then the function f is called nonsingular.

LEMMA 15. *Let α, β and b be real numbers such that $\alpha < \beta$. Let u be the function on $\mathbf{R}$ such that $u(t) = 0$ for $t < \alpha$, $u(t) = b(t - \alpha)$ for $\alpha \leq t < \beta$, and $u(t) = b(\beta - \alpha)$ for $t \geq \beta$. Then there exist numbers c_j and sets $X_j \in \mathscr{T}_2$,*

$j = 0, 1, 2, \ldots$, such that

$$\sum_{j=0}^{\infty} |c_j| \|P_{\mathbf{R}}^p(X_j)\| \leq 2D_p^2 \operatorname{var}(u)$$

and

$$\sum_{j=0}^{\infty} c_j X_j(t) = u(t)$$

for every $t \in \mathbf{R}$.

PROOF. Because $\operatorname{var}(u) = |b|(\beta - \alpha)$, by Corollary 14, the statement holds with $c_j = 2^{-j} b(\beta - \alpha)$, $j = 0, 1, 2, \ldots, X_0 = [\beta, \infty)$ and

$$X_j = \left\{ t \in \mathbf{R} \colon \exp\left(\frac{2^j \pi (t - \alpha)}{\beta - \alpha} i \right) \in \{\exp si \colon \pi \leq s < 2\pi\} \right\} \cap [\alpha, \beta),$$

$j = 1, 2, \ldots$.

PROPOSITION 16. *Let f be a real non-singular function of bounded variation on $\mathbf{R}$ such that $f(-\infty) = 0$. Then $f \in \mathscr{L}(P_{\mathbf{R}}^p)$ and*

$$P_{\mathbf{R}}^p(f) \leq 3D_p^2 \operatorname{var}(f), \tag{16}$$

for every $p \in (1, \infty)$.

PROOF. Let $f = f_1 + f_3$ for a function g, integrable on $\mathbf{R}$, such that $f_1(t) = \int_{-\infty}^t g(s)\, ds$, $t \in \mathbf{R}$, and a jump-function f_3 vanishing at $-\infty$. Then $\operatorname{var}(f) = \operatorname{var}(f_1) + \operatorname{var}(f_3)$. Moreover, there exist numbers c_j and intervals X_j, $j = 1, 2, 3, \ldots$, such that $f_3(t) = \sum_{j=1}^{\infty} c_j X_j(t)$, for every $t \in \mathbf{R}$, and $\operatorname{var}(f_3) = \sum_{j=1}^{\infty} |c_j|$. There also exist numbers b_j and bounded intervals Y_j, $j = 1, 2, \ldots$, such that, if $u_j(t) = \int_{-\infty}^t b_j Y_j(s)\, ds$, for every $t \in \mathbf{R}$ and $j = 1, 2, \ldots$, then

$$\sum_{j=1}^{\infty} \operatorname{var}(u_j) = \sum_{j=1}^{\infty} |u_j(\infty)| < \frac{3}{2} \int_{-\infty}^{\infty} |g(s)|\, ds = \frac{3}{2} \operatorname{var}(f_1)$$

and $f_1(t) = \sum_{j=1}^{\infty} u_j(t)$ for every $t \in \mathbf{R}$. Hence, by Lemma 15 and Proposition 4, $f \in \mathscr{L}(P_{\mathbf{R}}^p)$ and the inequality (16) holds.

This proposition points at the richness of the space $\mathscr{L}(P_{\mathbf{R}}^p)$. To be sure, this space also contains functions of bounded variation which do not vanish at $-\infty$ and many functions of unbounded variation. In fact, it also contains many functions of unbounded r-variation, for any $r > 1$, because already the characteristic functions of many sets from $\mathscr{T}_2$ are such. (In this context, see [7].) As $\mathscr{L}(P_{\mathbf{R}}^p) \subset \mathscr{M}^p(\mathbf{R})$, we have a large class of multiplier operators which are scalar in wider sense.

LEMMA 17. *Let r, α, β and b be real numbers such that $r \le \alpha < \beta \le r + 2\pi$. Let u be the function on $\mathbf{T}$ such that $u(\exp ti) = 0$ for $r \le t < \alpha$, $u(\exp ti) = b(t-\alpha)$ for $\alpha \le t < \beta$, and $u(\exp ti) = b(\beta - \alpha)$ for $\beta \le t < r + 2\pi$. Then there exist numbers c_j and sets $X_j \in \mathscr{K}_2(\mathbf{T})$, $j = 0, 1, 2, \dots$, such that*

$$\sum_{j=0}^{\infty} |c_j| \, \|P^p_{\mathbf{T}}(X_j)\| \le C_p^2 \operatorname{var}(u)$$

and

$$\sum_{j=1}^{\infty} c_j X_j(z) = u(z)$$

for every $z \in \mathbf{T}$.

PROOF. Let m be the largest integer such that $m(\beta - \alpha) \le 2\pi$. Let $\gamma = \alpha + 2\pi m^{-1}$. Note that $\operatorname{var}(u) = 2|b|(\beta - \alpha)$, $r \le \alpha < \beta \le \gamma \le r + 2\pi$ and $m(\gamma - \alpha) = 2\pi$. Hence, by Lemma 13, it suffices to take $c_0 = b(\beta - \alpha)$, $X_0 = \{\exp ti \colon \beta \le t < r + 2\pi\}$, $c_j = 2^{1-j}\pi b m^{-1}$ and

$$X_j = \{\exp(\gamma - t)i \colon \exp 2^{j-1} m t i \in \{\exp si \colon 0 < s \le \pi\}\} \cap \{\exp ti \colon \alpha \le t < \beta\}$$

for $j = 1, 2, \dots$.

PROPOSITION 18. *Let $r \in \mathbf{R}$ and let f be a real non-singular function of bounded variation on $\mathbf{T}$ such that $f(\exp ri) = 0$. Then $f \in \mathscr{L}(P^p_{\mathbf{T}})$ and*

$$P^p_{\mathbf{T}}(f) \le 2C_p^2 \operatorname{var}(f)$$

for every $p \in (1, \infty)$.

PROOF. It is analogous to that of Proposition 16 just Lemma 17 is used instead of Lemma 15.

COROLLARY 19. *Let $x \in G$, let u be a non-singular function of bounded variation on $\mathbf{T}$ and let $f(\xi) = u(\langle x, \xi \rangle)$, for every $\xi \in \Gamma$. Then $f \in \mathscr{L}(P^p_{\Gamma})$ for every $p \in (1, \infty)$.*

PROOF. A power of a character of a group is a character and all characters of $\mathbf{T}$ are powers of a single one, namely the identity function on $\mathbf{T}$. Interpreting G as the group of characters of Γ we see immediately that, for every $Y \in \mathscr{K}_n(\mathbf{T})$, the set $X = \{\xi \in \Gamma \colon \langle x, \xi \rangle \in Y\}$ belongs to $\mathscr{K}_n(\Gamma)$, $n = 1, 2, \dots$. So, Lemma 13 and Proposition 18 imply the result.

Now, each element, x, of the group G is interpreted as a function on Γ—the character it generates—that is, the function $\xi \mapsto \langle x, \xi \rangle$, $\xi \in \Gamma$. Then $x \in \mathscr{M}^p(\Gamma)$

and T_x is the operator of translation by x. By Corollary 19, $x \in \mathscr{L}(P^p_\Gamma)$ and

$$T_x = \int_\Gamma \langle x, \xi \rangle P^p_\Gamma(d\xi), \tag{17}$$

for every $x \in G$. For $p = 2$, this is an instance of Stone's theorem (see e.g. [9], 36E).

7

Some observations about the Stone formula (17) could be of interest because they could possibly have somewhat wider implications.

First, its proof shows that, for every $x \in G$, there exist numbers c_j and set $X_j \in \mathscr{K}_2(\Gamma)$, $j = 1, 2, \ldots$, which depend of course on x but not on p, such that

$$\sum_{j=1}^{\infty} |c_j| \| P^p_\Gamma(X_j) \| < \infty,$$

the equality

$$\langle x, \xi \rangle = \sum_{j=1}^{\infty} c_j X_j(\xi)$$

holds for every $\xi \in \Gamma$ and

$$T_x = \sum_{j=1}^{\infty} c_j P^p_\Gamma(X_j),$$

for every $p \in (1, \infty)$. Hence for each $p \in (1, \infty)$, the translation operator, T_x, is expressed as the sum of the same multiples of the projections $P^p_\Gamma(X_j)$, $j = 1, 2, \ldots$. These projections too are "the same" for each p, only the space, $E = L^p(G)$, in which they operate varies with p.

Also the fact that the sets X_j, $j = 1, 2, \ldots$, belong to the class $\mathscr{K}_2(\Gamma)$ may possibly be worth noting. The algebra $\mathscr{R}^p(\Gamma)$ contains of course also sets of much greater complexity than those belonging to $\mathscr{K}_2(\Gamma)$. It seems that it would contribute considerably to our understanding of multiplier operators to know what kind of sets, besides those belonging to the classes $\mathscr{K}_n(\Gamma)$, $n = 1, 2, \ldots$, are in the algebra $\mathscr{R}^p(\Gamma)$. The classes $\mathscr{T}_n$, $n = 1, 2, \ldots$, give us some indication in the case $\Gamma = \mathbf{R}$.

One can produce many multiplier operators by integration with respect to P^p_Γ. In fact, because $\mathscr{L}(P^p_\Gamma) \subset \mathscr{M}^p(\Gamma)$, each operator T_f with $f \in \mathscr{L}(P^p_\Gamma)$ is a multiplier operator which is scalar in wider sense. So, naturally, the question arises whether $\mathscr{L}(P^p_\Gamma) = \mathscr{M}^p(\Gamma)$. That is to say, whether each multiplier operator is scalar in wider sense. Or, otherwise expressed, whether each multiplier operator belongs to the Banach algebra generated by the idempotent multiplier operators (Proposition 5).

Then there is a question concerning the general theory of scalar operators. Any finite-dimensional scalar operator, T, can be expressed in the form

$$T = \sum_{j=1}^{n} c_j P_j,$$

where n is a positive integer, c_j are scalars and P_j, $j = 1, 2, \ldots, n$, pair-wise disjoint projections. If we admit $n = \infty$, then every compact scalar operator can be expressed in this form. Moreover, an arbitrary operator T which is scalar in the sense of Dunford can too be so expressed but, besides allowing $n = \infty$, the projections P_j, $j = 1, 2, \ldots$, may be not pair-wise disjoint although they belong to a Boolean σ-algebra of projection operators. As noted, also the translation operators, T_x, can be expressed in this form with the projections P_j, $j = 1, 2, \ldots$, belonging to a semisimple Boolean algebra of projections. Now, the question is whether there are operators scalar in wider sense which cannot be so expressed. That is, whether there are operators which can be approximated by linear combinations of projections from a semisimple Boolean algebra but cannot be expressed as the sum of a sequence of scalar multiples of projections belonging to this algebra.

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