

MONOTONIC NORMS IN ORDERED BANACH SPACES

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Abstract

Let B be an ordered Banach space with ordered Banach dual space. Let N denote the canonical half-norm. We give an alternative proof of the following theorem of Robinson and Yamamuro: the norm on B is α -monotone ($\alpha \geq 1$) if and only if for each f in B^* there exists $g \in B^*$ with $g \geq 0$, f and $\|g\| \leq \alpha N(f)$. We also establish a dual result characterizing α -monotonicity of B^* .

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Let $(B, B_+, \|\cdot\|)$ be an ordered Banach space with closed cone (thus $a \leq b$ in B means $b - a \in B_+$). Let B^* denote the Banach dual space, ordered by the dual cone $B_+^* = \{f \in B^*: f(a) \geq 0 \text{ for all } a \geq 0\}$. Let N denote the canonical half-norms in B or B^* . Thus

$$\begin{aligned} N(a) &:= \inf\{\|b\|: a \leq b \text{ in } B\} \quad (a \in B) \\ &= \sup\{f(a): f \in B_+^*, \|f\| \leq 1\} \end{aligned}$$

and

$$\begin{aligned} N(f) &:= \inf\{\|g\|: f \leq g \text{ in } B^*\} \quad (f \in B^*) \\ &= \sup\{f(a): a \in B_+, \|a\| \leq 1\}. \end{aligned}$$

The definition of N is given by W. Arendt, P. R. Chernoff and T. Kato in [1], and the characterizations of N are due to D. W. Robinson and S. Yamamuro [6, Theorems 2.1 and 3.5]. Theorem 1 below is essentially known:

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(a) It was proved by Robinson and Yamamuro [6, Theorem 3.8] in the special case when $\alpha = 1$.

(b) The equivalence (i) \leftrightarrow (iv) was proved also in [2, Proposition 5]. Our proof given below seems to be more direct.

THEOREM 1. *For $\alpha \geq 1$ the following conditions are equivalent:*

- (i) *The norm is α -monotone on B , that is, $0 \leq a \leq b$ in $B \rightarrow \|a\| \leq \alpha\|b\|$.*
- (ii) *$\|a\| \leq \alpha N(a) + (\alpha + 1)N(-a)$ for all a in B .*
- (iii) *For each f in B^* there exists $g \in B^*$ such that $0, f \leq g$ and $\|g\| \leq \alpha N(f)$.*
- (iv) *For each f in B^* there exists $g \in B^*$ such that $f, 0 \leq g$ and $\|g\| \leq \alpha\|f\|$.*

The verification for (i) \rightarrow (ii) is as in (1) \rightarrow (2) of Theorem 3.8 in [6]. Conversely, suppose (ii) holds and $0 \leq a \leq b$. Then $N(a) \leq \|b\|$ and $N(-a) = 0$ by definition of N , so

$$\|a\| \leq \alpha N(a) + (\alpha + 1)N(-a) \leq \alpha N(a) + 0 \leq \alpha\|b\|.$$

Therefore (i) \leftrightarrow (ii). That (iii) \rightarrow (iv) is trivial. Assuming (iv), let $f \in B^*$ with $\|f\| \leq 1$ and take g as in (iv). Then, whenever $0 \leq a \leq b$ in B , one has

$$f(a) \leq g(a) \leq g(b) \leq \alpha\|f\|\|b\| \leq \alpha\|b\|$$

and it follows from the Hahn-Banach theorem that $\|a\| \leq \alpha\|b\|$. Thus (iv) \rightarrow (i), and it remains to prove (i) \rightarrow (iii). We do this as in [4, Theorem 9.6]. Let $f \in B^*$. We define

$$\begin{aligned} q(a) &= \sup\{f(b) : 0 \leq b \leq a\} & (a \in B_+), \\ p(a) &= \alpha N(f)\|a\| & (a \in B). \end{aligned}$$

Then q is superlinear, p is sublinear and $q \leq p$ on B_+ . By Bonsall's generalization of the Hahn-Banach theorem (see [4, Theorem 1.15]), there exists a linear functional g on B such that $q \leq g$ on B_+ and $g \leq p$ on B . Then, for all $a \in B_+$,

$$0, f(a) \leq q(a) \leq g(a)$$

that is, $0, f \leq g$. Also, $g \leq p = \alpha N(f)\|\cdot\|$ on B so $\|g\| \leq \alpha N(f)$, proving (iii).

REMARK. We have not used the completeness of B , that is, Theorem 1 is valid for ordered normed spaces. However, for the following dual result of Theorem 1, the completeness will be essential. (Again, the equivalence (i) \leftrightarrow (iv) was known in the special case when $\alpha = 1$, see [2, Proposition 6].)

THEOREM 2. *For $\alpha \geq 1$ the following conditions are equivalent: (i) The norm is α -monotone on B^* .*

- (ii) *$\|f\| \leq \alpha N(f) + (\alpha + 1)N(-f)$.*
- (iii) *For each a in B and $\varepsilon > 0$ there exists $b \in B$ with $0, a \leq b$ and $\|b\| \leq \alpha N(a) + \varepsilon$.*

(iv) For each a in B and $\varepsilon > 0$ there exists $b \in B$ with $0, a \leq b$ and $\|b\| \leq \alpha\|a\| + \varepsilon$.

PROOF. We need only prove (i) \rightarrow (iii) as the other implications can be proved as in Theorem 1, or follow from Theorem 1 (applied to B^* instead of B). Let $a \in B$. Define

$$\begin{aligned} q(f) &= \sup\{g(a) : 0 \leq g \leq f \text{ in } B^*\} & (f \in B_+^*), \\ p(f) &= \alpha N(a)\|f\| & (f \in B^*). \end{aligned}$$

Then, as in the proof of [4, Theorem 9.7], we can verify that p and $-q$ are lower semi-continuous sublinear functionals (respectively on B^* and B_+^*) under the w^* -topology. By a result, dual to Bonsall's theorem (see [3, Theorem 3] or [4, Corollary 2.9]), for any $\varepsilon > 0$, there exists b in B such that $q(f) \leq f(b)$ and $g(b) \leq p(g) + \varepsilon\|g\|$ for all $f \in B_+^*$, and $g \in B^*$. Then $g(b) \leq \alpha N(a)\|g\| + \varepsilon\|g\|$ and it follows from the Hahn-Banach theorem that $\|b\| \leq \alpha N(a) + \varepsilon$. Also, for all $f \in B_+^*$, $0, f(a) \leq q(f) \leq f(b)$; since B_+ is assumed to be closed it follows that $0, a \leq b$.

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