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HARDY-LITTLEWOOD MAXIMAL FUNCTIONS ON SOME SOLVABLE LIE GROUPS

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Abstract

Let N be a nilpotent simply connected Lie group, and A a commutative connected d-dimensional Lie group of automorphisms of N which correspond to semisimple endomorphisms of the Lie algebra of N with positive eigenvalues. Form the split extension $S = N \times A \cong N \times \mathbf{a}$, $\mathbf{a}$ being the Lie algebra of A. We consider a family of "rectangles" $\mathcal{B}_r$ in S, parameterized by $r > 0$, such that the measure of $\mathcal{B}_r$ behaves asymptotically as a fixed power of r. One can construct the Hardy-Littlewood maximal function operator $f \to \mathcal{M}_f$ relative to left translates of the family $\{\mathcal{B}_r\}$. We prove that $\mathcal{M}$ is of weak type $(1,1)$. This complements a result of J.-O. Strömberg concerning maximal functions defined relative to hyperbolic balls in a symmetric space.

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Let G be a semi-simple connected non-compact Lie group with finite center and let $G = NAK$ be the Iwasawa decomposition of G. Let $S = G/K$ be the non-compact symmetric space. NA acts on S simply transitively and so there is a natural identification of the group NA (the group of translations of S) and S. We write

$$S = NA.$$

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The G-invariant metric ρ on S is thus a left-invariant metric on S and the G-invariant measure on S is the left invariant Haar measure μ_l on S. In this setting a theorem of J.-O. Strömberg [2] reads

THEOREM. *Let*
$$\mathbf{B}_r = \{s \in S : \rho(s, e) \le r\}.$$
The maximal function Mf *defined by*
$$Mf(s) = \sup_{r>0} \mu_l(\mathbf{B}_r)^{-1} \int_{s\mathbf{B}_r} f(s')\, d\mu_l(s')$$
is of weak type $(1,1)$.

The aim of this note is to show that a similar theorem is true for other families of balls $\{B_r\}_{r>0}$ on S (not K-invariant any more) and as a matter of fact, the proof is very easy and straightforward. As a simple calculation shows, the balls we consider and the balls with respect to the hyperbolic metric on the upper half-plane (identified with the '$ax + b$'-group as above) are not comparable in measure, so Strömberg's result and ours are not simple consequences of each other.

The setting of our theorem is as follows.

Let N be a nilpotent simply connected Lie group. Let A be a commutative connected d-dimensional Lie group of automorphisms of N which (as linear transformations on $\mathbf{n}$) are semi-simple with positive eigenvalues. We write $A = \{e^t : t \in \mathbf{a}\}$, $N \ni x \to e^t x \in N$ being the action of A on N. We then have $e^t \cdot e^{t'} x = e^{t+t'} x$.

Let
$$N \ni x \to |x| \in \mathbf{R}^+$$
be a continuous function on N with the property that for some positive constants c, C', Q
$$Cr^Q \le \text{measure}\{x : |x| \le r\} \le c' r^Q$$
for all $r > 0$. For $t \in \mathbf{a}$ let $|t|$ = norm of the operator t (acting on $\mathbf{n}$). We form the split extension of N by A:
$$S = NA = N \times \mathbf{a}$$
the multiplication being
$$(x, t)(x', t') = (x + e^{-t} x', t + t').$$
Then the left and right invariant Haar measures on S are,
$$d\mu_l(x, t) = e^{-\operatorname{Tr} t}\, dx\, dt,$$
$$d\mu_r(x, t) = dx\, dt,$$
respectively.

THEOREM. *Let*

$$B_r = \{s = (x,t) \colon |x| \le r, |t| \le r\}.$$

The maximal function $\mathcal{M}f$ *defined by*

$$\mathcal{M}f = \sup_{r>0} \mu_l(B_r)^{-1} \int_{sB_r} f(s')\, d\mu_l(s')$$

is of weak type $(1,1)$.

The proof follows [2] but is much simpler. In fact the theorem is an immediate consequence of the following two propositions.

Let

$$\mathcal{M}_0 f(s) = \sup_{r\le 1} \mu_l(B_r)^{-1} \int_{sB_r} f(s')\, d\mu_l(s'),$$

$$\mathcal{M}_\infty f(s) = \sup_{r\ge 1} \mu_l(B_r)^{-1} \int_{sB_r} f(s')\, d\mu_l(s').$$

PROPOSITION 1. $\mathcal{M}_0$ *is of weak type* $(1,1)$.

PROPOSITION 2. $\mathcal{M}_\infty f(s) \le |f| + \check{f}(s)$, *where* $\check{f} \in L^1(S, \mu_l)$.

Proposition 1 follows from the following two easy lemmas.

LEMMA 1. *Let* $E \subset S$ *and* $\mu_l(E) < +\infty$. *Suppose*

$$E \subset \bigcup_{s\in\sigma} sB_{r(s)}, \qquad r(s) \le 1.$$

Then there exists a subset $\{s_1, s_2, \dots\}$ *of* σ *such that if* $B_{r_j} = B_{r(s_j)}$, *then*

$$s_i B_{r_i} \cap s_j B_{r_j} = \varnothing \quad \textit{for } i \ne j$$

and

$$E \subset \bigcup_j s_j B_{r_j} B_{2r_j}^{-1} B_{2r_j}.$$

The proof is standard.

LEMMA 2. *There is a constant* C *such that*

$$\mu_l(B_r B_{2r}^{-1} B_{2r}) \le C \mu_l(B_r)$$

for all $r \le 1$.

PROOF OF PROPOSITION 2. We have

$$(1) \qquad \mu_l(B_r) \ge C r^Q \int_{|t|\le r} e^{-\operatorname{Tr} t}\, dt = C r^Q (\operatorname{sh} r)^d.$$

Following J.-L. Clerc and E. M. Stein [1], see also [1], we get

$$\varphi(x,t)=\begin{cases}\mu_l(B_{|x|})^{-1} & \text{if } |x|\geq \max\{|t|,1\},\\ \mu_l(B_{|t|})^{-1} & \text{if } |t|\geq \max\{|x|,1\},\\ 1 & \text{otherwise.}\end{cases}$$

and we note that for a constant C

$$\mu_l(B_r)^{-1}\chi_{B_r}(x,t)\leq C\varphi(x,t) \tag{2}$$

for all $r\geq 1$, where χ_E denotes the indicator function of E. In fact, it suffices to verify (2) for

$$r_0=\min\{r\colon (x,t)\in B_r\}$$

and for r_0 (2) is obvious.

By (1), we have

$$\varphi(x,t)\leq C(1+|x|^Q(sh|x|)^d+|t|^Q(sh|t|)^d)^{-1}=\tau(x,t).$$

Consequently, by (2),

$$\begin{aligned}\mathcal{M}_\infty f(s)&=\sup_{r\geq 1}\mu_l(B_r)^{-1}\int \chi_{sB_r}(s')f(s')\,d\mu_l(s')\\ &=\sup_{r\geq 1}\int \mu_l(B_r)^{-1}\chi_{B_r}(s'^{-1}s)f(s')\,d\mu_l(s')\\ &\leq |f|*\check{\tau}(x).\end{aligned}$$

But clearly

$$\int (1+|x|^Q(sh|x|)^d+|t|^Q(sh|t|)^d)^{-1}\,dt\,dx<+\infty$$

i.e.

$$\tau\in L^1(S,\mu_r)$$

whence $\check{\tau}\in L^1(S,\mu_l)$, and the proof is complete.

REMARK. It would perhaps be interesting to know whether a similar result holds for riemannian balls with respect to some left-invariant riemannian metric on a solvable Lie group S.

References

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