

ON THE INTEGRAL MODULUS OF CONTINUITY OF FOURIER SERIES

BABU RAM and SURESH KUMARI

(Received 23 July 1986)

Communicated by W. Moran

Abstract

For a wide class of sine trigonometric series we obtain an estimate for the integral modulus of continuity.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 42 A 32.

1. Introduction

Let $F(x)$ be a function of period 2π in L_p ($1 \leq p < \infty$). Then the integral modulus of continuity of order k of F in L_p is defined by

$$\omega_p^k(h; F) = \sup_{0 < |t| \leq h} \|\Delta_t^k F(x)\|_{L_p},$$

where

$$\Delta_t^k F(x) = \sum_{\alpha=0}^k (-1)^{k-\alpha} \binom{k}{\alpha} F(x + \alpha t)$$

and $\|\cdot\|_{L_p}$ denotes the norm in L_p .

Concerning the integral modulus of continuity of order 1 of a sine series whose coefficients form a quasiconvex null sequence, Izumi [2] and Teljakovskii [5] have obtained some interesting estimates. The class of quasiconvex null sequence has further been extended by Teljakovskii [6] in the following form.

Let

$$(1.1) \quad \sum_{k=1}^{\infty} a_k \sin kx$$

be a sine series satisfying $a_k = o(1)$, $k \rightarrow \infty$. If there exists a sequence $\langle A_k \rangle$ such that

$$(1.2) \quad A_k \downarrow 0, \quad k \rightarrow \infty,$$

$$(1.3) \quad \sum_{k=0}^{\infty} A_k < \infty,$$

$$(1.4) \quad |a_k - a_{k+1}| = |\Delta a_k| \leq A_k \quad \text{for all } k,$$

then we say that (1.1) belongs to the class S .

Setting $A_k = \sum_{m=k}^{\infty} |\Delta^2 a_m|$, we observe that every quasiconvex null sequence satisfies the condition S .

Let $g(x)$ be the sum of the sine series (1.1) belonging to the class S . Teljakovskii [6] showed that the condition

$$(1.5) \quad \sum_{k=1}^{\infty} \frac{|a_k|}{k} < \infty$$

is sufficient for the integration of the series (1.1) belonging to the class S .

The aim of this paper is to find an estimate for the integral modulus of continuity of order k of the series (1.1) belonging to the class S .

2. Results

We establish the following

THEOREM. *If (1.1) belongs to the class S and (1.5) holds, then*

$$\begin{aligned} \omega_1^k\left(\frac{1}{n}; g\right) &\leq B_k n^{-k} \log n \sum_{v=1}^n (v+1)^{k+1} \Delta A_v \\ &\quad + B_k \sum_{v=n+1}^{\infty} (v+1) \left(1 + \log \frac{v}{n}\right) \Delta A_v, \end{aligned}$$

where B_k is a constant depending upon k and not necessarily the same at each occurrence.

Letting $A_v = \sum_{m=v}^{\infty} |\Delta^2 a_m|$, the case $k = 1$ of our theorem yields

COROLLARY. If $\langle a_k \rangle$ is quasiconvex null sequence satisfying (1.5), then

$$\begin{aligned} \omega_1\left(\frac{1}{n}; g\right) &\leq B n^{-1} \log n \sum_{v=1}^n (v+1)^2 |\Delta^2 a_v| \\ &\quad + B \sum_{v=n+1}^{\infty} (v+1) \left(1 + \log \frac{v}{n}\right) |\Delta^2 a_v|. \end{aligned}$$

This result corresponds to a theorem of Izumi [2] as stated in Teljakovskii [5].

3. Proof of the theorem

Under the assumed hypothesis, g is integrable. Since the symmetry of the function implies $|\Delta_t^k g(-x)| = |\Delta_{-t}^k g(x)|$, therefore

$$\int_{-\pi}^{\pi} |\Delta_t^k g(x)| dx = \int_0^{\pi} |\Delta_{-t}^k g(x)| dx + \int_0^{\pi} |\Delta_t^k g(x)| dx.$$

Hence, to prove the theorem, it is sufficient to evaluate

$$\int_0^{\pi} |\Delta_{\pm t}^k g(x)| dx, \quad \text{for } 0 < t \leq \pi/n.$$

We write

$$\begin{aligned} (3.1) \quad \int_0^{\pi} |\Delta_{\pm t}^k g(x)| dx &= \int_0^{(k+1)\pi/n} + \int_{(k+1)\pi/n}^{\pi} \\ &= I_1 + I_2, \text{ say.} \end{aligned}$$

We first estimate I_1 . Denoting by $\tilde{D}_v(x)$ the kernel conjugate to the Dirichlet kernel, the use of partial summation yields

$$\begin{aligned} g(x) &= \sum_{v=1}^{\infty} \Delta a_v \tilde{D}_v(x) \\ &= \sum_{v=1}^{\infty} A_v \frac{\Delta a_v}{A_v} \tilde{D}_v(x) \\ &= \sum_{v=1}^{\infty} \Delta A_v \sum_{i=0}^v \frac{\Delta a_i}{A_i} \tilde{D}_i(x). \end{aligned}$$

Then

$$\begin{aligned} I_1 &\leq \sum_{v=1}^n \left[\Delta A_v \int_0^{(k+1)\pi/n} \sum_{i=0}^v |\Delta_{\pm t}^k \tilde{D}_i(x)| dx \right] \\ &\quad + \int_0^{(k+1)\pi/n} \left| \Delta_{\pm t}^k \sum_{v=n+1}^{\infty} \Delta A_v \sum_{i=0}^v \tilde{D}_i(x) \right| dx \\ &= I_{11} + I_{12}. \end{aligned}$$

If $\tilde{D}_i^{(k)}(x)$ denotes the k th derivative of $\tilde{D}_i(x)$, then to estimate I_{11} we use the equality (Aljančić [1], Ram [3])

$$(3.2) \quad |\tilde{D}_i^{(k)}(x)| = \begin{cases} B_k i^{k+1}, & 0 \leq x \leq \pi \\ B_k i^k x^{-1}, & 0 < x \leq \pi \end{cases} \quad (k = 1, 2, \dots)$$

and obtain

$$\begin{aligned} I_{11} &\leq B_k t^k \sum_{v=1}^n \Delta A_v \int_0^{(k+1)\pi/n} \left(\sum_{i=0}^v |\tilde{D}_i^{(k)}(x \pm \theta_i t)| \right) dx \\ &\quad (0 < \theta_i < k) \\ &\leq B_k n^{-k} \sum_{v=1}^n \Delta A_v (v+1)^{k+1}. \end{aligned}$$

To estimate I_{12} , we use the inequality (Timan [7])

$$\frac{1}{\pi} \int_0^{c/n} |\tilde{D}_v(x)| dx \leq \frac{2}{\pi} \log \frac{v}{n} + o(1), \quad c > 0, v \geq n,$$

and obtain

$$\begin{aligned} I_{12} &\leq B_k \left(\sum_{v=n+1}^{\infty} \Delta A_v \sum_{i=2}^v \left[\log \frac{i}{n} + o(1) \right] \right) \\ &= B_k \left(\sum_{v=n+1}^{\infty} \Delta A_v \left[(v+1) \log \frac{v}{n} + (v+1) \right] \right). \end{aligned}$$

It follows therefore that

$$\begin{aligned} (3.3) \quad I_1 &\leq B_k n^{-k} \sum_{v=1}^n (v+1)^{k+1} \Delta A_v \\ &\quad + B_k \left[\sum_{v=n+1}^{\infty} (v+1) \left(1 + \log \frac{v}{n} \right) \Delta A_v \right]. \end{aligned}$$

To estimate I_2 , we have

$$\begin{aligned} I_2 &= \int_{(k+1)\pi/n}^{\pi} \left| \Delta_{\pm t}^k g(x) \right| dx \\ &\leq \int_{(k+1)\pi/n}^{\pi} \left| \sum_{v=1}^n \Delta a_v \Delta_{\pm t}^k \tilde{D}_v(x) \right| dx \\ &\quad + \int_{(k+1)\pi/n}^{\pi} \left| \Delta_{\pm t}^k \sum_{v=n+1}^{\infty} \Delta a_v \tilde{D}_v(x) \right| dx \\ &= I_{21} + I_{22}. \end{aligned}$$

We now write

$$I_{21} \leq \sum_{m=1}^{n-1} \int_{(k+1)\pi/(m+1)}^{(k+1)\pi/m} \left| \sum_{v=1}^n \Delta a_v \Delta_{\pm t}^k \tilde{D}_v(x) \right| dx.$$

By virtue of $t \leq \pi/n$ and $x \geq (k+1)\pi/(m+1)$, it follows that

$$x - kt \geq \frac{k+1}{m+1} \pi - \frac{k}{n} \pi = \frac{\pi}{m+1} + k\pi \left(\frac{1}{m+1} - \frac{1}{n} \right) \geq \frac{\pi}{m+1}.$$

Therefore in the subinterval $[(k+1)\pi/(m+1), (k+1)\pi/m]$, using (3.2), we have

$$\begin{aligned} \left| \sum_{v=1}^n \Delta a_v \Delta_{\pm t}^k \tilde{D}_v(x) \right| &\leq B_k t^k \sum_{v=1}^n \left| A_v \frac{\Delta a_v}{A_v} \right| \max_{x-kt \leq \xi \leq x+kt} |\tilde{D}_v^{(k)}(\xi)| \\ &\leq B_k t^k \sum_{v=1}^m v^{k+1} \left| A_v \frac{\Delta a_v}{A_v} \right| + \frac{B_k t^k}{x-kt} \sum_{v=m+1}^n v^k \left| A_v \frac{\Delta a_v}{A_v} \right| \\ &\leq B_k t^k \sum_{v=1}^m v^{k+1} A_v + B_k t^k m \sum_{v=m+1}^n v^k A_v. \end{aligned}$$

But

$$\begin{aligned} \sum_{v=1}^m v^{k+1} A_v &= \sum_{v=1}^m \Delta A_v \sum_{i=0}^v i^{k+1} + A_{m+1} \sum_{i=0}^m i^{k+1} \\ &\leq \sum_{v=1}^m (v+1)^{k+2} \Delta A_v + m^{k+2} A_{m+1}, \end{aligned}$$

and

$$\begin{aligned} \sum_{v=m+1}^n v^k A_v &= \sum_{v=m+1}^n \Delta A_v \sum_{i=0}^v i^k + A_{n+1} \sum_{i=0}^n i^k - A_{m+1} \sum_{i=0}^m i^k \\ &\leq \sum_{v=m+1}^n (v+1)^{k+1} \Delta A_v + n^{k+1} A_{n+1}. \end{aligned}$$

Therefore

$$\begin{aligned}
 I_{21} &\leq B_k n^{-k} \left[\sum_{m=1}^{n-1} m^{-2} \left(\sum_{v=1}^m (v+1)^{k+2} \Delta A_v + m^{k+2} A_{m+1} \right) \right] \\
 &\quad + B_k n^{-k} \left[\sum_{m=1}^{n-1} m^{-1} \left(\sum_{v=m+1}^n (v+1)^{k+1} \Delta A_v + n^{k+1} A_{n+1} \right) \right] \\
 &= B_k n^{-k} \left[\sum_{m=1}^{n-1} m^{-2} \sum_{v=1}^m (v+1)^{k+2} \Delta A_v + \sum_{m=1}^{n-1} m^k A_{m+1} \right. \\
 &\quad \left. + \sum_{m=1}^{n-1} m^{-1} \sum_{v=m+1}^n (v+1)^{k+1} \Delta A_v + \sum_{m=1}^{n-1} m^{-1} n^{k+1} A_{n+1} \right].
 \end{aligned}$$

The first term in the square bracket is

$$\begin{aligned}
 \sum_{v=1}^{n-1} (v+1)^{k+2} \Delta A_v \left(\sum_{m=v}^{n-1} m^{-2} \right) &\leq \sum_{v=1}^{n-1} (v+1)^{k+2} \Delta A_v \left(\sum_{m=v}^{\infty} m^{-2} \right) \\
 &\leq B_k \sum_{v=1}^{n-1} (v+1)^{k+1} \Delta A_v,
 \end{aligned}$$

the second term is

$$\begin{aligned}
 \sum_{m=1}^{n-1} m^k A_{m+1} &= \sum_{m=1}^{n-1} \Delta A_{m+1} \sum_{i=0}^m i^k + A_n \sum_{i=0}^n i^k \\
 &\leq \sum_{m=1}^{n-1} m^{k+1} \Delta A_{m+1} + n^{k+1} A_n,
 \end{aligned}$$

and the third term is

$$\begin{aligned}
 \sum_{m=1}^{n-1} m^{-1} \sum_{v=m+1}^n (v+1)^{k+1} \Delta A_v &= \sum_{v=1}^{n-1} (v+1)^{k+1} \Delta A_v \sum_{m=1}^{v-1} m^{-1} \\
 &\leq B_k \sum_{v=2}^{n-1} (v+1)^{k+1} \Delta A_v \log v \\
 &\leq B_k \log n \sum_{v=1}^{n-1} (v+1)^{k+1} \Delta A_v.
 \end{aligned}$$

Therefore

$$I_{21} \leq B_k n^{-k} \log n \sum_{v=1}^n (v+1)^{k+1} \Delta A_v.$$

Lastly, making use of Abel's transformation and Fomin's Lemma (Ram [4], Lemma 1), we have

$$\begin{aligned}
 I_{22} &\leq \sum_{\alpha=0}^k \binom{k}{\alpha} \int_{(k+1)\pi/n \pm \alpha t}^{\pi \pm \alpha t} \left| \sum_{v=n+1}^{\infty} \Delta a_v \tilde{D}_v(x) \right| dx \\
 &\leq B_k \int_{\pi/n}^{\pi+k\pi/n} \left| \sum_{v=n+1}^{\infty} \Delta a_v \tilde{D}_v(x) \right| dx \\
 &\leq B_k \int_{\pi/n}^{(k+1)\pi} \left| \sum_{v=n+1}^{\infty} A_v \frac{\Delta a_v}{A_v} \tilde{D}_v(x) \right| dx \\
 &= B_k \int_{\pi/n}^{(k+1)\pi} \left[\left| \sum_{v=n+1}^{\infty} \Delta A_v \sum_{i=0}^v \alpha_i \tilde{D}_i(x) \right| + A_{n+1} \sum_{i=0}^n \alpha_i \tilde{D}_i(x) \right] dx \quad \left(\alpha_i = \frac{\Delta a_i}{A_i} \right) \\
 &\leq B_k \left[\sum_{v=n+1}^{\infty} \Delta A_v \int_0^{(k+1)\pi} \left| \sum_{i=0}^v \alpha_i \tilde{D}_i(x) \right| dx + A_{n+1} \int_0^{(k+1)\pi} \left| \sum_{i=0}^n \alpha_i \tilde{D}_i(x) \right| dx \right] \\
 &\leq B_k \left[\sum_{v=n+1}^{\infty} (v+1) \Delta A_v + (n+1) A_{n+1} \right] \\
 &\leq B_k \sum_{v=n+1}^{\infty} (v+1) \Delta A_v.
 \end{aligned}$$

Hence

$$(3.4) \quad I_2 \leq B_k \left[n^{-k} \log n \sum_{v=1}^n (v+1)^{k+1} \Delta A_v + \sum_{v=n+1}^{\infty} (v+1) \Delta A_v \right].$$

The conclusion of the Theorem follows from (3.1), (3.3), and (3.4).

References

- [1] S. Aljančić, 'Sur le module des séries de Fourier particulières et sur le module des séries de Fourier transformées par des types divers', *Bull. Acad. Serbe Sci. Arts* **30** (6) (1967), 13–38.
- [2] M. Izumi, and S. Izumi, 'Modulus of continuity of functions defined by trigonometric series,' *J. Math. Anal. Appl.* **24** (1968), 564–581.
- [3] B. Ram, 'On the integral modulus of continuity of Fourier series', *J. Analyse Math.* **28** (1975), 78–85.
- [4] B. Ram, 'Convergence of certain cosine sums in the metric space L ', *Proc. Amer. Math. Soc.* **66** (1977), 258–260.

- [5] S. A. Teljakovskii, 'The integral modulus of continuity of functions with quasiconvex Fourier coefficients', *Sibirsk. Mat. Ž.* **11** (1970), 1140–1145.
- [6] S. A. Teljakovskii, 'A sufficient condition of Sidon for the integrability of trigonometric series', *Mat. Zametki* **14** (1973), 317–328.
- [7] A. F. Timan, *Theory of approximation of functions of real variables* (Hindustan Publishing Corporation, India, 1966).

Department of Mathematics
Maharshi Dayanand University
Rohtak-124001
India