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DUALITY PROPERTIES OF SPACES OF NON-ARCHIMEDEAN VALUED FUNCTIONS

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Abstract

Let $C(X, F)$ be the space of all continuous functions from the ultraregular compact Hausdorff space X into the separated locally K-convex space F; K is a complete, but not necessarily spherically complete, non-Archimedean valued field and $C(X, F)$ is provided with the topology of uniform convergence on X. We prove that $C(X, F)$ is K-barrelled (respectively K-quasibarrelled) if and only if F is K-barrelled (respectively K-quasibarrelled). This is not true in the case of $\mathbf{R}$ or $\mathbf{C}$-valued functions. No complete characterization of the K-bornological spaces $C(X, F)$ is obtained, but our results are, nevertheless, slightly better than the Archimedean ones. Finally, we introduce a notion of K-ultrabornological spaces for K non-spherically complete and use it to study K-ultrabornological spaces $C(X, F)$.

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1. Introduction

Let K be a complete, non-trivially non-Archimedean valued field. Let E be a separated locally convex space (s.l.c.s.) over K. Let E' be the topological dual of E (if K is not spherically complete, then E' may reduce to $\{0\}$: see [6, Théorème 2]).

For X an ultraregular compact Hausdorff space, Let $C(X, E)$ be the vector space of all continuous functions from X into E, provided with the topology of uniform convergence. Obviously, $C(X, E)$ is again a s.l.c.s. over K.

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We define K-quasibarrelled, K-bornological, K-barrelled and K-ultrabornological s.l.c.s. over K and investigate whether spaces of type $C(X, E)$ have such properties.

DEFINITION 1.1. A subset A of E is a *polar set* if it has the form $A = B^0$, where $B \subseteq E'$.

Equivalently, A is a polar set if and only if $A = A^{00}$, with polars being taken in the pairing $\langle E, E' \rangle$. Obviously, a polar set is K-convex in the sense of [8]. If K is spherically complete, then the polar sets are just the Γ-closed sets of [8]. Intersections of polar sets are again polar sets, and so also are the inverse images of polar sets under continuous linear transformations.

DEFINITION 1.2. E is *K-quasibarrelled* if every bornivorous polar set is a neighborhood.

Equivalently, E is K-quasibarrelled if and only if every family of continuous linear functionals on E which is bounded on bounded subsets of E is equicontinuous. An easy argument involving [8, Théorème 4.14 and Théorème 4.15] shows that, for K spherically complete, E is K-quasibarrelled if and only if every closed K-convex bornivorous set is a neighborhood. This justifies the terminology.

Let E'_b denote the dual of E, provided with the strong topology. Obviously, E is K-quasibarrelled if and only if every bounded set in E'_b is equicontinuous.

Complemented subspaces of K-quasibarrelled spaces are easily seen to be K-quasibarrelled; and so also is any space that contains a dense K-quasibarrelled subspace.

Ultrametrizable s.l.c. spaces are K-quasibarrelled.

DEFINITION 1.3. E is *K-barrelled* if every absorbing polar set is a neighborhood.

Obviously, E is K-barrelled if and only if every pointwise bounded family of continuous linear functionals on E is equicontinuous. If K is spherically complete, then E is K-barrelled if and only if every closed K-convex absorbing set is a neighborhood. This justifies the terminology.

Complemented subspaces of K-barrelled spaces are K-barrelled; and so also is any space that contains a dense K-barrelled subspace.

Complete ultrametrizable s.l.c.s. are K-barrelled.

A K-barrelled space is K-quasibarrelled.

DEFINITION 1.4. E is *K-bornological* if every K-convex bornivorous set is a neighborhood.

Obviously, E is K-bornological if and only if every linear transformation L: $E \to F$, where F is any other s.l.c.s., which is bounded on bounded sets is continuous.

Complemented subspaces of K-bornological spaces are K-bornological.

A K-bornological space is K-quasibarrelled.

DEFINITION 1.5. A subset $A \subseteq E$ is *compactoid* if, for every zero-neighborhood U in E, there is a finite set $X \subseteq E$ such that $A \subseteq \mathrm{Co}(X) + U$, where $\mathrm{Co}(X)$ is the closed K-convex hull of X [7, page 134]. A bounded K-convex subset A of E is *completing* if $([A], m_A)$ is a Banach space, where m_A is the Minkowski functional of A on the linear span $[A]$ of A. A subset A of E is *K-compact* if it is K-convex, compactoid and completing. E is *K-ultrabornological* if every K-convex set in E that absorbs all K-compact sets is a neighborhood.

By straightforward arguments, the image of a K-compact set under a continuous linear transformation is K-compact.

Every n.A. Banach space is K-ultrabornological since a zero-sequence in it is easily seen to be contained in a K-compact set.

Complemented subspaces of K-ultrabornological spaces are K-ultrabornological.

A K-ultrabornological space is both K-barrelled and K-bornological.

REMARK 1.6. In the definition of compactoid, it is often useful to know that the definition is independent of the linear subspace in which A lies. For normed spaces E, this follows frmo [7, Theorem 4.37]. The case of general locally convex spaces is easily reduced to the normed case.

2. Spaces of continuous vector-valued functions

For F a s.l.c.s. with Γ as system of seminorms, the tensor product $C(X) \otimes F$ is algebraically identified in the usual way with $\{f \in C(X; F): f(X)$ lies in a finite-dimensional subspace of $F\}$. The π-tensor product of two n.A. locally convex spaces is defined in [6]. In the case of spherically complete K, the next result is in [6]; we prove it for arbitrary K.

PROPOSITION 2.1. *If X is a compact T_2-space, F a s.l.c.s., and $p \in \Gamma$, then on $C(X) \otimes F$ the seminorms π_p and $\|\cdot\|_p$ coincide, where*

$$\pi_p(z) := \inf_{z=\sum_i \varphi_i \otimes y_i} \sup_i \|\varphi_i\|_\infty p(y_i),$$

$$\|z\|_p := \sup_{x \in X} p\Big(\sum_i \varphi_i(x) y_i\Big).$$

PROOF. If $z = \sum \varphi_i \otimes y_i$, then obviously $\|z\|_p := \sup_{x \in X} p(\sum_i \varphi_i(x) y_i) \leqslant \sup_{x \in X} \sup_i |\varphi_i(x)| p(y_i) \leqslant \sup_i \|\varphi_i\|_\infty p(y_i)$. So $\|z\|_p \leqslant \pi_p(z)$.

On the other hand, let $f \in C(X) \otimes F$ and let any t with $0 < t < 1$ be given. Let $[f(X)]$ be the linear span of $f(X)$ and let $[f(X)]_p$ be the quotient space of $[f(X)]$ modulo $\{y\colon p(y) = 0\}$. By [7, Theorem 3.15] we may find a t-orthogonal basis $\{y_1, y_2, \ldots, y_m\}$ of $[f(X)]_p$. Choose $y_{n+1}, \ldots, y_n$ in $[f(X)] \cap \{y\colon p(y) = 0\}$ such that $\{y_1, \ldots, y_n\}$ is a basis of $[f(X)]$. Let $\varphi_1, \ldots, \varphi_n \in C(X)$ be such that $f = \sum_i \varphi_i y_i$.

Then

$$\begin{aligned} \|f\|_p &= \sup_{x \in X} p(f(x)) = \sup_{x \in X} p\left(\sum_{i=1}^{n} \varphi(x) y_i\right) \\ &= \sup_{x \in X} p\left(\sum_{i=1}^{m} \varphi_i(x) y_i\right) \geqslant t \sup_{x \in X} \sup_{i} |\varphi_i(x)| p(y_i) \\ &\geqslant t \sup_i p(y_i) \sup_{x \in X} |\varphi_i(x)| \geqslant t \sup_i \|\varphi_i\|_\infty p(y_i) \geqslant t \pi_p(f). \end{aligned}$$

This holds for all $0 < t < 1$; hence $\|f\|_p \geqslant \pi_p(f)$.

By routine arguments one sees that $C(X) \otimes_\pi F$ is a dense subspace of $C(X, F)$. Moreover, every bounded subset of $C(X, F)$ is contained in the closure of a bounded subset of $C(X) \otimes_\pi F$. (Note that $(C(X) \otimes_\pi F) \cap C(X, B)$ is dense in $C(X, B)$ whenever B is bounded in F.) Also, F is clearly contained as a complemented subspace in $C(X) \otimes_\pi F$ as well as in $C(X, F)$.

An important special case of a space $C(X, F)$ is $c_0(F)$, the space of zero-sequences in F, which is isomorphic to $C(\mathbb{N}^*, F)$ where $\mathbb{N}^*$ is the one-point compactification of the natural numbers. Indeed, $c_0(F)$ is isomorphic to $c(F)$, the space of converging sequences in F.

REMARK 2.2. A survey of what is known about spaces of real valued or real vector space valued continuous functions can be found in [9] and [11].

REMARK 2.3. In studying $C(X, F)$ for compact Hausdorff X, there is no loss of generality if we assume X to be ultraregular; indeed, we may always replace the topology of X by the weak topology induced by the functions in $C(X, F)$ (and, if necessary, we can consider a Hausdorff quotient space).

3. K-quasibarrelled and K-barrelled spaces $C(X, F)$

The following result is basic.

PROPOSITION 3.1. *If E is a normed space and F a K-quasibarrelled l.c. space, then $E \otimes_\pi F$ is K-quasibarrelled.*

Proof. Let D be a bounded subset of $(E \otimes F)'_b$; let $b(E)$ and $b(E'_b)$ denote the unit balls in E and E'_b, respectively. If B is bounded in F, then $b(E) \otimes B$ is bounded in $E \otimes_\pi F$ and so $\sup\{|L(x \otimes y)|\colon L \in D,\ x \in b(E),\ y \in B\} < \infty$. For any $L \in (E \otimes F)'$ and $y \in F$, the map $E \to K$, $x \nrightarrow L(x \otimes y)$, belongs to E'_b and has norm $\leqslant p(y)$ if $L(x_0 \otimes y_0) \leqslant \|x_0\| p(y_0)$ for $x_0 \in E$, $y_0 \in F$. Hence the map $T\colon (E \otimes_\pi F)' \to (F, E'_b)$, $L \nrightarrow (y \nrightarrow (x \nrightarrow L(x \otimes y)))$, is well defined. Clearly, it is also one-to-one. Consider $Z = \bigcap_{L \in D} T(L)^{-1}(b(E'_b))$. Then Z is a bornivorous polar set in F and so a neighborhood. Since $\sup\{|L(x \otimes y)|\colon L \in D,\ x \in b(E),\ y \in Z\} < 1$, D is an equicontinuous subset of $(E \otimes_\pi F)'_b$. Hence $E \otimes_\pi F$ is K-quasibarrelled.

Proposition 3.2. *If F is a s.l.c.s. and X a nonempty compact T_2-space, then $C(X, F)$ is K-quasibarrelled if and only if F is K-quasibarrelled.*

Proof. The "if" part follows from Proposition 2.1, from Proposition 3.1, and from the fact that $C(X) \otimes_\pi F$ is dense in $C(X, F)$; the "only if" part is true because F is a complemented subspace of $C(X, F)$.

Remark 3.3. The Archimedean analogue of Proposition 3.2 does not hold, as was first remarked in [3] and [10].

Proposition 3.4. *If F is K-barrelled and X a nonempty compact T_2-space, then $C(X, F)$ is K-barrelled.*

Proof. By Proposition 3.2, $C(X, F)$ is K-quasibarrelled. Let $A \subseteq C(X, F)$ be a polar set that absorbs the points of $C(X, F)$. It is enough to prove that A absorbs the bounded sets of $C(X, F)$, i.e. the sets of type $C(X, B)$, where B is bounded and K-convex in F.

First we treat the case that X is ultrametrizable. Let B be bounded in F and assume that A does not absorb $C(X, B)$. For every clopen subset Y of X, define

$$C_Y(X, B) := \{f \in C(X, B)\colon f = 0 \text{ on } X \setminus Y\}.$$

Then it is easily seen that there exists a sequence $Y_1 \supseteq Y_2 \supseteq \cdots$ of clopen subsets of X such that A does not absorb any $C_{Y_n}(X, B)$, and such that $\operatorname{diam} Y_n \to 0$. Obviously $\bigcap_k Y_k$ contains just one point, say t.

We claim that A does not absorb any of the sets $\{f \in C_{Y_n}(X, B)\colon f(t) = 0\}$. Indeed, choose $\varphi \in C(X, K)$ such that $\varphi(t) = 1$, $|\varphi| \leqslant 1$ everywhere, and $\varphi \equiv 0$ outside Y_n. If A absorbs $\{f \in C_{Y_n}(X, B)\colon f(t) = 0\}$, then it absorbs $\{f - f(t)\varphi\colon f \in C_{Y_n}(X, B)\}$ as well as $\{f(t)\varphi\colon f \in C_{Y_n}(X, B)\}$, and so the whole of $C_{Y_n}(X, B)$.

Choose any $\lambda \in K$, $|\lambda| > 1$. By induction we may construct a sequence $(f_n)_n$ in $C(X, B)$ such that $f_n \notin \lambda^n A$ for all n, and such that $\{n\colon f_n(x) \neq 0\}$ is finite for all $x \in X$.

Now define Φ: $c_0 \to C(X, F)$, $(\lambda_i)_i \mapsto \Sigma_i \lambda_i f_i$. This definition is possible, and Φ maps the unit ball of c_0 into $C(X, B)$; so Φ is continuous.

The set $\Phi^{-1}(A)$ is a polar set and absorbs all points. Since c_0 is K-barrelled, there is a $\lambda_0 \in K \setminus \{0\}$ such that $(\lambda_i)_i \in \lambda_0 \Phi^{-1}(A)$ whenever $\|(\lambda_i)_i\|_\infty \leqslant 1$. In particular, $f_i \in \lambda_0 A$ for all i, which contradicts the fact that $f_n \notin \lambda^n A$ for all n.

Now let X be arbitrary. Since A is closed, by the remarks following Proposition 2.1 it is enough to show that A absorbs $(C(X) \otimes_\pi F) \cap C(X, B)$. Of course, it is also sufficient to prove that A absorbs any given sequence in $(C(X) \otimes_\pi F) \cap C(X, B)$. There is, however, an ultra-semimetrizable topology on X, weaker than the original topology, with respect to which all functions in such a sequence are continuous. By passing, if necessary, to a quotient space of X, we are reduced to the first case.

REMARK 3.5. The Archimedean analogue of Proposition 3.4 does not hold (again, see [3] and [10]). However, if $C(X, F)$ is quasibarrelled and F is barrelled, then $C(X, F)$ is barrelled in the Archimedean case [4].

4. K-bornological spaces $C(X, F)$

The following result is fundamental.

PROPOSITION 4.1. *If E is a normed space and F a K-bornological s.l.c.s., then $E \otimes_\pi F$ is K-bornological.*

PROOF. Let G be a l.c.s., and let L: $E \otimes_\pi F \to G$ be bounded on all bounded sets of $E \otimes_\pi F$. Define L: $F \to L(E,G)$, $y \mapsto (x \mapsto L(x \otimes y))$. If B is bounded in F, the $b(E) \otimes B$ is bounded in $E \otimes_\pi F$, and so $\sup\{ p(L(x \otimes y))$: $x \in b(E)$, $y \in B\} < \infty$ for every continuous seminorm p on G. Hence $\hat{L}$ is well-defined and maps bounded sets onto bounded sets. The fact that F is K-bornological implies that, for every neighborhood U in G, there is a neighborhood V in F such that $L(x \otimes y) \in U$ if $x \in b(E)$ and $y \in V$. So L is continuous.

COROLLARY 4.2. *If X is a compact Hausdorff space and F a K-bornological s.l.c. space, then $\{f \in C(X, F)$: $[f(X)]$ is finite-dimensional$\}$ is K-bornological.*

REMARK 4.3. If F is a locally K-convex space with the finest locally convex topology, then $C(X, F)$ is K-bornological for every compact Hausdorff space X (indeed, then $C(X) \otimes_\pi F = C(X, F)$). In the Archimedean case, even $c_0(F)$ will not be bornological if F has uncountable dimension [3, 10].

Lemma 4.4. *Let E, F be s.l.c.s. and E a bornological subspace of F. Assume that for every $y \in F$ there is a bornological s.l.c.s. G and a continuous linear Φ: $G \to F$, as well as a net $(x_i)_{i \in I}$ in G that converges to $x \in G$, such that $\Phi(x_i) \in E$ for all i, and such that $\Phi(x) = y$. Then F is bornological.*

Proof. Let H be another s.l.c.s. and L: $F \to H$ a linear transformation that is bounded on the bounded sets of F. Since E is K-bornological, there is for every continuous seminorm p on H a continuous seminorm q on F such that $p(Lx) \leqslant q(x)$ whenever $x \in E$. Let $y \in F$ be arbitrary and let g, Φ, $(x_i)_i$, and x be as the assumptions. Then $L \circ \Phi$ is continuous, since G is K-bornological. Hence $p(L(y)) = p(L \circ \Phi(x)) = \lim_i p(L \circ \Phi(x_i)) \leqslant \lim_i q(\Phi(x_i)) = q(y)$.

Proposition 4.5. *Let F be K-bornological and X any compact Hausdorff space. Assume that for every $f \in C(X, F)$ there is a s.l.c.s. F_f and a continuous linear function Φ_f: $F_f \to F$ with $C(X, F_f)$ bornological and with $f \in \Phi_f \circ C(X, F_f)$. Then $C(X, F)$ is bornological.*

Proof. This is an easy consequence of Corollary 4.2 and Lemma 4.4.

Proposition 4.6. *Let F be a sequentially complete s.l.c.s. If $c_0(F)$ is K-bornological, then so also is $C(X, F)$, provided that X is a compact Hausdorff space for which $f(X)$ is ultrametrizable for all $f \in C(X, F)$.*

Proof. By Corollary 4.2 and Lemma 4.4 it is enough to prove that every $f \in C(X, F)$ is contained in a bornological subspace G of $C(X, F)$ such that f belongs to the closure of $G \cap (C(X) \otimes_\pi F)$. Let $\mathscr{T}_f$ be the weak topology induced on X by f. We put $G = C((X, \mathscr{T}_f), F)$. It is enough to prove that G is K-bornological. By [7, Corollary 5.26], $C(X, \mathscr{T}_f)$ is isomorphic either to c_0 or to a finite product of K. In the first case, $c_0 \otimes_\pi F \cong C(X, \mathscr{T}_f) \otimes_\pi F$, so that $c_0(F) \cong G$ by taking sequential closures on both sides. In the second case, G is a finite product of F.

Remark 4.7. An Archimedean analogue of 4.5 is known [1] but requires an additional assumption, namely, that F is an inductive limit $F = \mathrm{ind}_\lambda F_\lambda$ of compactly regular type. No Archimedean analogue of Proposition 4.6 seems to be known. It is meant to cover both the (in itself rather trivial) case that F is ultrametrizable as well as the case that X is ultrametrizable. (The continuous image of a compact ultrametrizable space is compact and ultrametrizable.)

5. K-ultrabornological spaces $C(X, F)$

The next result is a non-Archimedean analogue of [2, Proposition 1.2]; it shows, incidentally, that our definition of K-ultrabornological spaces in Section 1 was a natural one.

PROPOSITION 5.1. *Let X be a nonempty compact Hausdorff space in which all points form G_δ-sets. Then $C(X, F)$ is K-ultrabornological if and only if F is K-ultrabornological and $C(X, F)$ is K-bornological.*

PROOF. Essentially, we have to prove that, for every K-ultrabornological space F, every K-convex subset A of $C(X, F)$ which absorbs all K-compact sets absorbs all bounded sets.

Assume that A does not absorb $C(X, B)$, where B is bounded in F. Since every point in X is a G_δ-set, and since c_0 is K-ultrabornological, an argument exactly like the one in the proof of Proposition 3.4 shows that every point $x \in X$ has a clopen neighborhood U_x such that A absorbs $\{f \in C(X, B)\colon f = 0 \text{ on } X \setminus U_x \text{ and } f(x) = 0\}$.

Define $U := \bigcup_{x \in X}(U_x \setminus \{x\})$ and let $S := X \setminus U$. Then U is open, and S is finite (since S has no accumulation point in X). If V is a clopen subset of X which is contained in U, then A absorbs $\{f \in C(X, B)\colon f = 0 \text{ on } X \setminus V\}$.

Now write $S = \{s_1, s_2, \ldots, s_n\}$. Choose $(\varphi_i)_{i=1}^n \in C(X, K)$ such that $|\varphi_i| \leqslant 1$, $\varphi_i(s_i) = 1$, and $\varphi_i = 0$ outside U'_{s_i}, where $(U'_{s_i})_{i=1}^n$ constitute clopen sets such that $U'_{s_i} \cap U'_{s_j} = \varnothing$ if $i \neq j$.

Since F^n is K-ultrabornological, there is a $\lambda \in K \setminus \{0\}$ such that A absorbs every function $\varphi_i y_i$ if $y_i \in B$. The argument can now be completed easily, since every $f \in C(X, B)$ may be written in the form

$$f = f\,|_{X \setminus \bigcup_i U'_{s_i}} + \sum_{i=1}^{n} \left(f\chi_{U'_{s_i}} - f(s_i)\varphi_i\right) + \sum_{i=1}^{n} f(s_i)\varphi_i.$$

COROLLARY 5.2. *$c_0(F)$ is K-ultrabornological if and only if F is K-ultrabornological and $c_0(F)$ is K-bornological.*

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