

INSCRIBED CENTERS, REFLEXIVITY, AND SOME APPLICATIONS

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Abstract

We first define an inscribed center of a bounded convex body in a normed linear space as the center of a largest open ball contained in it (when such a ball exists). We then show that completeness is a necessary condition for a normed linear space to admit inscribed centers. We show that every weakly compact convex body in a Banach space has at least one inscribed center, and that admitting inscribed centers is a necessary and sufficient condition for reflexivity. We finally apply the concept of inscribed center to prove a type of fixed point theorem and also deduce a proposition concerning so-called Klee caverns in Hilbert spaces.

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1. Definitions and terminology

Let X be a normed linear space and B a bounded convex body in X (that is, B has a non-empty interior in X). Let us define the (*nearest*) *distance* of each point $b \in B$ from the complement cB of B by

$$d(b, cB) = \inf_{x \in cB} \|b - x\|.$$

We define the *inscribed radius* of B by

$$\rho(B) = \sup_{b \in B} d(b, cB).$$

We shall also say that the bounded convex body B has an *inscribed center* if there exists some $c_0 \in B$ such that $d(c_0, cB) = \rho(B)$. When such a c_0 exists we call the open ball $B^0(c_0, \rho(B))$ an *inscribed ball* of B . If each bounded convex body in X has at least one inscribed center, we call X a *normed linear space admitting inscribed centers*.

2. Results

First of all, we point out that even in Euclidean spaces a bounded convex body may have lots of inscribed centers. For instance, in the Euclidean plane each point $(x, 0)$ with $-1 \leq x \leq 1$ is an inscribed center for the rectangle with vertices $(-2, 1)$, $(-2, -1)$, $(2, -1)$, and $(2, 1)$. Also, as an example to show that in general the notion of “Chebyshev center” for a bounded convex body (see [2] for the definition) differs from that of “inscribed center”, let us choose B to be the right triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$ in the Euclidean plane. Then clearly $(1, 2)$ is the Chebyshev center of B , whereas B has $(\sqrt{3} - 1, \sqrt{3} - 1)$ as its inscribed center.

The following theorem shows that completeness is a necessary condition for a normed linear space to admit inscribed centers.

THEOREM 1. *Let X be an incomplete normed linear space. Then X contains a closed convex body with no inscribed center.*

PROOF. We let D denote the virtual ball of radius 1 in X , as constructed in the proof of Theorem 1 in [2], that is

$$D = \{x \in X : r(x) \leq 1\},$$

where $r(x) = \lim_n \|x - a_n\|$, and where $\{a_n\}$ is a fixed non-convergent Cauchy sequence in X with $\lim_n \|a_n\| = 1$. Since D equals the intersection with X of the ball $B(\lim_n a_n, 1)$ in the completion of X , D is dense in $B(\lim_n a_n, 1)$. Therefore $\rho(D) = \rho(B(\lim_n a_n, 1))$, and the only possible inscribed center for D would be $\lim_n a_n$ (the inscribed center of $B(\lim_n a_n, 1)$), which is not in X . Hence X admits no inscribed center for D .

The above theorem confines our attention to Banach spaces for this study of admitting inscribed centers. The following theorem has the corollary that reflexivity is a sufficient condition for a Banach space to admit inscribed centers.

THEOREM 2. *Let B be a weakly compact convex body in a Banach space X . Then B has at least one inscribed center in X .*

PROOF. For sufficiently large n let us define $C_n \subset X$ as follows:

$$C_n = c \left[cB + \left(\rho(B) - \frac{1}{n} \right) U(X) \right],$$

where $U(X)$ denotes the closed unit ball of X . Since cB (the complement of B in X) is open, it follows that each C_n is closed, and since

$$(1) \quad x \in C_n \Leftrightarrow x + \left(\rho(B) - \frac{1}{n} \right) U(X) \subset B,$$

it follows that for each n we have $C_n \subset B$. On the other hand, for each n , we have

$$cB + \left(\rho(B) - \frac{1}{n} \right) U(B) \subset cB + \left(\rho(B) - \frac{1}{n+1} \right) U(X),$$

from which we deduce that $C_{n+1} \subset C_n$. That is, $\{C_n\}$ is a decreasing sequence. We now show that each C_n is convex. To this end, let $b_1, b_2 \in C_n$ and let $0 < t < 1$. By (1), for each $u \in U(X)$, we have

$$b_1 + \left(\rho(B) - \frac{1}{n} \right) u, \quad b_2 + \left(\rho(B) - \frac{1}{n} \right) u \in B.$$

Therefore, by the convexity of B , for each $u \in U(X)$, we have

$$\begin{aligned} tb_1 + (1-t)b_2 + \left(\rho(B) - \frac{1}{n} \right) u \\ = t \left[b_1 + \left(\rho(B) - \frac{1}{n} \right) u \right] + (1-t) \left[b_2 + \left(\rho(B) - \frac{1}{n} \right) u \right] \in B. \end{aligned}$$

Hence $tb_1 + (1-t)b_2 + (\rho(B) - 1/n)U(X) \subset B$, and, by (1), $tb_1 + (1-t)b_2 \in C_n$.

Now each C_n (being closed and convex) is weakly closed [3, Theorem 13, page 422]. Also, by the weak compactness of B and by the fact that $C_n \subset B$, it follows that each C_n is weakly compact. Since $\{C_n\}$ is a decreasing sequence of weakly compact subsets of B , we deduce from a theorem of Smullian (cf. [3, Theorem 2, page 433]) that $C = \bigcap_n C_n$ is non-empty. If now $c_0 \in C$, it follows from (1) that for each n we have

$$c_0 + \left(\rho(B) - \frac{1}{n} \right) U(X) \subset B.$$

Therefore each ball $B(c_0, \rho(B) - 1/n)$ is contained in B . Therefore

$$B^0(c_0, \rho(B)) = \bigcup_n B\left(c_0, \rho(B) - \frac{1}{n}\right) \subset B.$$

We deduce that $d(c_0, cB) \geq \rho(B)$. Hence $d(c_0, cB) = \rho(B)$, and the result follows.

COROLLARY 1. *The set consisting of all inscribed centers of a bounded convex body B in a normed linear space X is closed convex, and nowhere dense in X .*

PROOF. We may assume without loss of generality that B is closed. Then it is enough to observe that the closed convex subsets $C_n \subset B$ (and hence $C = \bigcap_n C_n$) in the proof of Theorem 2 may be constructed, even if X is an arbitrary normed linear space. Hence, with the convention that the empty set is convex, the set $C \subset B$ is closed and convex. The nowhere density of C is obvious; for otherwise C would contain a ball $B(c, \delta)$, and then we would have $B(c, \rho(B) + \delta) \subset B$, which is absurd.

Our next theorem shows that admitting inscribed centers characterises reflexivity. To prove this we shall use Theorem 2 above, a well known theorem of R. C. James in [4], and also the following lemma.

LEMMA 1. *Let X be a normed linear space and f a continuous linear functional of norm 1 on X . Then the inscribed radius of the (upper) half unit ball $B = \bigcup(X) \cap f^{-1}([0, \infty))$ equals $\frac{1}{2}$.*

PROOF. We first show that $\rho(B) \geq \frac{1}{2}$. To do so, let $\varepsilon > 0$ be arbitrary, and choose $z \in B$ such that $\|z\| = 1$ and such that $1 - \varepsilon < f(z) < 1$. Then

$$(1) \quad B\left(\frac{z}{2}, \frac{1}{2} - \frac{\varepsilon}{2}\right) \subset B\left(\frac{z}{2}, \frac{1}{2}\right) \subset U(X).$$

Also we have

$$(2) \quad B\left(\frac{z}{2}, \frac{1}{2} - \frac{\varepsilon}{2}\right) \subset f^{-1}([0, \infty)),$$

for if $y \in B\left(\frac{z}{2}, \frac{1}{2} - \frac{\varepsilon}{2}\right)$, then $f\left(\frac{z}{2}\right) - f(y) \leq \left\|\frac{z}{2} - y\right\| \leq \frac{1}{2} - \frac{\varepsilon}{2}$, and hence $f(y) \geq f\left(\frac{z}{2}\right) - \frac{1}{2} + \frac{\varepsilon}{2} > 0$. From (1) and (2) we get

$$B\left(\frac{z}{2}, \frac{1}{2} - \frac{\varepsilon}{2}\right) \subset B.$$

Now since $\varepsilon > 0$ was arbitrary, we deduce that $\rho(B) \geq \frac{1}{2}$. We end the proof of the lemma by showing that $\rho(B) > \frac{1}{2}$ is impossible. If $\rho(B) > \frac{1}{2}$, then there would exist $b \in B$ such that $d(b, cB) > \frac{1}{2}$, and hence for some $\alpha > 0$ we would have

$$(3) \quad \frac{1}{2} + \alpha < d(b, cB) = \inf_{z \in cB} \|b - z\| \leq \|b\|.$$

On the other hand (3) implies that

$$B(b, \tfrac{1}{2} + \alpha) \subset U(X).$$

Now, observing that

$$b' = \left(1 + \frac{1}{2\|b\|}\right)b \in B(b, \tfrac{1}{2} + \alpha) \subset U(X),$$

we get from (3) that

$$1 \geq \|b''\| = \|b\| + \tfrac{1}{2} > \tfrac{1}{2} + \alpha + \tfrac{1}{2} = 1 + \alpha.$$

This contradiction shows that we must have $\rho(B) = \tfrac{1}{2}$.

THEOREM 3. *For a Banach space X the following conditions are equivalent:*

- (i) X is reflexive;
- (ii) X admits inscribed centers.

PROOF. (i) \Rightarrow (ii). If X is reflexive, and if $B \subset X$ is a bounded convex body, then its norm closure \bar{B} is weakly closed. By reflexivity of X , \bar{B} is weakly compact [3, Corollary 8, page 425]. Hence by Theorem 2, \bar{B} , and therefore B , has an inscribed center in X .

(ii) \Rightarrow (i). To prove this we only need to show that every continuous linear functional f on X attains its supremum on $U(X)$ (cf. Theorem 5 in [4]). Let f be an arbitrary continuous linear functional on X . We assume without loss of generality that $\|f\| = 1$. Let $B = U(X) \cap f^{-1}([0, \infty))$ be the (upper) half unit ball determined by f . By Lemma 1, $\rho(B) = \tfrac{1}{2}$, and by hypothesis there exists $c \in B$ such that

$$d(c, cB) = \rho(B) = \tfrac{1}{2}.$$

We now claim that $\|c\| \leq \tfrac{1}{2}$. For otherwise (as in the proof of Lemma 1), the inequality $\tfrac{1}{2} < \|c\|$ together with the relationships

$$c'' = \left(1 + \frac{1}{2\|c\|}\right)c \in B(c, \tfrac{1}{2}) \subset U(X)$$

imply the following contradiction:

$$1 \geq \|c''\| = \|c\| + \tfrac{1}{2} > 1.$$

Hence $\|d\| \leq \tfrac{1}{2}$. On the other hand

$$\begin{aligned} \tfrac{1}{2} &= \rho(B) = d(c, cB) = \inf_{z \in cB} \|c - z\| \leq \inf\{\|c - z\| : z \in f^{-1}(\{0\}) \cap U(X)\} \\ &= f(c) \leq \|c\| \leq \tfrac{1}{2}. \end{aligned}$$

Therefore $f(c) = \|c\| = \frac{1}{2}$, and $f(2c) = \|2c\| = 1 = \|f\|$. We deduce that f attains its supremum on $U(X)$, and this completes the proof of the theorem.

COROLLARY 2. *In every non-reflexive Banach space X there exists a partition of the unit ball $U(X)$ into two half balls, neither of which contains a ball of radius $\frac{1}{2}$. These half balls are $B_1 = \bigcup (X) \cap f^{-1}([0, \infty))$ and $B_2 = -B_1$, where f is a continuous linear functional on X which does not attain its supremum on $U(X)$.*

EXAMPLE. In c_0 , the Banach space of all real sequences (x_n) converging to 0, the subsets $B_1 = \{(x_n): \|x_n\| \leq 1; 0 \leq \sum_{n=1}^{\infty} x_n/2^n\}$ and $B_2 = -B_1$ are two half balls which do not contain a largest ball (of radius $\frac{1}{2}$). This follows since the continuous linear functional f defined by $f(x_n) = \sum_{n=1}^{\infty} x_n/2^n$ on c_0 , does not attain its supremum on $U(c_0)$ (see [5, Example 18.8, page 173]).

3. Applications

In this section we point out two applications of the concept of inscribed centers. The first application is to deduce the following fixed point theorem. In this theorem $\text{Inscr}(B)$ denotes the set of all inscribed centers of a given bounded convex body.

THEOREM 4. *Let X be a normed linear space, and let $B \subset X$ be a bounded convex body with $\text{Inscr}(B) \neq \emptyset$. Let $K: B \rightarrow [1, \infty)$ be a given function, and assume that $T: B \rightarrow B$ is a map such that for each $x \in B$ and $y \in cB$ we have*

$$(1) \quad d(x, cB) \leq K(x) \|y - Tx\|.$$

Then T leaves $\text{Inscr}(B)$ invariant. In particular, if $\text{Inscr}(B)$ is a singleton, then its only member is a fixed point for T .

PROOF. We only need to prove the first assertion of the theorem. Let $z \in \text{Inscr}(B)$ be given. By (1), for each $y \in cB$ we have

$$d(z, cB) \leq K(z) \|y - Tz\|.$$

Taking the infimum over cB in the right side of this inequality and noting that $Tz \in B$, we get

$$\rho(B) = d(z, cB) \leq K(z) d(Tz, cB) \leq \rho(B).$$

Therefore $d(Tz, cB) = \rho(B)$ and $Tz \in \text{Inscr}(B)$. Hence the result follows.

We may recall that under the conditions of the above theorem the map T may not have a fixed point if $\text{Inscr}(B)$ contains more than one point. As an example,

let B be the rectangle with vertices $(-2, 1)$, $(-2, -1)$, $(2, -1)$, and $(2, 1)$ in the Euclidean plane. As we mentioned at the beginning of Section 2, $\text{Inscr}(B) = \{(a, 0) : -1 \leq a \leq 1\}$. If we consider the map $T: B \rightarrow B$ defined by $T(0, 0) = (1, 0)$, and $T(a, b) = (-a, b/2)$ for $(a, b) \neq (0, 0)$, then (for $K = 1$) T satisfies the conditions of Theorem 4 (since for each $(a, b) \in B$, $d((a, b), cB) \leq d(T(a, b), cB)$), while clearly T has no fixed point in B .

As our next application of the concept of inscribed center, we point out the following proposition concerning so-called Klee caverns in Hilbert spaces. Recall that a subset K of a normed linear space X is called *Chebyshev* if K admits a unique nearest point to each point of X . Chebyshev subsets of Hilbert spaces whose complements are bounded and convex have been called *Klee caverns* by Asplund in [1]. Asplund showed that Klee caverns exist, provided that non-convex Chebyshev sets exist (see [1, page 239]).

PROPOSITION 1. *If a Hilbert space H contains a non-convex Chebyshev subset, then H contains a Klee cavern whose complement has a unique inscribed center.*

PROOF. We adopt the notations and the details stated in [1, pages 238–239]. Thus, let K be a non-convex Chebyshev subset of H and let G be the subset (with the unique farthest point property) of H obtained from K by Ficken's method of inversion (see [1, page 238]). Let y denote the unique Chebyshev center of G . Then the subset $C = \{x \in H : t(x) \geq t(y) + 1\}$, where $t(x) = \sup_{z \in G} \|x - z\|$, is a Klee cavern. If b is the metric projection onto C , then for each $x \notin C$ the following equality holds:

$$t(x) + \|x - b(x)\| = t(y) + 1.$$

The above equation with its constant right hand side reveals that as $t(x)$ decreases to reach its greatest lower bound over cC (the complement of C), $\|x - b(x)\|$ increases to reach its least upper bound. Since the only point at which $t(x)$ takes its minimum is y , it follows that y is the unique point in cC for which $\|x - b(x)\|$ takes its maximum. Therefore y is at the same time the Chebyshev center of G and the unique inscribed center of cC , and the proposition follows.

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