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FREE PRODUCTS OF TOPOLOGICAL GROUPS WITH A CLOSED SUBGROUP AMALGAMATED

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Abstract

It is shown that if $\{G_n: n = 1, 2, \ldots\}$ is a countable family of Hausdorff k_ω-topological groups with a common closed subgroup A, then the topological amalgamated free product $*_A G_n$ exists and is a Hausdorff k_ω-topological group with each G_n as a closed subgroup. A consequence is the theorem of La Martin that epimorphisms in the category of k_ω-topological groups have dense image.

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1. Introduction

Let $\{G_n: n = 1, 2, \ldots\}$ be a countable family of k_ω-topological groups, each having a fixed topological group A as a closed subgroup. We show that $*_A G_n$, the free topological product of $\{G_n\}$ with A amalgamated, exists, is a (Hausdorff) k_ω-group, and contains G_n as a closed subgroup for each n.

Katz and Morris [2, 3, 4] have already shown that an amalgamated product $G *_A H$ of k_ω-groups is k_ω whenever A is in a class of closed subgroups, including those which are normal and those which are the product of a compact subgroup and a central subgroup. Our theorem clearly contains these results, and moreover yields another proof of La Martin's theorem that epimorphisms in the category of k_ω-groups have dense range ([6]; see also [9] and [11]).

The proof of our theorem has similarities to Ordman's proof [10] that the free k-group on a t_2 k-space exists and is a t_2 k-group, and more especially to the proof of Brown and Hardy [1] that the universal topological groupoid on a k_ω-groupoid exists and is k_ω.

2. The theorem

Recall that a Hausdorff space X is a *k_ω-space* if it has the weak topology with respect to some increasing sequence of compact subsets $X_1 \subseteq X_2 \subseteq \cdots$ with union X; then we say that $\bigcup X_n$ is a *k_ω-decomposition* of X. A topological group is a *k_ω-group* if as a topological space it is k_ω. The appendix of [1] contains a useful list of the properties of k_ω-spaces.

Let $\{G_\lambda: \lambda \in \Lambda\}$ be a family of topological groups. Then we say that $(A, \{i_\lambda\})$ is a *common subgroup* of the G_λ if A is a topological group and, for each $\lambda \in \Lambda$, i_λ is a topological isomorphism of A onto a subgroup of G_λ. We denote $i_\lambda(A)$ by A_λ, and the isomorphism $i_\mu i_\lambda^{-1}$: $A_\lambda \to A_\mu$, where λ, $\mu \in \Lambda$, is denoted by $i_{\lambda,\mu}$. The common subgroup is *closed* if A_λ is closed in G_λ for each λ.

The above isomorphisms, of course, simply serve to identify the various copies of A in the G_λ. In purely algebraic arguments involving the amalgamated product it is often convenient to suppress these maps, and to regard A as a subgroup of each G_λ (cf. Chapter III, 12 of [8]); this can be done with advantage in the lemma below. In topological arguments, on the other hand, it is desirable to use the maps explicitly.

DEFINITION (cf. [2, 3, 4]). Let $(A, \{i_\lambda\})$ be a common subgroup of the topological groups G_λ, $\lambda \in \Lambda$. A topological group $G = *_A G_\lambda$ is the *free product of* $\{G_\lambda\}$ *with A amalgamated* if

(i) G_λ is a topological subgroup of G for each λ,

(ii) $\bigcup_\lambda G_\lambda$ generates G algebraically, and

(iii) for any topological group H and any collection of continuous homomorphisms ϕ_λ: $G_\lambda \to H$ which agree on A (that is $\phi_\lambda i_\lambda = \phi_\mu i_\mu$ for all λ and μ), there exists a continuous homomorphism Φ: $G \to H$ which extends each ϕ_λ.

THEOREM. *If $(A, \{i_n\})$ is a common closed subgroup of the k_ω-groups G_n, $n \in \mathbf{N}$, then $*_A G_n$ exists and is a (Hausdorff) k_ω-group, with each G_n as a closed subgroup.*

Note that A is also necessarily a k_ω-group.

The proof of the theorem occupies almost the remainder of the paper.

Let $U = \sqcup_n G_n$ and $W = \sqcup_n U^n = \bigcup_n W_n$, where $W_n = \sqcup_{i=1}^n U^i$ (here $\sqcup$ denotes disjoint union (or the coproduct in the category of topological spaces), and U^n denotes the Cartesian product $U \times \cdots \times U$ of n copies of U). Clearly U and W are k_ω-spaces. Let G be the abstract amalgamated free product $*_A G_n$ of the G_n with the A_n amalgamated, and give G the quotient topology under the map $p\colon W \to G$ which sends $(g_1, \ldots, g_n)$ to the product of $g_1, \ldots, g_n$ in G. We shall show that G has all the properties required by the definition. The key to doing this is to show first that G has a (Hausdorff) k_ω-topology, and for this we need the definition and lemma below.

For convenience, first define $\Omega\colon U \to \mathbf{N}$ by setting $\Omega(g)$, for $g \in U$, equal to the (unique) $n \in \mathbf{N}$ for which $g \in G_n$.

Definition. An n-tuple $(g_1, \ldots, g_n) \in W$ is *reduced* if $g_j \in G_{\Omega(g_j)} \setminus A_{\Omega(g_j)}$, $j = 1, \ldots, n$, and if $\Omega(g_j) \neq \Omega(g_{j+1})$, $j = 1, \ldots, n-1$.

Lemma. *Let $(g_1, \ldots, g_n)$ and $(h_1, \ldots, h_m)$ be reduced elements of W. Then, writing $\omega(j) = \omega(g_j)$ for $j = 1, \ldots, n$, we have $p(g_1, \ldots, g_n) = p(h_1, \ldots, h_m)$ if and only if*

(i) $n = m$,

(ii) $\Omega(h_j) = \omega(j)$, $j = 1, \ldots, n$, *and*

(iii) $h_1^{-1} g_1 \in A_{\omega(1)}$,

$h_2^{-1} i_{\omega(1),\omega(2)}(h_1^{-1} g_1) g_2 \in A_{\omega(2)}$,

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$h_{n-1}^{-1} i_{\omega(n-2),\omega(n-1)}(h_{n-2}^{-1} \cdots g_{n-2}) g_{n-1} \in A_{\omega(n-1)}$, *and*

$h_n^{-1} i_{\omega(n-1),\omega(n)}(h_{n-1}^{-1} \cdots g_{n-1}) g_n = 1$.

Moreover, (i), (ii) *and* (iii) *together imply that $p(g_1, \ldots, g_n) = p(h_1, \ldots, h_m)$, whether or not $(g_1, \ldots, g_n)$ and $(h_1, \ldots, h_m)$ are reduced.*

Proof. Suppose $p(g_1, \ldots, g_n) = p(h_1, \ldots, h_m)$. By Chapter I of [8], we see that $p(g_1, \ldots, g_n)$ and $p(h_1, \ldots, h_m)$ have lengths n and m, respectively, in G, so that $n = m$, proving (i).

Let S_k ($k \in \mathbf{N}$) be a complete set of left coset representatives for A_k in G_k, with the representative of A_k always taken to be 1. Recall that for all $k, l \in \mathbf{N}$, $i_k(a)$ and $i_l(a)$ are identified as elements of G, for each $a \in A$. In the group $G_{\omega(1)}$, set

$$g_1 = s_1 a_1 \qquad (s_1 \in S_{\omega(1)} \setminus \{1\},\ a_1 \in A_{\omega(1)}), \tag{1}$$

and in the group $G_{\omega(j)}$, $j = 2, \ldots, n$, set

$$i_{\omega(j-1),\omega(j)}(a_{j-1}) g_j = s_j a_j \qquad (s_j \in S_{\omega(j)} \setminus \{1\},\ a_j \in A_{\omega(j)}). \tag{2}$$

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Then from the well-known algebraic structure of G [8], we see that, in the group G, $g_1 g_2 \cdots g_n = s_1 s_2 \cdots s_n a_n$, and that the latter is the (uniquely-defined) normal form of $g_1 g_2 \cdots g_n$. Computing the normal form of $h_1 h_2 \cdots h_n$ similarly, we see that (writing $\Omega'(j) = \Omega(h_j)$, $j = 1, \ldots, n$) we have

$$(3) \qquad h_1 = s_1' a_1' \qquad \left(s_1' \in S_{\omega'(1)} \backslash \{1\},\, a_1 \in A_{\omega'(1)}\right)$$

and, for $j = 2, \ldots, n$,

$$(4) \qquad i_{\omega'(j-1),\omega'(j)}\left(a_{j-1}'\right) h_j = s_j' a_j' \qquad \left(s_j' \in S_{\omega'(j)} \backslash \{1\},\, a_j' \in A_{\omega'(j)}\right),$$

so that $h_1 h_2 \cdots h_n$ has normal form $s_1' s_2' \cdots s_n' a_n'$. Since each element of G has a unique normal form, we must have $s_j = s_j'$, $j = 1, \ldots, n$, and $a_n = a_n'$, and so $\omega(j) = \omega'(j)$ for each j, proving (ii).

Combining (1) and (3) then shows that $h_1^{-1} g_1 = (a_1')^{-1} a_1 \in A_{\omega(1)}$, and from repeated combination of (2) and (4) it follows that $h_j^{-1} i_{\omega(j-1),\omega(j)}(h_{j-1}^{-1} \cdots g_{j-1}) g_j = (a_j')^{-1} a_j \in A_{\omega(j)}$, $j = 2, \ldots, n$. Thus (noting that $(a_n')^{-1} a_n = 1$) we see that (iii) is true.

The remainder of the proof of the lemma follows along similar lines, again using the normal form, and the details are left to the reader.

Proposition. *The graph Γ of the equivalence relation defined by p (that is, the set $\{(w, w') \in W \times W\colon p(w) = p(w')\}$) is closed in $W \times W$.*

Proof. Clearly $W \times W$ has the weak topology with respect to the sets $W_n \times W_n$, and it suffices to show that $\Gamma_n = \Gamma \cap (W_n \times W_n)$ is closed in $W_n \times W_n$ for each n. The proof is by induction on n. We point out that the proof will not make use of the fact that the G_n are k_ω; Hausdorffness is the only topological condition required.

Now $W_1 \times W_1 = U \times U = \bigsqcup_{j,k} G_j \times G_k$, and it is clear that

$$\Gamma \cap \left(G_j \times G_k\right) = \begin{cases} \left\{(g, g)\colon g \in G_j\right\}, & j = k, \\ \left\{\left(i_j(a), i_k(a)\right)\colon a \in A\right\}, & j \neq k, \end{cases}$$

which is closed for all j and k, as each G_i is Hausdorff and A_i is closed in each G_i. Hence Γ_1 is closed in $W_1 \times W_1$.

Suppose that Γ_{n-1} is closed in $W_{n-1} \times W_{n-1}$ for some $n \geqslant 2$. We proceed to show that Γ_n is closed in $W_n \times W_n$. This will be done by decomposing $W_n \times W_n$ into a disjoint union of smaller subspaces, and by showing that the intersection of Γ_n with each of these is closed. To this end, we introduce some definitions.

For $k_1, \ldots, k_n \in \mathbf{N}$ define $K(k_1, \ldots, k_n)$ to be the set of $(g_1, \ldots, g_n, h_1, \ldots, h_n) \in G_{k_1} \times \cdots \times G_{k_n} \times G_{k_1} \times \cdots \times G_{k_n}$ such that $g_1, \ldots, g_n, h_1, \ldots, h_n$ satisfy all the conditions listed in (iii) of the lemma. It is straightforward to check that $K(k_1, \ldots, k_n)$ is a closed subset of the above product.

Also for $k_1, \dots, k_n \in \mathbf{N}$, we define certain classes of functions from subsets of $G_{k_1} \times \cdots \times G_{k_n}$ into W_{n-1} as follows. First, if for any $p (1 \leqslant p \leqslant n-1)$, k_p and k_{p+1} are equal, we define $\lambda_p^{(k_1, \dots, k_n)} \equiv \lambda_p$ by $\lambda_p(g_1, \dots, g_n) = (g_1, \dots, (g_p g_{p+1}), \dots, g_n) \in W_{n-1}$ for each $(g_1, \dots, g_n) \in G_{k_1} \times \cdots \times G_{k_n}$, the multiplication taking place in G_{k_p}. And second, if $(g_1, \dots, g_n)$ is such that g_p lies in A_{k_p}, we define $\mu_p^{(k_1, \dots, k_n)} \equiv \mu_p$ by

$$\mu_p(g_1, \dots, g_n) = \left(g_1, \dots, \left(i_{k_p, k_{p+1}}(g_p) g_{p+1}\right), \dots, g_n\right)$$

for $p = 1, \dots, n-1$, and $\nu_p^{(k_1, \dots, k_n)} \equiv \nu_p$ by

$$\nu_p(g_1, \dots, g_n) = \left(g_1, \dots, \left(g_{p-1} i_{k_p, k_{p-1}}(g_p)\right), \dots, g_n\right) \quad \text{for } p = 2, \dots, n.$$

By means of these three classes of functions we can describe all possible reductions of a non-reduced n-tuple in W_n to a (reduced or non-reduced) $(n-1)$-tuple. Further, it is clear that, for each p (and each $k_1, \dots, k_n$), each λ_p, μ_p and ν_p has closed domain and is continuous.

Now we see easily from the definition of W_n that $W_n \times W_n = \bigsqcup G_{l,m}^{i,j}$, where the disjoint union is over all $i, j \leqslant n$ and (for each fixed i and j) all positive integers $l_1, \dots, l_i$, $m_1, \dots, m_j$, and where $G_{l,m}^{i,j}$ is shorthand for $(G_{l_1} \times \cdots \times G_{l_i}) \times (G_{m_1} \times \cdots \times G_{m_j})$ (with l standing for $(l_1, \dots, l_i)$ and m for $(m_1, \dots, m_j)$). To show that $\Gamma_n = \Gamma \cap (W_n \times W_n)$ is closed in $W_n \times W_n$, it therefore suffices to show that $\Gamma_{l,m}^{i,j} = \Gamma \cap G_{l,m}^{i,j}$ is closed in $G_{l,m}^{i,j}$ for all i, j, l, m. We need to distinguish four cases: (a) $i, j < n$; (b) $i = n$, $j < n$; (c) $i < n$, $j = n$; and (d) $i = j = n$.

In case (a), $G_{l,m}^{i,j}$ in fact lies in $W_{n-1} \times W_{n-1}$, so that $\Gamma_{l,m}^{i,j} = \Gamma \cap G_{l,m}^{i,j} = \Gamma_{n-1} \cap G_{l,m}^{i,j}$, which is closed in $G_{l,m}^{i,j}$ by the inductive assumption.

In case (b), we claim that $\Gamma_{l,m}^{i,j} = \bigcup_\sigma (\sigma \times \iota)^{-1}(\Gamma_{n-1})$, where ι is the identity on $G_{m_1} \times \cdots \times G_{m_j}$, and where σ runs through all the functions of $\{\mu_p \colon p = 1, \dots, n-1\}$, of $\{\nu_p \colon p = 2, \dots, n\}$, and of $\{\lambda_p \colon p \text{ satisfies } l_p = l_{p+1}\}$ (with the superscripts $(l_1, \dots, l_n)$ assumed). For if $(w, w') \in \Gamma_{l,m}^{i,j}$ (with $i = n$, $j < n$), then w must be non-reduced, since $p(w)$ and $p(w')$ have the same length; and then one of the functions σ just listed, when applied to w, gives $w'' \in W_{n-1}$ satisfying $p(w) = p(w'')$, so that $(\sigma \times \iota)(w, w') = (w'', w') \in \Gamma_{n-1}$. Thus $\Gamma_{l,m}^{i,j} \subset \bigcup (\sigma \times \iota)^{-1}(\Gamma_{n-1})$. Conversely, if $(\sigma(w), w') \in \Gamma_{n-1}$ for some $(w, w') \in G_{l,m}^{i,j}$, then we must have $p(w) = p(\sigma(w))$, so that $(w, w') \in \Gamma_{l,m}^{i,j}$. Hence $\Gamma_{l,m}^{i,j} = \bigcup_\sigma (\sigma \times \iota)^{-1}(\Gamma_{n-1})$, as claimed. Since all the functions $\sigma \times \iota$ are continuous on closed subsets of $G_{l,m}^{i,j}$, and since Γ_{n-1} is closed in $W_{n-1} \times W_{n-1}$ by assumption, it follows that $\Gamma_{l,m}^{i,j}$ is a finite union of closed sets, and is therefore closed in $G_{l,m}^{i,j}$. Case (c) is obviously dealt with similarly.

Finally, consider case (d). If $(w, w') \in \Gamma_{l,m}^{i,j}$, then since the lengths of $p(w)$ and $p(w')$ are equal, w and w' are either both reduced or both non-reduced. If, for any p $(1 \leqslant p \leqslant n)$, we have $l_p \neq m_p$, then it is clear from the lemma that $\Gamma_{l,m}^{i,j}$ can contain no pairs (w, w') in which w and w' are reduced. An argument like that for case (b) then shows that $\Gamma^{i,j}_{l,m} = \bigcup_{\sigma,\tau}(\sigma \times \tau)^{-1}(\Gamma_{n-1})$, where σ runs through the functions specified in case (b), and τ runs through a set of functions specified analogously, with (assumed) superscripts $(m_1, \ldots, m_n)$. It follows (with the assumption $l_p \neq m_p$ for some p) that $\Gamma^{i,j}_{l,m}$ is closed. Now suppose that $l_p = m_p$ for $p = 1, \ldots, n$. We claim that under this assumption $\Gamma^{i,j}_{l,m} = K(l_1, \ldots, l_n) \cup \bigcup_{\sigma,\tau}(\sigma \times \tau)^{-1}(\Gamma_{n-1})$, with σ and τ as above. To prove this, consider $(w, w') \in \Gamma^{i,j}_{l,m}$. If w and w' are reduced, then the lemma shows that $(w, w') \in K(l_1, \ldots, l_n)$, while if w and w' are not reduced, then $(w, w') \in (\sigma \times \tau)^{-1}(\Gamma_{n-1})$ for suitable σ and τ, as earlier. Conversely, if $(w, w') \in K(l_1, \ldots, l_n)$, then the last part of the lemma shows that $(w, w') \in \Gamma^{i,j}_{l,m}$, while $(w, w') \in (\sigma \times \tau)^{-1}(\Gamma_{n-1})$ implies that $(w, w') \in \Gamma^{i,j}_{l,m}$, much as in case (b). Therefore $\Gamma^{i,j}_{l,m}$ is again a finite union of closed sets, and hence is closed.

Thus $\Gamma^{i,j}_{l,m}$ is closed in $G^{i,j}_{l,m}$ for all i, j, l, m, whence Γ_n is closed in $W_n \times W_n$. The proposition now follows by induction.

From Proposition 4.25 of [7] (or Proposition A.1 of [1]), we may immediately deduce the following result, using the fact that W is a k_ω-space.

COROLLARY. *With the quotient topology determined by p, G is a (Hausdorff) k_ω-space.*

Continuity of the group operations in G now follows by a standard argument (cf. [1], [7]) which uses the facts that $p\colon W \to G$ and $p \times p\colon W \times W \to G \times G$ are both quotient maps of k_ω-spaces ([1], [7]). It also follows routinely that G has the universal property required of it by the definition. It thus remains only to show that the restriction of p to G_j is a closed embedding, for each j. This is achieved by a simple inductive argument, modelled on that given above, which shows that if C is a closed subset of G_j for any j, then $p^{-1}(p(C))$ is closed in W, so that $p(C)$ is closed in G. The outline of this argument is as follows. Write $\Delta = p^{-1}(p(C))$. Now $\Delta \cap W_1 = \bigsqcup_m(\Delta \cap G_m)$, and clearly $\Delta \cap G_m$ is C if $m = j$, and is $i_{j,m}(C \cap A_j)$ otherwise. Therefore $\Delta \cap W_1$ is closed. We now assume that $\Delta \cap W_{n-1}$ is closed for some $n \geqslant 2$ and show that $\Delta \cap W_n$ is closed. To do this, it suffices to show that $\Delta \cap (G_{k_1} \times \cdots \times G_{k_n})$ is closed for every choice of $k_1, \ldots, k_n \in \mathbf{N}$. But it is easy to see that $\Delta \cap (G_{k_1} \times \cdots \times G_{k_n}) = \bigcup_\sigma \sigma^{-1}(\Delta \cap W_{n-1})$, with the functions σ as defined earlier, and so the result follows.

This completes the proof of the theorem.

As mentioned in the introduction, we can now provide a new proof of the following result of La Martin [6]; our proof is a topologized version of the original proof ([5]; see also [9]) that epimorphisms of groups are surjective.

COROLLARY. *Epimorphisms in the category of (Hausdorff)* k_ω*-groups have dense image.*

PROOF. Let $f\colon H \to G$ be an epimorphism of k_ω-groups, and let A be the closure of $f(H)$ in G; thus A is a k_ω-group. Now let ϕ_1, ϕ_2 be the two natural topological isomorphisms from G into the topological amalgamated free product $G *_A G$. Clearly ϕ_1 and ϕ_2 agree on, and only on, A. Hence $\phi_1 f = \phi_2 f$, and so, since f is an epimorphism, $\phi_1 = \phi_2$. This implies that $A = G$, that is, that $f(H)$ is dense in G.

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