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SPECIAL SERIES OF UNITARY REPRESENTATIONS OF GROUPS ACTING ON HOMOGENEOUS TREES

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Abstract

Let G be a group acting faithfully on a homogeneous tree of order $p+1$, $p>1$. Let $\mathscr{K}^0$ be the space of functions on the Poisson boundary Ω, of zero mean on Ω. When p is a prime, G is a discrete subgroup of $PGL_2(\mathbf{Q}_p)$ of finite covolume. The representations of the special series of $PGL_2(\mathbf{Q}_p)$, which are irreducible and unitary in an appropriate completion of $\mathscr{K}^0$, are shown to be reducible when restricted to G. It is proved that these representations of G are algebraically reducible on $\mathscr{K}^0$ and topologically irreducible on $\mathscr{K}^0$ endowed with the weak topology.

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1. Introduction

Let G be a group acting isometrically and simply transitively on a homogeneous tree of order $p+1$, $p>1$. Every such group is isomorphic to the free product G_{rs} of r copies of $\mathbf{Z}$ and s copies of $\mathbf{Z}_2$, with $2r+s=p+1$ [1]. Following [3], we denote by Ω the Poisson boundary of G with respect to an isotropic nearest neighbour random walk, by ν the corresponding Poisson measure on Ω and by $P(x,\omega)=d\nu(x^{-1}\omega)/d\nu(\omega)$ the associated Poisson kernel.

Let $\mathscr{K}(\Omega)$ be the space of continuous simple functions on Ω, endowed with the weak topology defined by the functionals

$$F(\xi)=(\xi,\eta),\qquad \xi,\eta\in\mathscr{K}(\Omega).$$

For each $z\in\mathbf{C}$, we consider the representation π_z of G on $\mathscr{K}(\Omega)$, defined by

$$\pi_z(x)\xi(\omega)=p^z(x,\omega)\xi(x^{-1}\omega),\qquad \xi\in\mathscr{K}(\Omega),\ x\in G.$$

Let $\Upsilon = \{z \in \mathbb{C}: z = h\pi i/\log p,\ h \in \mathbb{Z}\}$. In the case when G is a free group, then π_z is topologically irreducible on $\mathscr{K}(\Omega)$, whenever $z,\ 1 - z \notin \Upsilon$ [4, Proposition 3.2]. On the other hand, this representation is algebraically reducible on $\mathscr{K}(\Omega)$ [4, Proposition 3.3].

The purpose of this note is to extend these results to the representation π_z of G for $1 - z \in \Upsilon$ (the remaining case $z \in \Upsilon$ is trivial). The argument of the proof works in exactly the same way for all groups G_{rs}. For the sake of simplicity of notation, from now on we restrict attention to the case $G = G_{0s}$; all the results that we prove also hold in the general case.

The subspace $\mathscr{K}^0(\Omega)$ of $\mathscr{K}(\Omega)$ defined by

$$\mathscr{K}^0(\Omega) = \{\xi \in \mathscr{K}(\Omega),\ (\xi, \mathbb{1}) = 0\}$$

is invariant under the representation π_z, $1 - z \in \Upsilon$; we endow $\mathscr{K}^0(\Omega)$ with the weak topology defined by the functionals

$$F(\xi) = (\xi, \eta), \qquad \xi, \eta \in \mathscr{K}^0(\Omega),$$

and call it the $\mathscr{K}^0$-topology. Then we prove that the representation π_z, $1 - z \in \Upsilon$, is topologically irreducible on $\mathscr{K}^0(\Omega)$, but algebraically reducible.

A preliminary step in the irreducibility proof consists in finding a finite set of functions ψ_j, $j = 1,\dots,p + 1$, in $\mathscr{K}^0(\Omega)$, such that the linear span of $\{\pi_z(x)\psi_j$: $x \in G, j = 1,\dots,p + 1\}$ is the whole of $\mathscr{K}^0(\Omega)$. Then the argument proceeds by constructing operators $T_n^{(j)}$, $n \in \mathbb{N}$, $j = 1,\dots,p + 1$, such that for any $j = 1,\dots,p + 1$, $\xi \in \mathscr{K}^0(\Omega)$, $T_n^{(j)}\xi$ converges weakly in $\mathscr{K}^0(\Omega)$ to $(\xi, \psi_j)\psi_j$, as $n \to +\infty$.

By way of contrast, we show that the representation π_z, $1 - z \in \Upsilon$, is topologically reducible on the Hilbert space where it acts unitarily.

2. Principal results

Let G be the free product $\mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$, $(p + 1)$-times, with generators a_j, $a_j^2 = 1$, $j = 1,\dots,p + 1$. Given any $x \in G$, let $E(x) = \{\omega \in \Omega: \omega^{(n)} = x\}$, where $\omega^{(n)}$ denotes the first n letters of the infinite reduced word ω.

We define, for $1 \leqslant j \leqslant p + 1$,

$$\psi_j(\omega) = \psi_{a_j}(\omega) = \begin{cases} 1, & \omega \in E(a_j), \\ -1/p, & \omega \notin E(a_j), \end{cases}$$

and for any $x \in G$, $|x| = n > 1$,

$$\psi_x(\omega) = \begin{cases} 1, & \omega \in E(x), \\ -1/(p - 1), & \omega \in E(x^{(n-1)}) \setminus E(x), \\ 0, & \text{otherwise.} \end{cases}$$

Linear combinations of ψ_x, $x \in G$, exhaust $\mathscr{K}^0(\Omega)$; moreover we have the following result.

PROPOSITION 1. *Let* $1 - z \in \Upsilon$. *Then* $\mathscr{K}^0(\Omega)$ *is the linear span of* $\{\pi_z(x)\psi_j\colon x \in G, j = 1,\ldots,p+1\}$.

PROOF. It suffices to show that for every $x \in G$, ψ_x is a linear combination of functions of the type $\pi_z(y)\psi_j$, for some $y \in G$, $j = 1,\ldots,p+1$. Let x be the reduced word $x = x_1 \cdots x_n$, $n > 1$. It follows by explicit calculations that

$$\psi_x = (p^2/p^2 - 1)p^{(1-z)n}p^{-n}\left[p^z\pi_z(x^{(n-1)})\psi_{x_n} - \pi_z(x^{(n-2)})\psi_{x_{n-1}}\right].$$

Let $N(\omega, \omega')$ be the largest integer n such that $\omega^{(n)} = \omega'^{(n)}$. The representation π_z, $1 - z \in \Upsilon$, acts unitarily on $\mathscr{K}^0(\Omega)$ with respect to the inner product defined by

$$(\xi, \eta)_1 = 2\log p\int_\Omega d\omega \int_\Omega d\omega' N(\omega, \omega')\xi(\omega)\overline{\eta(\omega')}, \qquad \xi, \eta \in \mathscr{K}^0(\Omega).$$

Moreover the following fact holds.

LEMMA 2. *For any* $\xi \in \mathscr{K}^0(\Omega)$ *and* $x \in G$, *we have*

$$(\xi, \psi_x)_1 = c_x(\xi, \psi_x)$$

where $c_x = \log p \cdot p^{2-|x|}/(p+1)^2$.

PROOF. This is obvious from the definitions.

To prove that the representation π_z is topologically irreducible on $\mathscr{K}^0(\Omega)$ with respect to the $\mathscr{K}^0$-topology, we build, for any generator a_j, a sequence $\{\nu_n^{(j)}\}_{n\in\mathbb{N}}$ of measures on G such that $(\pi_z(\nu_n^{(j)})\psi_x, \psi_y)$ tends to $(\psi_x, \psi_j)(\psi_j, \psi_y)$ as $n \to \infty$, for any x, $y \in G$.

For any fixed $j = 1,\ldots,p+1$ and any large integer n, let $\nu_n^{(j)}$ be the measure supported on the words of length n, defined by

$$\nu_n^{(j)}(x_1 \cdots x_n) = \left[p^{(z-1)n}(p+1)/p\Gamma\right]\begin{cases} \gamma_{11}, & x_1 = x_n = a_j, \\ \gamma_{10}, & x_1 = a_j, x_n \neq a_j, \\ \gamma_{01}, & x_1 \neq a_j, x_n = a_j, \\ \gamma_{00}, & x_1 \neq a_j, x_n \neq a_j, \end{cases}$$

where $\Gamma = \gamma_{11} - \gamma_{10} - \gamma_{01} + \gamma_{00} \neq 0$, and γ_{11}, γ_{10}, γ_{01} and γ_{00} are fixed.

LEMMA 3. *Let* $1 - z \in \Upsilon$. *For any* $j = 1,\ldots,p+1$ *and for every* x, $y \in G$

$$\lim_{n\to\infty}\left(\pi_z(\nu_n^{(j)})\psi_x, \psi_y\right) = (\psi_x, \psi_j)(\psi_j, \psi_y).$$

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PROOF. The proof is based on direct calculations, which require some precision but follow straight from the definitions. We give only the end results of these calculations.

Let $x, y \in G$, $n > |x| + |y|$. For $|x| = |y| = 1$ we have

$$\left(\pi_z\left(\nu_n^{(j)}\right)\psi_x, \psi_y\right) = \begin{cases} p^{-2} + O(p^{-n}), & x = y = a_j, \\ p^{-4} + O(p^{-n}), & x \neq a_j, y \neq a_j, \\ -p^{-3} + O(p^{-n}), & \text{otherwise,} \end{cases}$$

and

$$(\psi_x, \psi_j)(\psi_j, \psi_y) = \begin{cases} p^{-2}, & x = y = a_j, \\ p^{-4}, & x \neq a_j, y \neq a_j, \\ -p^{-3}, & \text{otherwise.} \end{cases}$$

In the other cases ($|x||y| > 1$) we have

$$\left(\pi_z\left(\nu_n^{(j)}\right)\psi_x, \psi_y\right) = O(p^{-n})$$

and

$$(\psi_x, \psi_j)(\psi_j, \psi_y) = 0.$$

Using the above lemmas we can prove that for $1 - z \in \Upsilon$, $\mathscr{H}^0(\Omega)$ has no nontrivial invariant subspace with respect to $\pi_z(x)$, which is closed in the $\mathscr{H}^0$-topology.

THEOREM 4. *Let* $1 - z \in \Upsilon$.
(i) *For* $j = 1, \ldots, p + 1$ *and* n *a large integer, let*

$$T_n^{(j)} = \pi_z\left(\nu_n^{(j)}\right);$$

then

$$\lim_{n \to \infty} \left(T_n^{(j)}\xi, \eta\right) = (\xi, \psi_j)(\psi_j, \eta), \qquad \xi, \eta \in \mathscr{H}^0(\Omega);$$

(ii) *if* $\mathscr{M}$ *is a subspace invariant with respect to* $\pi_z(x)$, *and closed in the* $\mathscr{H}^0$*-topology, then either* $\mathscr{M} = \{0\}$ *or* $\mathscr{M} = \mathscr{H}^0(\Omega)$.

PROOF. (i) This follows from Lemma 3. (ii) Let $\mathscr{M}$ be an invariant subspace with respect to $\pi_z(x)$ and closed in the $\mathscr{H}^0$-topology. If $(\xi, \psi_j) = 0$ for every $\xi \in \mathscr{M}$ and all $j = 1, \ldots, p + 1$, then for every $\xi \in \mathscr{M}$, $x \in G$, and all $j = 1, \ldots, p + 1$, we have $(\pi_z(x)\xi, \psi_j) = 0$. This implies, by Lemma 2, that

$$\left(\pi_z(x)\xi, \psi_j\right)_1 = \left(\xi, \pi_z(x^{-1})\psi_j\right)_1 = 0,$$

for every $\xi \in \mathscr{M}$, $x \in G$ and all $j = 1,\dots,p+1$. So $\mathscr{M} = \{0\}$, by Proposition 1. Otherwise take $\xi \in \mathscr{M}$ and ψ_k with $(\xi, \psi_k) \neq 0$. Since $\mathscr{M}$ is closed, we deduce from (i) that $\psi_k \in \mathscr{M}$. But $(\psi_k, \psi_j) \neq 0$ for every j, and therefore $\psi_j \in \mathscr{M}$ for all j. So $\mathscr{M} = \mathscr{K}^0(\Omega)$.

Finally we prove that the representation π_z, for $1 - z \in \Upsilon$, is algebraically reducible on $\mathscr{K}^0(\Omega)$. For any $j = 1,\dots,p+1$, we denote by $\mathscr{M}_j$ the linear span of $\{\pi_z(x)\psi_j\colon x \in G\}$.

THEOREM 5. *Let $1 - z \in \Upsilon$. For any $j = 1,\dots,p+1$, $\mathscr{M}_j$ is a nontrivial proper invariant subspace of $\mathscr{K}^0(\Omega)$ with respect to the representation $\pi_z(x)$, $x \in G$.*

PROOF. Fix $j = 1,\dots,p+1$. It is enough to prove that, for $i \neq j$, $\psi_i \notin \mathscr{M}_j$. Let φ be an element of $\mathscr{M}_j$. Without loss of generality φ can be written as

$$(*) \qquad \varphi = \sum_{n=1}^{N} \sum_{\substack{|x|=n\\ x_n = a_j}} C_x \pi_z(x)\psi_j,$$

where $x = x_1 \cdots x_n$ and C_x depends only on $x \in G$. If $N = 1$, it is obvious that $\varphi \neq \psi_i$, whenever $i \neq j$. Indeed in this case $\varphi = C_{a_j}\pi_z(a_j)\psi_j = -C_{a_j}\psi_j$, which cannot be equal to ψ_i, if $i \neq j$. Suppose now there exists a function φ of type $(*)$ where $N > 1$, and such that $\varphi = \psi_i$. Since φ is of type $(*)$, then for any $y \in G$, $|y| = N$ and $y = y_1 \cdots y_{N-1}a_j$, there exists a constant K_y such that, for $\omega \in E(y^{(N-1)})$, we have

$$\varphi(\omega) = \begin{cases} -p^{N-1}C_y + K_y, & \omega \in E(y),\\ p^{N-2}C_y + K_y, & \omega \in E(y^{(N-1)}) \setminus E(y). \end{cases}$$

On the other hand, $\varphi = \psi_i$, and φ must be constant on $E(y^{(N-1)})$. So necessarily $C_y = 0$, and φ reduces to

$$\varphi = \sum_{n=1}^{N-1} \sum_{\substack{|x|=n\\ x_n = a_j}} C_x \pi_z(x)\psi_j.$$

By the same argument we prove that $C_x = 0$ for all x such that $|x| > 1$, and this contradicts the assumption that $N > 1$.

3. Concluding remarks

Let H be the isometry group of the tree associated with G [7, 6]. Let $\mathscr{H}_1(\Omega)$ be the completion of $\mathscr{K}^0(\Omega)$ with respect to the inner product $(\ ,\ )_1$ defined in Section 2. The representations of the special series of H are unitaries on $\mathscr{H}_1(\Omega)$

and their restrictions to G coincide with π_z, $1 - z \in \Upsilon$. In particular, if p is a prime, then the representations π_z, $1 - z \in \Upsilon$, are restrictions to G of the special series of $PGL_2(\mathbf{Q}_p)$ [8]. The topological reducibility of π_z on $\mathscr{H}_1(\Omega)$ is now immediate, as the following argument shows. Indeed each representation of the special series of H is a subrepresentation of the regular representation of H [8]; therefore the representations π_z, $1 - z \in \Upsilon$, are subrepresentations of the regular representation of G. Since G is a discrete group, it is in particular a non-compact SIN group [2]. Hence it has no minimal projections in $L^2(G)$ [2, Corollary 4.2]. In view of the correspondence between minimal projections in $L^2(G)$ and topologically irreducible subrepresentations of the regular representations of G, this result implies that the representations π_z, $1 - z \in \Upsilon$, are topologically reducible on $\mathscr{H}_1(\Omega)$.

It would be interesting now, in view of [5], to characterize all the discrete subgroups Γ of H of finite covolume, which have the property that the spherical representations restrict irreducibly to Γ.

References

[1] W. Betori and M. Pagliacci, 'Harmonic analysis for groups acting on trees', *Boll. Un. Mat. Ital.* **6** 3-*B* (1984), 333–349.

[2] C. Cecchini and A. Figà-Talamanca, 'Projections of uniqueness for $L^p(G)$', *Pacific J. Math.* **51** (1974), 37–47.

[3] A. Figà-Talamanca and M. A. Picardello, 'Spherical functions and harmonic analysis on free groups', *J. Funct. Anal.* **47** (1982), 281–304.

[4] A. Figà-Talamanca and M. A. Picardello, *Harmonic analysis on free groups* (Lecture Notes in Pure and Applied Math., Vol. 87, Marcel Dekker, New York, 1983).

[5] A. Figà-Talamanca and M. A. Picardello, 'Restriction of spherical representations of $PGL_2(\mathbf{Q}_p)$ to a discrete subgroup', *Proc. Amer. Math. Soc.* **91** (1984), 405–408.

[6] G. I. Ol'shanskii, 'Classification of irreducible representations of groups of automorphisms of Bruhat-Tits trees', *Functional Anal. Appl.* **11** (1977), 26–34.

[7] J. P. Serre, 'Arbres, amalgames, SL_2', *Astérisque* **46** (1977).

[8] A. J. Silberger, *PGL_2 over the p-adics: its representations, spherical functions, and Fourier analysis* (Lecture Notes in Math. 166, Springer-Verlag, Berlin, Heidelberg, New York, 1970).

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