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A NOTE ON GENERALISED WREATH PRODUCT GROUPS

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Abstract

Generalised wreath products of permutation groups were discussed in a paper by Bailey and us. This note determines the orbits of the action of a generalised wreath product group on m-tuples ($m \geqslant 2$) of elements of the product of the base sets on the assumption that the action on each component is m-transitive. Certain related results are also provided.

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1. Introduction

In an earlier paper with R. A. Bailey [3] we discussed a number of properties of the generalised wreath product group (over a poset (I, ρ)), denoted by $(G, \Delta) = \prod_{(I,\rho)}(G_i, \Delta_i)$, and, in particular, determined the orbits of the action of G on $\Delta \times \Delta$. These orbits take a particularly simple form if (G_i, Δ_i) is 2-transitive for each $i \in I$. One of the purposes of this note is to derive the corresponding result for G acting on Δ^m for $m \geqslant 2$, under the assumption that (G_i, Δ_i) is m-transitive for each $i \in I$. We go on to discuss the action of certain subgroups of G on certain subsets of the orbits so determined.

The results of this note are required for a discussion of cumulants and k-statistics, of order higher than 2, of families of random variables labelled by the index sets which arise in complicated analyses of variance.

2. Preliminaries

The notation and terminology of Bailey *et al.* [3] will be used without comment. The poset $(I, \leqslant)$ is assumed finite throughout this note. For any natural number m, we write $\mathfrak{m} = \{1, \ldots, m\}$ and, if $h\colon \mathfrak{m} \to S$ is any map defined on $\mathfrak{m}$, we write $\ker h$ for the partition of $\mathfrak{m}$ induced by h, i.e. x and y in $\mathfrak{m}$ are in the same block of $\ker h$ if and only if $xh = yh$. The lattice of all partitions of $\mathfrak{m}$ is denoted by $\mathscr{P}(\mathfrak{m})$; see Aigner [1] for many properties of these lattices.

We write $\mathrm{Hom}(I, \mathscr{P}(\mathfrak{m}))$ for the set of all monotone maps $\phi\colon I \to \mathscr{P}(\mathfrak{m})$; this is a lattice under the pointwise operations. Now, any map $h\colon \mathfrak{m} \to \Delta$ defines an element $\phi^h \in \mathrm{Hom}(I, \mathscr{P}(\mathfrak{m}))$ by the formula $\phi^h(i) = \bigwedge_{j \geqslant i} \ker h_j$, where $h_j = h\pi_j$. Note that

(a) for all $x, y \in \mathfrak{m}$, we have that x and y are in the same block of $\phi^h(i)$ if and only if $xh \sim_{A[i]} yh$,

(b) $\phi^h(i) = \ker h\pi^i \wedge \ker h_i$,

(c) we have $\phi^h = \phi^k$ if and only if $\bigwedge_{j \in J} \ker h_j = \bigwedge_{j \in J} \ker k_j$ for all ancestral sets J.

For $\phi \in \mathrm{Hom}(I, \mathscr{P}(\mathfrak{m}))$, we write $\mathscr{O}_\phi = \{h \in \Delta^{\mathfrak{m}}\colon \phi^h = \phi\}$.

3. The main result

Our main result is the following.

THEOREM. *If (G_i, Δ_i) is m-transitive for each $i \in I$, then $\{\mathscr{O}_\phi\colon \phi \in \mathrm{Hom}(I, \mathscr{P}(\mathfrak{m}))\}$ is exactly the set of orbits of the generalized wreath product group G acting on $\Delta^{\mathfrak{m}}$.*

The proof is contained in the following lemmas.

LEMMA 1. *$\mathscr{O}_\phi$ is G-invariant.*

PROOF. For each $i \in I$ and $h \in \Delta^{\mathfrak{m}}$, Theorem B of [2] shows that, if $x, y \in \mathfrak{m}$,

$$xh \underset{A[i]}{\sim} yh \quad \text{if and only if} \quad xhf \underset{A[i]}{\sim} yhf.$$

Thus, by note (a) above, $\phi^h = \phi^{hf}$.

LEMMA 2. *If (G_i, Δ_i) is m-transitive for each $i \in I$, then G acts transitively on $\mathscr{O}_\phi$.*

PROOF. Fix $i \in I$ and $h, k \in \mathcal{O}_\phi$, and suppose that $\ker h\pi^i$ has blocks $B_1, \ldots, B_s$. Then, for all $r \leqslant s$ and $x, y \in B_r$, we have $xh\pi^i = yh\pi^i$ if and only if $xk\pi^i = yk\pi^i$ and, consequently, $xh\pi_i = yh\pi_i$ if and only if $xk\pi_i = yk\pi_i$. Since each $|B_r| \leqslant m$, our assumptions imply that, for all $r \leqslant s$, there exists $g_r \in G_i$ such that, for all $x \in B_r$, we have $(xh)_i g_r = (xk)_i$. Also, by the definition of $\ker h\pi^i$, there is a map $f_i\colon \Delta^i \to G_i$ such that, for all $r \leqslant s$ and $x \in B_r$, we have $(xh\pi^i)f_i = g_r$.

Carrying out this process for each $i \in I$ produces an element $f = (f_i) \in G$ such that $h^f = k$.

The proof of Lemma 2 shows more, namely that if, for each $i \in I$, we have G_i being m_i-transitive with $m_i \geqslant \sup\{|B|\colon B \text{ is a block of } \phi(i)\}$, then G is transitive on $\mathcal{O}_\phi$.

These two lemmas show that, when all the (G_i, Δ_i) are m-transitive, the orbits of G on $\Delta^{\mathfrak{m}}$ are labelled by the elements of $\mathrm{Hom}(I, \mathscr{P}(\mathfrak{m}))$ (a result which is well known when $|I| = 1$), as follows: the $|\mathcal{O}_\phi|$ are disjoint, and each is non-empty since, for $\phi \in \mathrm{Hom}(I, \mathscr{P}(\mathfrak{m}))$, we can define an $h\colon \mathfrak{m} \to \Delta$ such that $\phi^h = \phi$ by arbitrarily choosing its component maps $h_i\colon \mathfrak{m} \to \Delta_i$ subject only to $\ker h_i = \phi(i)$ for each $i \in I$.

The following reformulation of the definition of ϕ^h is of some interest.

LEMMA 3. $\phi^h = \bigvee\{\phi \in \mathrm{Hom}(I, \mathscr{P}(\mathfrak{m}))\colon (\forall\, i \in I)(\phi(i) \leqslant \ker h_i)\}$.

PROOF. Denote the right-hand side of the above expression by ψ^h. If $i \leqslant j$, then $\psi^h(i) \leqslant \psi^h(j) \leqslant \ker h_j$ and thus, if x and y belong to the same block of $\psi^h(i)$, then $xh_j = yh_j$ for all $j \geqslant i$. But this means that $xh\pi^i = yh\pi^i$ and so $\psi^h \leqslant \phi^h$. On the other hand, $\phi^h \leqslant \psi^h$ by definition, and so $\phi^h = \psi^h$.

REMARK. When $m = 2$, the lattice $\mathscr{P}(\mathfrak{m})$ is just the 2-element chain and in this case $\mathrm{Hom}(I, \mathscr{P}(\mathfrak{m}))$ is isomorphic to the distributive lattice of all ancestral sets (i.e. dual ideals or filters) of I. Thus these conclusions are consistent with Theorem C of Bailey *et al.* [2].

As an illustration of our conclusion for $m > 2$, we depict in Figure 1 the lattice $\mathrm{Hom}(I, \mathscr{P}(\mathfrak{m}))$ where I is the poset $\{1, 2\colon 2 \leqslant 1\}$ and $m = 3$. This lattice labels the orbits of $S_n \operatorname{wr} S_k$ acting on triples of elements from $\mathfrak{n} \times \mathfrak{k}$.

In Speed and Bailey [5] it was shown that the poset (I, ρ) defines an association scheme on Δ. Theorem C of Bailey *et al.* [3] shows that the associate classes coincide with the orbits of G if each G_i is 2-transitive. The above proof gives the following stronger result: if each G_i is 2-transitive then the poset-defined association scheme is t-transitive for all t, in the sense of Cameron [4, p. 103], and hence t-regular for all t, in the sense of Babai [2, p. 2].

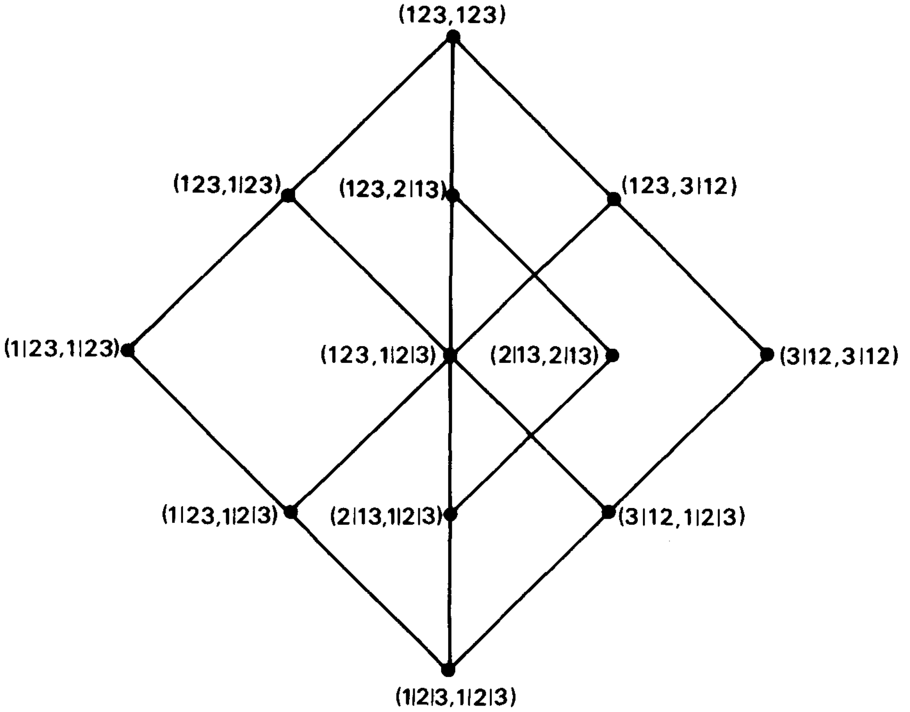

Figure 1. A lattice $\mathrm{Hom}(I, \mathscr{P}(\mathfrak{m}))$

4. Related results

For certain results in statistics, which will be published elsewhere, it is necessary to have information concerning the actions of some subgroups of G on certain subsets of $\mathcal{O}_\phi$.

Let $j \in I$ be fixed and write (G^j, Δ) for the generalized wreath product $\Pi_{(I, \leqslant)}(\tilde{G}_i, \Delta_i)$ where, for $i \neq j$, the group $\tilde{G}_i$ contains the identity permutation alone, whilst $\tilde{G}_j = G_j$. Thus G^j is the subgroup of G corresponding to an action which moves only the jth coordinate; see Lemma 4 below. For $h \in \Delta^{\mathfrak{m}}$ and $\phi \in \mathrm{Hom}(I, \mathscr{P}(\mathfrak{m}))$, we write

$$\mathcal{O}_\phi^{h,j} = \left\{k \in \mathcal{O}_\phi : k_i = h_i \text{ for all } i \neq j\right\},$$

$$\mathcal{Q}_\phi^{h,j} = \left\{k \in \mathcal{O}_\phi : k_i = h_i \text{ for all } i > j\right\}.$$

LEMMA 4. *If $f \in G^j$ and $h \in \Delta^{\mathfrak{m}}$ then $(hf)_i = h_i$ for all $i \neq j$.*

Proof. This is an immediate consequence of the definition of G^j and the action of generalized wreath product groups: if $f = (f_i)$, where $f_i\colon \Delta^i \to \tilde{G}_i$ for each $i \in I$, and $x \in \mathfrak{m}$, then, for $i \neq j$,

$$(xhf)_i = (xh)_i\big((xh\pi^i)f_i\big) = (xh)_i 1_i = (xh)_i,$$

where we have denoted the identity permutation on Δ_i by 1_i.

Corollary. *G^j fixes both $\mathcal{O}_\phi^{h,j}$ and $\mathcal{Q}_\phi^{h,j}$ setwise.*

Lemma 5. *If G_j is m-transitive the G^j is transitive on $\mathcal{O}_\phi^{h,j}$.*

Proof. Take $k \in \mathcal{O}_\phi^{h,j}$. It is sufficient to find $f \in G^j$ so that $kf = h$, and by Lemma 4 we need only consider the jth coordinates.

We denote the blocks of $\ker h\pi^j$ by $B_1, \ldots, B_s$ and, by the reasoning in the proof of Lemma 2, we see that, for each $r = 1, \ldots, s$, we can choose $g_r \in G_j$ such that, for each $x \in B_r$, we have $(xk)_j g_r = (xh)_j$. Continuing the line of reasoning of Lemma 2, we choose f_j arbitrarily subject only to the requirement that, for each $r = 1, \ldots, s$ and $x \in B_r$, we have $(xh\pi^j)f_j = g_r$. The definition of f is now completed by defining f_i $(i \neq j)$ in the only way possible and we have found an f with $kf = h$.

Remark. The proof has in fact shown that, if $h, k \in \mathcal{O}_\phi$ and $h_i = k_i$ for all $i > j$, then there exists an element $f \in G^j$ such that $(kf)_i = h_i$ for all $i \geqslant j$. This shows that the orbits of G^j on $\mathcal{Q}_\phi^{h,j}$ are labelled by the elements of $\{\{k_i\colon i \ngeqslant j\}\colon k \in \mathcal{Q}_\phi^{h,j}\}$ and are exactly the sets

$$\{l \in \mathcal{O}_\phi\colon l_i = h_i, i > j, l_i = k_i, i \ngeqslant j\}.$$

Our final result shows that, for $h, k \in \mathcal{O}_\phi$, we can find an $f \in G$ such that $kf = h$, having the form

$$f = f_1 f_2 \cdots f_u, \qquad \text{with } f_t \in G^{j_t} \qquad (t = 1, \ldots, u), \tag{1}$$

where $I = \{j_1, \ldots, j_u\}$. Loosely speaking, we can "move over" $\mathcal{O}_\phi$ using elements from the subgroups G^j of G. This is the only result for which I *must* be finite.

Lemma 6. *If (G_i, Δ_i), for each $i \in I$, is m-transitive then, for $h, k \in \mathcal{O}_\phi$, there exists $f \in G$ of the form* (1) *such that $kf = h$.*

Proof. We number the elements of I, beginning with the maximal ones, in such a way that if $i > j$ in I, then the number that j is assigned is larger than that assigned to i.

By the remark following the proof of Lemma 5, we can find $f_1 \in G^{j_1}$ such that $(kf_1)_{j_1} = h_{j_1}$. Assume now that this has been done for $j_1, \ldots, j_{t-1}$, $t \geqslant 2$, and so $k(f_1 \cdots f_{t-1})$ agrees with h at $j_1, \ldots, j_{t-1}$. Then we have $k' = k(f_1 \cdots f_{t-1}) \in \mathscr{Q}_\phi^{h,j_t}$ and, by the last remark, once more there exists $f_t \in G^{j_t}$ which sends k' to $k(f_1 \cdots f_t) \in \mathscr{Q}_\phi^{h,j_t}$. Thus $k(f_1 \cdots f_t)$ agrees with h at $j_1, \ldots, j_t$ and the induction proof is complete.

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