J. Austral. Math. Soc. (*Series A*) 38 (1985), 394–407

CONTINUOUS TRACE C^*-ALGEBRAS WITH GIVEN DIXMIER-DOUADY CLASS

IAIN RAEBURN and JOSEPH L. TAYLOR

(Received 25 November 1983)

Communicated by W. Moran

Abstract

We give an explicit construction of a continuous trace C^*-algebra with prescribed Dixmier-Douady class, and with only finite-dimensional irreducible representations. These algebras often have non-trivial automorphisms, and we show how a recent description of the outer automorphism group of a stable continuous trace C^*-algebra follows easily from our main result. Since our motivation came from work on a new notion of central separable algebras, we explore the connections between this purely algebraic subject and C^*-algebras.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 46 L 05, 46 L 40; secondary 13 A 20, 55 N 05.

Let A be a continuous trace C^*-algebra with paracompact spectrum T. Dixmier and Douady [4] constructed a cohomology class $\delta(A) \in H^3(T, \mathbf{Z})$, now known as the Dixmier-Douady class of A, which vanishes exactly when A is the C^*-algebra defined by a continuous field of Hilbert spaces over T [3, 10.7.15]. This invariant has attracted considerable attention in recent years since in the case of separable algebras $\delta(A)$ determines A up to stable isomorphism. (This can easily be deduced from, for example, [2, Théorème 2], [8, Lemma 1.11] and [3, 10.8.4].)

Dixmier and Douady also showed in [4] that every class in $H^3(T, \mathbf{Z})$ is $\delta(A)$ for some A. Their proof of this uses Zorn's lemma and the fact that when H is an infinite-dimensional Hilbert space the sheaf of germs of $U(H)$-valued functions is

Research partially supported by the National Science Foundation of the U.S.A. and the Australian Research Grants Scheme.

soft, and all the irreducible representations of the resulting algebra are infinite-dimensional. We present here an explicit construction of a continuous trace C^*-algebra with prescribed Dixmier-Douady class. The irreducible representations of the C^*-algebra we construct are all finite-dimensional; as it is easy to see that $n\delta(A) = 0$ when A is n-homogeneous [5, Proposition 1.4] it follows that our algebra is in general far from homogeneous.

This construction is the content of our first section. In Section 2 we discuss the automorphism groups of the algebras we have built. For any continuous trace C^*-algebra A with spectrum T and any automorphism $\alpha \in \mathrm{Aut}_{C(T)} A$ there is a cohomology class $\zeta(\alpha) \in H^2(T, \mathbf{Z})$ which vanishes when α is implemented by a multiplier, and the main theorem of [8] asserts that when A is stable and separable every class in $H^2(T, \mathbf{Z})$ arises this way. The proof of this in [8] is modelled on the surjectivity argument of Dixmier-Douady, and is not constructive; however, for suitable algebras of the type in Section 1 we can write down automorphisms corresponding to given elements of $H^2(T, \mathbf{Z})$, and we use this to give a short proof of the theorem in [8].

The construction we describe here arose in connection with work on a notion of central separable algebra which does not require that the algebra have an identity [14], [11], and when the spectrum T is compact our C^*-algebras are also central separable algebras in this sense. In our third section we discuss the relationship between these central separable algebras and continuous trace C^*-algebras.

Our notation concerning continuous trace C^*-algebras will more or less conform to that of [3, Chapter 10]. If H is a continuous field of Hilbert spaces over T with continuous sections $\Gamma(H)$, we denote the corresponding field of elementary C^*-algebras by $\mathfrak{A}(H)$ and write $\Gamma(\mathfrak{A}(H))$ for the C^*-algebra defined by H. In Section 3 it will be crucial that we are working with purely algebraic tensor products, so we shall always write $\overline{\otimes}$ when we mean to take a completion.

1. The construction of continuous trace C^*-algebras with prescribed Dixmier-Douady class

Let T be a paracompact space, and let $\mathscr{R}$, $\mathscr{S}$ respectively denote the sheaves of germs of continuous real and S^1-valued functions on T. Then the covering map $t \to \exp 2\pi i t$ gives a short exact sequence

$$0 \to \mathbf{Z} \to \mathscr{R} \to \mathscr{S} \to 0$$

of sheaves, which in turn gives a long exact sequence of cohomology

$$\ldots \to H^2(T, \mathscr{R}) \to H^2(T, \mathscr{S}) \to H^3(T, \mathbf{Z}) \to H^3(T, \mathscr{R}) \to \ldots .$$

The sheaf $\mathscr{R}$ is fine, so $H^n(T, \mathscr{R}) = 0$ for $n > 0$, and the middle map is an isomorphism. The Dixmier-Douady class of a continuous trace C^*-algebra A with paracompact spectrum T is by definition the image of a class in $H^2(T, \mathscr{S})$ [3, Section 10.7], so we may as well start with a 2-cocycle $\lambda_{ijk}: N_{ijk} \to S^1$ relative to a locally finite open cover $\{N_i: i \in I\}$. We can always replace the cover by another $\{M_i: i \in I\}$ with $\overline{M_i} \subset N_i$, so we may also suppose that each λ_{ijk} is defined on the closure $\overline{N_{ijk}}$.

THEOREM 1. *Let $\{N_i: i \in I\}$ be a locally finite cover of a locally compact paracompact space T by relatively compact open sets, and suppose that λ_{ijk}: $\overline{N_{ijk}} \to S^1$ is a 2-cocycle. Let*

$$A_1 = \left\{ \sum_{j,k \in I} \phi_{jk} e_{jk} \middle| \phi_{jk} \in C(T), \phi_{jk} \equiv 0 \text{ outside } N_{jk} \right\}$$

have the obvious structure as a $C(T)$-module, multiplication defined by

$$(\phi_{jk} e_{jk})(\psi_{lm} e_{lm}) = \delta_{kl} \psi e_{jm},$$

where δ_{kl} is the Kronecker delta and ψ is given by

$$\psi(t) = \begin{cases} \overline{\lambda_{jkm}(t)}\, \phi_{jk}(t) \psi_{km}(t) & \text{for } t \in N_{jkm}, \\ 0 & \text{for } t \notin N_{jkm}, \end{cases}$$

and involution defined by

$$(\phi_{jk} e_{jk})^* = \overline{\phi_{jk}}\, e_{kj}.$$

For $t \in T$ let $I(t) = \{i \in I: t \in N_i\}$: note that $n_t = |I(t)|$ is finite. If $t \in N_{ij}$ then for the usual $n_t \times n_t$ matrix norm we have

$$\left\| \left(\lambda_{ikl}(t) \phi_{kl}(t)\right)_{k,l \in I(t)} \right\| = \left\| \left(\lambda_{jkl}(t) \phi_{kl}(t)\right)_{k,l \in I(t)} \right\|,$$

so that for each t we have a semi-norm $\|\cdot\|_t$ on A_1. Let A be the set of $a \in A_1$ such that $t \to \|a\|_t$ vanishes at infinity, and set $\|a\| = \sup \|a\|_t$. Then A is a continuous trace C^*-algebra with spectrum T whose Dixmier-Douady class $\delta(A) \in H^3(T, \mathbf{Z}) = H^2(T, \mathscr{S})$ is represented by the cocycle $\{N_i, \lambda_{ijk}\}$. The dimension of the irreducible representation corresponding to $t \in T$ is n_t.

PROOF. Simple calculations using the cocycle identity show that A is a $*$-algebra with the above operations, and that for $t \in N_i$

$$\pi_{i,t}\left(\sum \phi_{jk} e_{jk}\right) = \left(\overline{\lambda_{ijk}(t)}\, \phi_{jk}(t)\right)_{j,k \in I(t)}$$

defines a $*$-representation of A into $M_{n_t}(\mathbf{C})$. If $t \in N_{ij}$ and $D(\mu_k)$ is the diagonal matrix with entries μ_k, then the cocycle identity yields

$$D\left(\overline{\lambda_{ijk}(t)}\right)\left[\left(\overline{\lambda_{ikl}(t)}\, \phi_{kl}(t)\right)_{k,l \in I(t)}\right] D\left(\lambda_{ijl}(t)\right) = \left[\left(\overline{\lambda_{jkl}(t)}\, \phi_{kl}(t)\right)_{k,l \in I(t)}\right].$$

The diagonal matrices are unitary, so we deduce that the norms of the matrices $\pi_{i,t}(a)$ and $\pi_{j,t}(a)$ are always equal, and we have well-defined semi-norms

$$\|a\|_t = \|\pi_{i,t}(a)\| \quad \text{for } t \in N_i,$$

as claimed. The norm on A satisfies the C^*-condition $\|aa^*\| = \|a\|^2$ since each $\|\cdot\|_t$ does, and it is not hard to see that A is complete, so A is a C^*-algebra.

For each $t \in T$ we define an ideal in A_1 by

$$J_t = \left\{ \sum_{j,k} \phi_{jk} e_{jk} \in A \,|\, \phi_{jk}(t) = 0 \text{ for all } j, k \right\}.$$

We denote the quotient C^*-algebra A/J_t by $A(t)$, and we write $a(t)$ for the image of $a \in A_1$ in $A(t)$. Note that if $t \in N_i$ the representation $\pi_{i,t}$ induces an isomorphism of $A(t)$ onto M_{n_t}, so each $A(t)$ is an elementary C^*-algebra. In fact, $\mathfrak{A} = \{A(t), A_1\}$ is a continuous field of elementary C^*-algebras over T such that A is the C^*-algebra of continuous sections vanishing at infinity. For by definition $\{a(t) \colon a \in A\}$ is all of $A(t)$, and the continuity of

$$t \to \|a(t)\| = \|\pi_{i,t}(a)\|$$

follows from the continuity of the matrix norm (note that if $t_\alpha \to t$ then $I(t_\alpha)$ eventually contains $I(t)$). Further, if $x = (x(t)) \in \prod A(t)$ then there are unique scalars $\nu_{jk}(t)$ such that

$$\pi_{i,t}(x(t)) = \left(\overline{\lambda_{ijk}(t)}\, \nu_{jk}(t) \right)_{j,k \in I(t)}.$$

If x is locally uniformly approximable by elements of A_1, then standard arguments show that the ν_{jk} are continuous and vanish off N_{jk}, so that x is the section defined by $\sum \nu_{jk} e_{jk} \in A$. Thus $\mathfrak{A}$ is a continuous field as asserted and $A = \Gamma_0(\mathfrak{A})$ has spectrum T. It is easy to see that $\mathfrak{A}$ satisfies Fell's condition (for example, if $t \in N_i$, $\rho \equiv 1$ near t and $\rho \equiv 0$ off N_i, then $(\rho e_{ii})(s)$ is a rank one projection for s near t) and hence A has continuous trace.

To compute the Dixmier-Douady class of A we build fields H_i of Hilbert spaces over $\overline{N_i}$ and isomorphisms of the associated fields of elementary C^*-algebras $\mathfrak{A}(H_i)$ onto $\mathfrak{A}|_{\overline{N_i}}$. For $t \in \overline{N_i}$ we define

$$H_i(t) = \left\{ \sum_{k \in I(t)} \lambda_k e_k \colon \lambda_k \in \mathbf{C} \right\},$$

with the usual inner product $(e_k | e_l) = \delta_{kl}$, and take as our space of continuous sections

$$\Gamma(H_i) = \left\{ \sum_{k \in I_i} \phi_k e_k \colon \phi_k \in C(\overline{N_i}),\, \phi \equiv 0 \text{ off } N_k \right\},$$

where $I_i = \{k \colon N_k \cap \overline{N_i} \neq \{\varnothing\}\}$. It is routine to check that this does define a continuous field of Hilbert spaces. The corresponding field $\mathfrak{A}_i = \mathfrak{A}(H_i)$ of elementary C^*-algebras is that generated by fields of the form $e \otimes \bar{f}$ for $e, f \in \Gamma(H_i)$, where $x \otimes \bar{y}$ denotes the rank one operator $z \to (z|y)x$. We define a linear map from $\Gamma(H_i) \otimes \Gamma(\overline{H}_i)$ to $A|_{\overline{N}_i}$ by

$$h_i\left(\phi e_j \otimes \overline{\psi e_k}\right) = \theta e_{jk}, \text{ where } \theta(t) = \begin{cases} \lambda_{ijk}(t)\phi(t)\overline{\psi(t)} & \text{for } t \in \overline{N_{ijk}}, \\ 0 & \text{for } t \in \overline{N}_i \setminus N_{jk}; \end{cases}$$

a standard Urysohn's lemma argument shows that θe_{jk} is in fact the restriction of an element of A. Further, h_i is a $*$-homomorphism, is isometric from the usual norm on $\Gamma(\mathfrak{A}_i)$ to the given one on A, and is easily seen to be surjective; hence it extends to an isomorphism of $\mathfrak{A}_i$ onto $\mathfrak{A}|_{\overline{N}_i}$. (In fact, every θe_{jk} is the image of an elementary tensor so if the index set I is finite h_i defines an isomorphism of the algebraic tensor product $\Gamma(H_i) \otimes_{C(\overline{N}_i)} \Gamma(H_i)$ onto $A|_{\overline{N}_i}$. As the latter algebra is complete so is the algebraic tensor product, which therefore equals $\Gamma(\mathfrak{A}_i)$.) We now define isomorphisms g_{ij}: $H_j|_{\overline{N_{ij}}} \to H_i|_{\overline{N_{ij}}}$ by

$$g_{ij}(t)(\phi_k(t)e_k) = \begin{cases} \lambda_{ijk}(t)\phi_k(t)e_k & \text{if } t \in \overline{N_{ijk}}, \\ 0 & \text{if } t \in \overline{N_{ij}} \setminus N_k. \end{cases}$$

The induced isomorphism Ad g_{ij} of $\mathfrak{A}(H_j)$ into $\mathfrak{A}(H_i)$ is given on elementary tensors by

$$\begin{aligned} (\mathrm{Ad}\, g_{ij})(t)\left(\phi_k(t)e_k \otimes \overline{\psi_l(t)e_l}\right) &= g_{ij}(t)\phi_k(t)e_k \otimes \overline{g_{ij}(t)\psi_l(t)e_l} \\ &= \begin{cases} \lambda_{ijk}(t)\phi_k(t)e_k \otimes \overline{\lambda_{ijl}(t)\psi_l(t)e_l} & \text{if } t \in \overline{N_{ijkl}}, \\ 0 & \text{if } t \in \overline{N_{ij}} \setminus N_{kl}, \end{cases} \end{aligned}$$

so that routine calculations using the cocycle identity give

$$h_i(t) \circ (\mathrm{Ad}\, g_{ij})(t) = h_j(t) \quad \text{for } t \in \overline{N_{ij}}.$$

Thus g_{ij} defines the isomorphism $h_i^{-1}h_j$ as in [3, 10.7.11], and for $t \in \overline{N_{ijk}}$ we have

$$g_{ij}(t)g_{jk}(t) = \lambda_{ijk}(t)g_{ik}(t),$$

so that the class $\delta(A) = \gamma(\mathfrak{A})$ in $H^2(T, \mathscr{S})$ is represented by the cocycle $\{N_i, \lambda_{ijk}\}$ as claimed (see [3, 10.7.12–14]).

REMARKS. 1. If T has covering dimension n, then we can realise any class in $H^2(T, \mathscr{S})$ as a cocycle relative to a cover where at most $n + 1$ different sets intersect. Because the Dixmier-Douady class determines a separable continuous trace C^*-algebra up to stable isomorphism, our theorem implies that every such algebra with spectrum T is stably isomorphic to an algebra whose irreducible representations have dimension $\leqslant n + 1$. This has already been shown by Brown [1, Corollary 2.11] using different reasoning.

2. The last part of the proof could be simplified a bit by constructing the class $\delta(A)$ as in [9, 2.6–2.9] using local rank one projections and intertwining partial isometries rather than fields of Hilbert spaces and isomorphisms. However, in our present proof we also showed that, when T is compact, the algebraic tensor product $\Gamma(H_i) \otimes_{C(T)} \Gamma(\overline{H}_i)$ is complete, and this has some interesting algebraic consequences, which we shall discuss in Section 3.

3. The C^*-algebra A we construct in Theorem 1 can also be viewed as a twisted groupoid C^*-algebra. For simplicity we suppose T is compact and $\{N_i: i \in I\}$ is a finite cover. Then we let ψ be the local homeomorphism of the disjoint union $X = \bigcup_i N_i$ onto T, let $\mathscr{R}(\psi)$ be the equivalence relation induced on X by ψ as in [6, Section 4], and let G be the corresponding topological groupoid with left Haar system induced by counting measure on the fibres of ψ (see [12, Section 1.2]). Given a cocycle $\lambda_{ijk}: N_{ijk} \to S^1$ we define a 2-cocycle $\sigma: G^2 \to S^1$ by

$$\sigma((x, y), (y, z)) = \lambda_{i(x)i(y)i(z)}(\psi(x)),$$

where G^2 denotes the set of composable elements of G and $i: X \to I$ is defined by $x \in N_{i(x)}$. We define $\Phi: C_c(G) \to A$ by $\Phi f = \sum \phi_{jk} e_{jk}$, where

$$\phi_{jk}(t) = \begin{cases} f(x, y) & \text{if } x \in N_j, y \in N_k \text{ and } \psi(x) = \psi(y) = t, \\ 0 & \text{if } t \notin N_{jk}. \end{cases}$$

It is routine to verify that Φ defines a $*$-monomorphism of the convolution algebra $C_c(G, \sigma)$ (see [12, Section 4.1]) onto a dense subalgebra of A, and hence gives an isomorphism of $C^*(G, \sigma)$ with A.

In particular, when $\lambda_{ijk} \equiv 1$ the algebra A is the C^*-algebra $C^*(\psi)$ associated by Kumjian [6] to the local homeomorphism ψ. This can also be seen directly: his imprimitivity bimodule $l^2(\psi)$ is isomorphic to

$$H = \left\{ \sum_k \phi_k e_k : \phi_k \in C(T), \phi_k \equiv 0 \text{ outside } N_k \right\}$$

with $C(T)$-valued inner product given by

$$\left(\sum \phi_k e_k \middle| \sum \psi_l e_l\right) = \sum \overline{\phi_k} \psi_l,$$

and A acts on H by

$$\left(\sum \phi_{jk} e_{jk}\right)\left(\sum \psi_l e_l\right) = \sum_j \left(\sum_k \phi_{jk} \psi_k\right) e_j.$$

This is, of course, the same construction as we carried out locally to prove our theorem, modulo changes in convention regarding inner products.

2. Automorphisms

For any continuous trace C^*-algebra A with paracompact spectrum T there is an exact sequence

$$0 \to \operatorname{Inn} A \to \operatorname{Aut}_{C(T)} A \xrightarrow{\zeta_A} H^2(T, \mathbf{Z}),$$

where Inn A denotes the group of A determined by multipliers of A (see [13, Section 5]). We shall now investigate the range of the homomorphism ζ_A for the algebra A constructed in Theorem 1.

So let $\{N_i, \lambda_{ijk}\}$ and A be as in Theorem 1, and let $c \in H^2(T, \mathbf{Z})$ be given. If c can be represented by a 1-cocycle $\{N_i, \mu_{ij}\}$ with values in $\mathscr{S}$ relative to the same cover $\{N_i\}$, then we can define an automorphism α of A by

$$\alpha\left(\sum \phi_{jk} e_{jk}\right) = \sum \psi_{jk} e_{jk} \text{ where } \psi_{jk}(t) = \begin{cases} \mu_{jk}(t)\phi_{jk}(t) & \text{if } t \in N_{jk}, \\ 0 & \text{otherwise.} \end{cases}$$

This is easily seen to be a $C(T)$-module automorphism: we compute its class $\zeta(\alpha)$ in $H^1(T, \mathscr{S}) \cong H^2(T, \mathbf{Z})$. Let $\{M_i\}$ be an open cover of T with $\overline{M_i} \subset N_i$, and choose continuous functions $\rho_i \colon T \to [0, 1]$ such that $\rho_i \equiv 1$ on M_i and $\rho_i \equiv 0$ off N_i. We can now define multipliers m_i of A by

$$m_i = \sum_{j \in I} \rho_i \overline{\mu_{ij}} e_{jj};$$

note that although $\rho_i \mu_{ij}$ is not defined on all of T, whenever we have $\phi \equiv 0$ off N_{jk} the function $\rho_i \overline{\mu_{ij}} \phi$ does extend to be continuous on T, and simple calculations show that under the usual multiplication rule this gives a multiplier of A. Further, the cocycle identity shows that for $t \in M_i$

$$\alpha\left(\sum \phi_{jk} e_{jk}\right)(t) = \left(m_i\left(\sum \phi_{jk} e_{jk}\right) m_i^*\right)(t),$$

so that m_i implements α over M_i. Moreover, the same cocycle identity also gives

$$\mu_{ij}(t) m_j(t) = m_i(t) \quad \text{for } t \in M_{ij},$$

and we deduce that $\zeta(\alpha)$ is represented by the cocycle $\{M_i, \mu_{ij}\}$ (see [13, Section 5]). This defines the same class as the one we started with, and therefore $\zeta(\alpha) = c$.

Of course, we cannot expect to represent an arbitrary class in $H^2(T, \mathbf{Z})$ relative to a fixed open cover (and we will come back to this question later), but if we start with classes $d \in H^3(T, \mathbf{Z})$ and $c \in H^2(T, \mathbf{Z})$ then we can always represent them as S^1-valued cocycles relative to the same cover. Hence the argument in the preceding paragraph proves the following result:

PROPOSITION 2. *Let T be a locally compact paracompact space, and let $d \in H^3(T, \mathbf{Z})$, $c \in H^2(T, \mathbf{Z})$. Then there are a continuous trace C^*-algebra A with spectrum T, with $\delta(A) = d$, and with only finite-dimensional irreducible representations, and an automorphism $\alpha \in \operatorname{Aut}_{C(T)} A$ such that $\zeta_A(\alpha) = c$.*

REMARK. Alex Kumjian has noticed independently that, given a cocycle μ_{ij}: $N_{ij} \to S^1$, one can write down an automorphism α of the C^*-algebra $C^*(\psi)$ associated to the local homeomorphism $\psi: \bigcup_i N_i \to T$ such that $\zeta(\alpha)$ is represented by $\{N_i, \mu_{ij}\}$. In fact, it was his observation that alerted us to the realisation of A as the twisted groupoid C^*-algebra $C^*(G, \sigma)$ (see Remark (3) in Section 1). The automorphism α can be conveniently viewed in this realisation too: define a continuous 1-cocycle $c: G \to S^1$ by

$$c(x, y) = \mu_{i(x)i(y)}(\psi(x)),$$

and then the automorphism α of $C^*(G, \sigma)$ is defined by

$$\alpha(f)(x, y) = c(x, y)f(x, y) \quad \text{for } f \in C_c(G),$$

as in [12, Proposition II.5.1].

COROLLARY 3 ([8, Theorem 2.1]). *Let A be a separable stable continuous trace C^*-algebra with spectrum T. Then the homomorphism ζ_A is surjective.*

PROOF. Let $c \in H^2(T, \mathbf{Z})$. Then by the proposition there are an algebra B and an automorphism $\alpha \in \operatorname{Aut}_{C(T)} B$ such that $\delta(B) = \delta(A)$ and $\zeta_B(\alpha) = c$. Since the Dixmier-Douady class determines a separable continuous trace C^*-algebra up to stable isomorphism, we have

$$A \cong A \mathbin{\overline{\otimes}} K(H) \cong B \mathbin{\overline{\otimes}} K(H),$$

and we may assume that this isomorphism induces the identity map from $T = \hat{A}$ to $T = \hat{B} = (B \mathbin{\overline{\otimes}} K(H))\hat{}$ (see [10, Lemma 4.3]). If $m \in M(B)$ implements α over N then

$$m \otimes 1 \in M(B) \mathbin{\overline{\otimes}} M(K(H)) \subset M\big(B \mathbin{\overline{\otimes}} K(H)\big)$$

implements $\alpha \otimes \mathrm{id}$ over N, so $\zeta_{B \otimes K(H)}(\alpha \otimes \mathrm{id}) = \zeta_B(\alpha) = c$, and the corresponding automorphism β of A therefore satisfies $\zeta_A(\beta) = c$.

REMARK. Corollary 3 holds for arbitrary C^*-algebras with paracompact spectrum T [10, Corollary 3.12]. However, the proof given there involves establishing surjectivity for $A = C_0(T, K(H))$ first, and this seems to be more complicated than our Proposition 2.

As we have seen above, if A is the C^*-algebra of Theorem 1 corresponding to a cocycle $\lambda_{ijk}: \overline{N_{ijk}} \to S^1$ then the range of ζ_A contains the subgroup $H^1(\mathscr{N}, \mathscr{S})$ of $H^1(T, \mathscr{S}) \cong H^2(T, \mathbf{Z})$ consisting of those classes realisable on the cover $\mathscr{N} = \{N_i\}$. It is quite easy to see that, while this range need not be all of $H^2(T, \mathbf{Z})$, it may contain more than $H^1(\mathscr{N}, \mathscr{S})$. First of all, take the one set cover $\{T\}$ of a space with $H^2(T, \mathbf{Z}) \neq 0$; then $A \cong C_0(T)$ and the range of ζ_A is $\{0\} \neq H^2(T, \mathbf{Z})$.

This example is more general than it appears, since if Y is a compact set contained in only one member of the cover, then $A|_Y \cong C(Y)$ and no element of $H^2(T, \mathbf{Z})$ whose image in $H^2(Y, \mathbf{Z})$ is non-zero can come from a $C(T)$-automorphism of A. On the other hand, if we take the trivial cover $N_1 = N_2 = T$ consisting of two sets, then $A = C(T, M_2)$ and there can be automorphisms α for which $\zeta(\alpha) \neq 0$ and hence does not belong to $H^1(\mathcal{N}, \mathcal{S})$. This argument is also more general than it first appears, since if Y is a compact subset of $N_1 \cap N_2$ which meets no other N_i, then $A|_Y \cong C(Y, M_2)$ and there could be automorphisms of A which do not trivialise over $N_1 \cap N_2$.

3. Central separable algebras

Let R be a commutative ring with identity, let A be an R-algebra (not necessarily with an identity) and let $Z(A)$ denote the ring of $A - A$ bimodule endomorphisms of A. There is always a natural map i: $R \to Z(A)$ and we call A central if this is an isomorphism. Following [14, Section 2] we say A is separable if $A^2 = A$, A is projective as an $A - A$ bimodule, and for each maximal ideal M of $Z(A)$ we have $MA \neq A$. An immediate property of such algebras is that the multiplication map: $A \otimes_R A \to A$ is split as an $A - A$ bimodule homomorphism. Our main theorem can be strengthened as follows.

PROPOSITION 4. *Let T be a compact Hausdorff space and let $d \in H^3(T, \mathbf{Z})$. Then there is a continuous trace C^*-algebra A with $\delta(A) = d$ which is also a central separable $C(T)$-algebra.*

PROOF. Let $\{N_i\}$ be a finite open cover of T such that d is represented by a cocycle λ_{ijk}: $\overline{N_{ijk}} \to S^1$, and let A be the algebra constructed in Theorem 1. Choose another cover $\{M_i\}$ with $\overline{M_i} \subset N_i$, and functions $\rho_i \in C_0(N_i)$ with $\rho_i \equiv 1$ on $\overline{M_i}$. We define $A - A$ bimodule homomorphisms ω_i on $A \otimes_{C(T)} A$ by interchanging the two copies of $\Gamma(\overline{H_i})$ in

$$A \otimes_{C(T)} A|_{\overline{N_i}} \cong \Gamma(H_i) \otimes_{C(T)} \Gamma(\overline{H_i}) \otimes_{C(T)} \Gamma(H_i) \otimes_{C(T)} \Gamma(\overline{H_i})$$

and multiplying by ρ_i. Let $\tilde{\Omega}$ be the module of $A - A$ homomorphisms generated by the ω_i, and let tr: $A \otimes_{C(T)} \tilde{\Omega} \to Z(A)$ be as in [14, page 174]. We have $Z(A) = C(T)$, since $Z(A)$ is by definition the centre of the multiplier algebra of A. Thus it will follow from [14, Proposition 3.8] that A is central separable if we can prove that the range of tr is not contained in any maximal ideal of $C(T)$. However, a straightforward calculation shows that if $t \in M_i$ then $\mathrm{tr}(\rho_i e_{ii} \otimes \omega_i)$ cannot vanish at t, and the result is proved.

This last result raises an obvious question: what is the relationship between the classes of C^*-algebras with compact spectrum T and central separable algebras over $C(T)$? It is well-known that the central separable $C(T)$-algebras with identity are precisely the locally homogeneous C^*-algebras with spectrum T, but the situation for algebras without identity is rather more complicated. For example, the algebra $FR(H)$ of finite rank operators on a separable (!) Hilbert space H is central separable over $\mathbf{C}$ (we have $FR(H) = H \otimes_{\mathbf{C}} \overline{H}$) but it is not a C^*-algebra unless H is finite-dimensional. Our next proposition gives an answer to this question—and shows, among other things, that Proposition 4 does say more than Theorem 1.

PROPOSITION 5. (1) *Let A be a C^*-algebra with compact Hausdorff spectrum T which is also a central separable $C(T)$-algebra for the natural action of $C(T)$ on A. Then A is a continuous trace C^*-algebra whose irreducible representations have finite, bounded dimensions.*

(2) *There are continuous trace C^*-algebras A with compact spectrum and $\{\dim \pi: \pi \in \hat{A}\}$ bounded which are not central separable algebras.*

The proof of this result will depend on a series of simple lemmas. We begin with a purely algebraic result which is implicit in [14].

LEMMA 6. *Let B be a central separable algebra over a commutative ring R, and suppose that $p \in B$ is a rank one idempotent (i.e., $pBp = Rp$). Then the map $ap \otimes pb \to apb$ induces an isomorphism of $Bp \otimes_R pB$ onto B.*

PROOF. This is a consequence of the proofs of Proposition 4.2 and 4.3 of [14] with $N = Bp$, $M = pB$ and λ: $N \otimes_R M \to B$ given by the multiplication in B; the regularity of N, M follows from [14, Propositions 1.1 and 1.6]. The proof of Proposition 4.2 shows that if $A = M \otimes_B N$, then λ induces an isomorphism of $N \otimes_A M$ onto B. However, the last argument in the proof of Proposition 4.3 shows that the multiplication also induces an isomorphism of $A = pB \otimes_B Bp$ onto pBp, which is just Rp since p is rank one. We therefore deduce that $B \cong Bp \otimes_R pB$ as claimed.

LEMMA 7. *Let A be a C^*-algebra which is also a central separable $C(T)$-algebra. Then every closed 2-sided ideal in A is regular, and has the form IA for some 2-sided ideal I in $C(T)$.*

PROOF. The multiplication map $A \otimes_{C(T)} A \to A$ is split and A is therefore a regular 2-sided A-module. If M is a 2-sided ideal in A, then we have $MA = M$

(see, for example, [7, 1.4.5]) and M is regular by [14, Proposition 1.6]. The result now follows from [14, Proposition 3.5].

LEMMA 8. *Let A be a C^*-algebra which is central separable over* $\mathbf{C}$. *Then* $A \cong M_n(\mathbf{C})$.

PROOF. Let π: $A \to B(H)$ be an irreducible representation; by Lemma 7 A has no non-trivial ideals so π must be faithful. By [14, Proposition 4.8] A must contain a rank one idempotent p: we claim that $P = \pi(p) \in B(H)$ is also rank one. For suppose $\xi \in PH$, $\xi \neq 0$ and $\eta \in P^*H$ satisfies $(\xi|\eta) = 0$. Then for any $a \in A$

$$(\eta|\pi(a)\xi) = (P^*\eta|\pi(a)P\xi) = (\eta|\pi(pap)\xi) \in \mathbf{C}(\eta|\xi) = 0.$$

Since $P^*H = ((I - P)H)^{\perp}$ this says that

$$\eta \perp \xi, \eta \perp (I - P)H \Rightarrow \eta \perp \pi(A)\xi$$

and because π is irreducible it follows that ξ and $(I - P)H$ span H. Thus $PH = \mathbf{C}\xi$ and P is rank one. The irreducibility of π implies that $\pi(A) \supset K(H)$, and as A has no ideals $\pi(A) = K(H)$. But the latter consists of finite rank operators so H must be finite-dimensional. This will be a $*$-isomorphism if P is chosen so that $P^* = P$.

PROOF OF PROPOSITION 5(1). Let π: $A \to B(H)$ be irreducible. By Lemma 7 $\ker \pi$ is a regular ideal of the form IA for some ideal I in $C(T)$. The extension $\bar{\pi}$ of π to the multiplier algebra restricts to a representation of $C(T)$ in $\pi(A)' = \mathbf{C}1$, and hence $\ker \bar{\pi}$ is the ideal I_t of functions vanishing at some point t of T. We then have $I_tA \subset \ker \pi$ and the maximality of I_t shows that $I_tA = \ker \pi$. Thus by [14, Propositions 2.7 and 3.5] $A/\ker \pi$ is central separable over $C(T)/I_t \cong \mathbf{C}$, hence isomorphic to $M_n(\mathbf{C})$ by Lemma 8, and π is finite-dimensional.

We now prove that A satisfies Fell's condition. Let $\pi \in \hat{A}$ and choose $a \in A$ such that $\pi(a)$ is a rank one projection. The map $\rho \to \|\rho(a)\|$ is continuous on $\hat{A}$, so $\|\rho(a)^2 - \rho(a)\|$ is small for ρ near π, and if $f \equiv 1$ near 1, $f \equiv 0$ near 0 then $p = f(a)$ will satisfy $\rho(p)^2 = \rho(p) = \rho(p)^*$ for ρ near π. If N is a compact neighbourhood of π then $pAp|_N = A/I_NA$ is central separable over $C(N)$ and has an identity; therefore by shrinking N we may suppose $pAp|_N \cong C(N, M_n)$ for some n. As $\pi(p)$ is rank one, $n = 1$ and $\rho(p)$ is rank one throughout N. Thus A has continuous trace.

Suppose now that $\pi_n \in \hat{A}$ and $\dim \pi_n \geqslant n$ for all n. As $\hat{A}$ is compact, we may assume $\pi_n \to \pi$ (technically, we might have to pass to a subnet, but the idea's the same). Pick $a_n \in A$ with $a_n \geqslant 0$ and rank $\pi_n(a_n) \geqslant n$, and let $a = \sum_{k=1}^{\infty} 2^{-k} a_k$.

Then $a \geqslant 0$ and we have

$$\operatorname{rank} \pi_n(a) = \operatorname{rank}\left(\sum_{k=1}^{\infty} \pi_n(2^{-k}a_k) \right) \geqslant \operatorname{rank} \pi_n(2^{-n}a_n) \geqslant n.$$

By restricting to a compact neighbourhood N of π we may suppose that A has a rank one idempotent p, and then by Lemma 6 multiplication gives an isomorphism μ: $Ap \otimes_{C(N)} pA \to A$. In particular, we can write

$$a = \mu\left(\sum_{i=1}^{m} a_i p \otimes pb_i \right) = \sum_{i=1}^{m} a_i pb_i;$$

but this is impossible since the rank of $\pi_n(\Sigma a_i pb_i)$ is at most m for each n.

The proof of the second part of Proposition 5 consists of building an example. We are grateful to Shaun Disney for providing the following topological lemma.

LEMMA 9. *Let L_n be the canonical complex line bundle over complex projective space $\mathbf{C}P^n$. Then any n sections of L_n have a common zero.*

PROOF. Let $\xi_1, \ldots, \xi_n$ be n sections of $L = L_n$, and suppose they do not simultaneously vanish. Then the direct sum nL of n copies of L has a non-vanishing section, and so can be decomposed as $nL \cong \mathbf{1} \oplus F$, where $\mathbf{1}$ denotes the trivial line bundle. The first Chern class of $\mathbf{1}$ is 0, so

$$c_1(L)^n = c_n(nL) = c_1(\mathbf{1})c_{n-1}(F) = 0.$$

But the cohomology ring $H^*(\mathbf{C}P^n, \mathbf{Z})$ is a truncated polynomial ring generated by $c_1(L)$, and in particular $H^{2n}(\mathbf{C}P^n, \mathbf{Z}) \cong \mathbf{Z}$ is generated by $c_1(L)^n$. We therefore have a contradiction, and the ξ_i must vanish simultaneously.

COROLLARY 10. *Let L_n be the field of Hilbert spaces over $\mathbf{C}P^n$ obtained by putting a Hermitian structure on the canonical line bundle, and let $A_n = \Gamma(\mathfrak{A}(L_n))$ be the C^*-algebra defined by L_n. Then the identity in A_n cannot be written in the form $\sum_{i=1}^{m} \xi_i \otimes \bar{\eta}_i$ for $\xi_i, \eta_i \in \Gamma(L_n)$ unless $m > n$. (In fact $A_n \cong C(\mathbf{C}P^n)$, but this is not important here.)*

PROOF OF PROPOSITION 5(2). Let L_n be as in Corollary 10, and define a field of Hilbert spaces K over the disjoint union $X = \bigcup_{n=1}^{\infty} \mathbf{C}P^n$ by taking $K = L_n$ on $\mathbf{C}P^n$. We now define a field H over the compactification $T = X \cup \{\infty\}$ by

$$H(x) = \mathbf{C} \oplus K(x) \cong \mathbf{C}^2, \qquad H(\infty) = \mathbf{C}, \qquad \Gamma(H) = C(T) \oplus \Gamma_0(K).$$

Let A be the C^*-algebra $\Gamma(\mathfrak{A}(H))$ defined by H; we claim that A is not just $\Gamma(H) \otimes_{C(T)} \Gamma(\overline{H})$. For A contains the closure of $\Gamma_0(K) \otimes_{C(T)} \Gamma_0(K)$, which is the c_0-direct sum of the C^*-algebras A_n. If we define $f \in A_n$ by

$$f(x) = \frac{1}{n} 1_{K(x)} \quad \text{if } x \in \mathbf{C}P^n,$$

then f cannot be written in the form $\sum_{i=1}^m \xi_i \otimes \bar{\eta}_i$ for any finite m, and so does not belong to the algebraic tensor product $\Gamma_0(K) \otimes \Gamma_0(\bar{K})$; this justifies the claim. The algebra A contains the idempotent $p = 1_{C(T)} \otimes 1_{C(T)}$, and it is easy to see that $pAp = C(T)$, so p is rank one, and if A were central separable we would have $A \cong Ap \otimes_{C(T)} pA$ by Lemma 6. However, $Ap = \Gamma(H)$ so we have just shown this is not the case.

Finally, we observe that, although continuous trace C^*-algebras with compact spectrum are not in general central separable, they do always have a dense ideal which is. For any continuous trace C^*-algebra with spectrum T can be constructed from a cover $\{N_i\}$ of T, fields of Hilbert spaces H_i over N_i, and isomorphisms $h_{ij}: H_j|_{N_{ij}} \to H_i|_{N_{ij}}$ which satisfy

$$\operatorname{Ad} h_{ij}(t) \circ \operatorname{Ad} h_{jk}(t) = \operatorname{Ad} h_{jk}(t) \quad \text{for } t \in N_{ijk}$$

[3, 10.7.11]. For convenience we suppose N_i is compact. Then the algebraic tensor product $A_i = \Gamma(H_i) \otimes_{C(N_i)} \Gamma(\bar{H}_i)$ is a central separable $C(N_i)$-algebra, and the isomorphisms $\operatorname{Ad} h_{ij}$ map $A_j|_{N_{ij}}$ onto $A_i|_{N_{ij}}$. We can therefore use them to piece together a central separable algebra (cf. the proof of Proposition 4) which is clearly dense in A. Conversely, if A is a C^*-algebra with Hausdorff spectrum T and A contains a dense central separable $C(T)$-subalgebra then as in the proof of Proposition 5 it is not hard to see that A satisfies Fell's condition. Thus this property characterises continuous trace C^*-algebras.

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School of Mathematics
University of New South Wales
Post Office Box 1
Kensington, NSW, 2033
Australia

(usual address of J. L. Taylor:
Department of Mathematics
University of Utah
Salt Lake City
Utah 84112, U.S.A.)