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THE POSITIVE PART OF A FOURIER TRANSFORM

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Abstract

We consider the function u whose Fourier transform is the positive part of the Fourier transform of a function f on $\mathbf{R}^n$. If $n \leqslant 2$ and f satisfies simple regularity conditions (in particular if f is in the Schwartz space $\mathscr{S}(\mathbf{R}^n)$), then u lies in $L^1(\mathbf{R}^n)$. If $n \geqslant 3$, then simple counterexamples exist; for example, if $f(x) = |x|^2 \exp(-|x|^2)$, then u does not lie in $L^1(\mathbf{R}^n)$.

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Let $\mathscr{F}$ denote the Fourier transformation; for $f \in L^1(\mathbf{R}^n)$,

$$\hat{f}(\xi) = \mathscr{F}f(\xi) = \int_{\mathbf{R}^n} f(x)e^{-2\pi i x\xi}\,dx, \qquad \xi \in \mathbf{R}^n.$$

Under appropriate hypotheses on $\hat{f}$, f can be recovered from $\hat{f}$ by the inverse transformation $\mathscr{F}^{-1}$: $(\mathscr{F}^{-1}g)(x) = \mathscr{F}g(-x)$. On the other hand, $\hat{f}$ may be decomposed: $\hat{f}(\xi) = r^+(\xi) - r^-(\xi) + ij^+(\xi) - ij^-(\xi)$ where, for instance, $r^+ = \mathrm{Re}(\hat{f})_+$.

Such a decomposition is natural in many contexts; for instance, if we think of $\hat{f}$ as a measure, or if f acts as an operator on (say) $L^2(\mathbf{R}^n)$ by convolution. Here we consider $\mathscr{F}^{-1}(r^+)$ and ask if $\mathscr{F}^{-1}(r^+) \in L^1(\mathbf{R}^n)$ provided f lies in some nice sub-class of $L^1(\mathbf{R}^n)$, for instance $\mathscr{S}(\mathbf{R}^n)$ or $\mathscr{D}(\mathbf{R}^n)$.

The answer depends on n: if $n \leqslant 2$ then mild conditions on f assure that $\mathscr{F}^{-1}(r^+) \in L^1(\mathbf{R}^n)$, while if $n \geqslant 3$ $\mathscr{F}^{-1}(r^+) \notin L^1(\mathbf{R}^n)$ unless rather odd conditions are imposed (the zeroes of $\mathrm{Re}(\hat{f})$ have to be of order greater than 1 (roughly speaking) and it is not obvious how this can be read off from f).

In what follows we maintain the notation r^+ for the positive part of $\hat{f}$; also r will be the real part of $\hat{f}$.

Case 1. The case $n \geqslant 3$.

If $f \in L^1(\mathbb{R}^n)$ and f is radial then ([1], page 35) $\hat{f}$ is continuously differentiable on $\mathbb{R}^n \setminus \{0\}$; r^+ will be continuously differentiable only if $r \geqslant 0$ or if the zeroes of r are of order $\geqslant 2$ and so, in general, $\mathscr{F}^{-1}(r^+) \notin L^1(\mathbb{R}^n)$.

Case 2. The case $n \leqslant 2$.

This case is somewhat subtler. We enunciate and prove our theorem after establishing a preliminary lemma, due to Michael Cowling.

LEMMA. *Suppose that* $r\colon \mathbb{R} \to \mathbb{R}$ *is twice continuously differentiable and that* $r, r' \in C_0(\mathbb{R})$. *If* $u(x) = \int_{\mathbb{R}} d\xi r^+(\xi) e^{2\pi i x\xi}$, *then*

$$|u(x)| \leqslant \frac{3}{8\pi^2 x^2} \int_{\mathbb{R}} d\xi |r''(\xi)|.$$

PROOF. Let $D = \{\xi \in \mathbb{R}\colon r(\xi) > 0\}$. Then

$$u(x) = \int_D d\xi r(\xi) e^{2\pi i x\xi}.$$

We may break up D into a countable union of disjoint open intervals $I_n = (a_n, b_n)$ which we assume are all finite for the moment. Then

$$u(x) = \sum_n \int_{I_n} d\xi r(\xi) e^{2\pi i x\xi}.$$

Integration by parts, together with the fact that $r(a_n) = r(b_n) = 0$, shows that

$$\begin{aligned} u(x) &= -\sum_n (2\pi i x)^{-1} \int_{I_n} d\xi r'(\xi) e^{2\pi i x\xi} \\ &= -\sum_n (2\pi i x\xi)^{-2} \int_{I_n} d\xi r'(\xi) (e^{2\pi i x\xi})' \\ &= \sum_n (2\pi x)^{-2} \left[r'(b_n) - r'(a_n) - \int_{I_n} d\xi r''(\xi) e^{2\pi i x\xi} \right]. \end{aligned}$$

So

$$|u(x)| \leqslant \sum_n (2\pi x)^{-2} \left[|r'(b_n)| + |r'(a_n)| + \int_{I_n} d\xi |r''(\xi)| \right].$$

It is enough to show that

$$\sum_n \left[|r'(b_n)| + |r'(a_n)| \right] \leqslant \frac{1}{2} \int_{I_n} d\xi |r''(\xi)|$$

to conclude the proof.

Since $r(a_n) = 0$ and $r(a_n + \varepsilon) > 0$ for small ε, $r'(a_n) \geqslant 0$. Similarly $r'(b_n) \leqslant 0$. If $r'(a_n) = 0$, put $a'_n = a_n$ and otherwise let $a'_n = \sup\{\xi \in \mathbb{R}\colon \xi < a_n, r'(\xi) = 0\}$.

Similarly if $r'(b_n) = 0$, put $b'_n = b_n$, and otherwise let $b'_n = \inf\{\xi \in \mathbb{R}: \xi > b_n, r'(\xi) = 0\}$. Further, by Rolle's theorem, there exists c_n in (a_n, b_n) such that $r'(c_n) = 0$. Let $I'_n(a'_n, b'_n)$. By construction the intervals I'_n are disjoint, and so it will suffice to show that

$$|r'(a_n)| + |r'(b_n)| \leqslant \frac{1}{2}\int_{I'_n} d\xi |r''(\xi)|.$$

This is easy:

$$r'(a_n) = \int_{a'_n}^{a_n} d\xi r''(\xi) = -\int_{a_n}^{c_n} d\xi r''(\xi)$$

so

$$2|r'(a_n)| \leqslant \int_{a'_n}^{a_n} d\xi |r''(\xi)| + \int_{a_n}^{c_n} d\xi |r''(\xi)|,$$

and

$$2|r'(b_n)| \leqslant \int_{c_n}^{b_n} d\xi |r''(\xi)| + \int_{b_n}^{b'_n} d\xi |r''(\xi)|$$

analogously. If some I_n is infinite, the above arguments no longer make sense but the necessary modifications are easy.

Now we can prove our theorems.

THEOREM 1. *If the function $x \to (1 + |x|) f(x)$ is in $L^2(\mathbb{R})$ and if $\hat{f}(\xi) = r^+(\xi) - r^-(\xi) + ij^+(\xi) - ij^-(\xi)$ is the decomposition of f into positive and negative real and imaginary parts, then $\mathscr{F}^{-1}(r^+)$ is in $L^1(\mathbb{R})$ and*

$$\|\mathscr{F}^{-1}(r^+)\|_1 \leqslant 2^{-1/2}\left[\int_{-\infty}^{\infty} dx(1 + 4\pi^2x^2)|f(x)|^2\right]^{1/2}.$$

PROOF. Let $D = \{\xi \in \mathbb{R}: r(\xi) > 0\}$, where $r = \operatorname{Re}(\hat{f})$. Distributionally, $r^+ = \chi_D r$ and $(r^+)' = (\chi_D)'r + \chi_D r'$. Now $f \in L^1(\mathbb{R})$ (by Cauchy-Schwarz) so r is continuous and, in particular, $r = 0$ where $\chi'_D \neq 0$. Thus $(r^+)' = \chi_D r'$. Now

$$\begin{aligned}
\int_{\mathbb{R}} dx |\mathscr{F}^{-1}(r^+)| &\leqslant \left[\int_{\mathbb{R}} dx(1 + 4\pi^2x^2)^{-1}\right]^{1/2} \\
&\quad \cdot \left[\int_{\mathbb{R}} dx(1 + 4\pi^2x^2)|\mathscr{F}^{-1}(r^+)(x)|^2\right] \\
&\leqslant \left(\frac{1}{2}\right)^{1/2}\left[\int_{\mathbb{R}} d\xi |r^+(\xi)|^2 + |(r^+)'(\xi)|^2\right]^{1/2} \\
&\leqslant 2^{-1/2}\left[\int_{\mathbb{R}} d\xi |r(\xi)|^2 + |r'(\xi)|^2\right]^{1/2} \\
&\leqslant 2^{-1/2}\left[\int_{\mathbb{R}} d\xi |f(\xi)|^2 + |f'(\xi)|^2\right]^{1/2} \\
&= 2^{-1/2}\left[\int_{\mathbb{R}} dx(1 + 4\pi^2x^2)|f(x)|^2\right]^{1/2},
\end{aligned}$$

by Cauchy-Schwarz' and Plancherel's theorems.

Theorem 2. *Suppose that the functions* $(x, y) \to (1 + x^2 + y^2)f(x, y)$, $(\partial/\partial x)x^2 f(x, y)$ *and* $(\partial/\partial y)y^2 f(x, y)$ *are in* $L^2(\mathbb{R}^2)$ *and that* $\hat{f}(\xi, \eta) = r^+(\xi, \eta) - r^-(\xi, \eta) + ij^+(\xi, \eta) - ij^-(\xi, \eta)$ *is the decomposition of* $\hat{f}$ *into positive and negative real and imaginary parts. Then* $\mathscr{F}^{-1}(r^+) \in L^1(\mathbb{R}^2)$ *and*

$$\|\mathscr{F}^{-1}(r^+)\|_1 \leqslant 2\|f\|_2 + 6\left\|x^2 f + \frac{\partial}{\partial x}(x^2 f)\right\|_2 + 6\left\|y^2 f + \frac{\partial}{\partial y}(y^2 f)\right\|_2.$$

Proof. It is obvious that

$$\int_{-1}^{+1} dx \int_{-1}^{+1} dy |\mathscr{F}^{-1}(r^+)(x, y)| \leqslant 2\left[\int_{-1}^{+1} dx \int_{-1}^{+1} dy |\mathscr{F}^{-1}(r^+)(x, y)|^2\right]^{1/2}$$
$$\leqslant 2\|\mathscr{F}^{-1}(r^+)\|_2 = 2\|r^+\|_2 \leqslant 2\|\hat{f}\|_2 = 2\|f\|_2.$$

We shall show now that

$$\int_1^\infty dx \int_{-x}^{x} dy |\mathscr{F}^{-1}(r^+)(x, y)| \leqslant 3\left\|x^2 f + \frac{\partial}{\partial x}(x^2 f)\right\|_2.$$

By symmetry, analogous estimates hold for the three integrals

$$\int_{-\infty}^{-1} dx \int_{x}^{-x} dy |\mathscr{F}^{-1}(r^+)(x, y)|, \quad \int_1^\infty dy \int_{-y}^{y} dx |\mathscr{F}^{-1}(r^+)(x, y)|$$

and

$$\int_{-\infty}^{-1} dy \int_{y}^{-y} dx |\mathscr{F}^{-1}(r^+)(x, y)|,$$

whence the theorem follows.

Observe that

$$\int_1^\infty dx \int_{-x}^{x} dy |\mathscr{F}^{-1}(r^+)(x, y)| \leqslant \int_1^\infty dx (2x)^{1/2}\left[\int_{-x}^{x} dy |\mathscr{F}^{-1}(r^+)(x, y)|^2\right]^{1/2}$$
$$\leqslant \int_1^\infty dx (2x)^{1/2}\left[\int_{-\infty}^{\infty} dy |\mathscr{F}^{-1}(r^+)(x, y)|^2\right]^{1/2}$$
$$= \int_1^\infty dx (2x)^{1/2}\left[\int_{-\infty}^{\infty} d\eta |u(x, \eta)|^2\right]^{1/2},$$

where $u(x, \eta) = \int_{-\infty}^{\infty} d\xi r^+(\xi, \eta) e^{2\pi i x \xi}$. The hypotheses of the theorem imply that $xf \in L^1(\mathbb{R}^2)$ and that $f \in L^1(\mathbb{R}^2)$, so $\hat{f}$ and $\partial/\partial\xi(\hat{f}) \in C_0(\mathbb{R}^2)$. By the lemma

$$|u(x, \eta)| \leqslant \frac{3}{8\pi^2 x^2} \int_{\mathbb{R}} d\xi \left|\frac{\partial^2}{\partial \xi^2} r(\xi, \eta)\right|,$$

so

$$\int_1^\infty dx \int_{-x}^x dy |\mathscr{F}^{-1}(r^+)(x, y)|$$
$$\leqslant \int_1^\infty dx \frac{3\sqrt{2}}{8\pi^2} x^{-3/2} \left[\int_{-\infty}^\infty d\eta \left\{ \int_{-\infty}^\infty d\xi \left| \frac{\partial^2}{\partial \xi^2} r(\xi, \eta) \right| \right\}^2 \right]^{1/2}$$
$$\leqslant \frac{3\sqrt{2}}{4\pi^2} \left[\int_{-\infty}^\infty d\eta \left(\frac{1}{2} \right) \left\{ \int_{-\infty}^\infty d\xi |1 + 4\pi^2 \xi^2| \left| \frac{\partial^2}{\partial \xi^2} r(\xi, \eta) \right|^2 \right\} \right]^{1/2}$$

by the Cauchy-Schwarz inequality and the explicit result that

$$\int_{-\infty}^\infty d\xi |1 + 4\pi^2 \xi^2|^{-1} = \frac{1}{2}.$$

Thus

$$\int_1^\infty dx \int_{-x}^x dy |\mathscr{F}^{-1}(r^+)(x, y)|$$
$$\leqslant \frac{3}{4\pi^2} \left[\int_{-\infty}^\infty d\eta \int_{-\infty}^\infty d\xi \left| (1 - 2\pi i \xi) \frac{\partial^2}{\partial \xi^2} \hat{f}(\xi, \eta) \right|^2 \right]^{1/2}$$
$$= 3 \left[\int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \left| \left(1 + \frac{\partial}{\partial x} \right) x^2 f(x, y) \right|^2 \right]^{1/2}$$

as required.

References

[1] H. Reiter, *Classical harmonic analysis and locally compact groups* (Oxford University Press, Oxford, 1968).

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