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SOME GROUPS WITH T_1 PRIMITIVE IDEAL SPACES

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Abstract

Let G be a second countable locally compact group possessing a normal subgroup N with G/N abelian. We prove that if G/N is discrete then G has T_1 primitive ideal space if and only if the G-quasiorbits in Prim N are closed. This condition on G-quasiorbits arose in Pukanzky's work on connected and simply connected solvable Lie groups where it is equivalent to the condition of Auslander and Moore that G be type R on N (-nilradical). Using an abstract version of Pukanzky's arguments due to Green and Pedersen we establish that if G is a connected and simply connected Lie group then Prim G is T_1 whenever G-quasiorbits in $[G, G]\hat{}$ are closed.

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1. Introduction

Let G be a second countable locally compact group. We denote by $C^*(G)$ the C^*-algebra of the group G and by Prim G the primitive ideal space of $C^*(G)$ (see [3] and [9] for definitions). We are interested in the question of when G has T_1 primitive ideal space, or more precisely, if G has a normal subgroup N with G/N abelian we ask: under what conditions does G have T_1 primitive ideal space? Posed in this generality this is a difficult problem.

The motivation for this note is our observation that implicit in a result of Pukanzky ([11], Lemma 30) is the statement that for a connected solvable Lie group G with $N =$ nilradical, G has T_1-primitive ideal space whenever the G-quasiorbits in Prim N are closed. (A G-quasiorbit in Prim N is an equivalence class under the relation $I_1 \sim I_2$ if I_1 is in the closure of the G-orbit through I_2 and I_2 is in the closure of the G-orbit through I_1.) In this case of course N is CCR and so if G is type I this proves that G is CCR. (Pukanzky also shows that this condition on G-quasiorbits in $\hat{N}$ is equivalent to the condition of Auslander and Moore [1] that G be type R on its nilradical.)

The conjecture suggested by these results of Pukanzky is clearly whether G has T_1 primitive ideal space if and only if G-quasiorbits in Prim N are closed. Without conditions on N and G/N it is unlikely that the conjecture is true. The simplest possible case is covered by

THEOREM 1.1. *If G/N is discrete abelian then G has T_1 primitive ideal space if and only if the G-quasiorbits in* Prim N *are closed.*

The proof of this theorem relies on a number of results of Green [6] and uses some ideas exploited by us in another context [2]. The first step in the proof is to "localise" the problem. If ρ is a representation of $C^*(G)$ with kernel some primitive ideal J then ρ determines a representation of $C^*(N)$ (by restricting ρ from G to N), written $\rho|_{C^*(N)}$. This representation may be written as a direct integral over Prim N of representations of $C^*(N)$ with respect to some measure which is concentrated on a G-quasiorbit θ. This quasiorbit is independent of ρ and we say that J restricts to θ (see [5] for more details). If the G-quasiorbits in Prim N are closed then the subset of Prim G consisting of primitive ideals which restrict on N to θ is closed in the hull-kernel topology on Prim G because restriction is a continuous map. Hence it suffices to show that primitive ideals of G are maximal in the subset of Prim G lying over θ. As θ is closed there is an ideal I of $C^*(N)$ such that $A \triangleq C^*(N)/I$ has θ as its primitive ideal space. Moreover the subset of Prim G which restricts to θ is the primitive ideal space of the twisted covariance algebra $C^*(G, A, \tau_N)$ (see [6], Proposition 12). We refer to [6] for the definition and properties of $C^*(G, A, \tau_N)$ although in Section 2 where this C^*-algebra is used to define it by a faithful representation which is rather easily described. Let A be a separable C^*-algebra such that the twisted covariance algebra $C^*(G, A, \tau_N)$ may be defined. Then this localisation argument makes Theorem 1.1 a special case of

THEOREM 1.2. *If G/N is discrete abelian then $C^*(G, A, \tau_N)$ has T_1-primitive ideal space if and only if G-quasiorbits in* Prim A *are closed.*

In Section 2 we establish the "only if" part of the theorem. In the other direction the discussion preceding Theorem 1.2 means that it is sufficient to establish that if A has no G-invariant non-trivial ideals (that is, A is G-simple) then $C^*(G, A, \tau_N)$ has T_1 primitive ideal space (provided of course G/N is discrete abelian).

In Section 2 we establish

PROPOSITION 1.3. *If G/N is abelian then there is a natural dual action of $(G/N)\hat{}$ on $C^*(G, A, \tau_N)$ with the property that when $(G/N)\hat{}$ is compact $C^*(G, A, \tau_N)$ is $(G/N)\hat{}$ simple whenever A is G-simple.*

A by-product of the proof of this result is an extension of Takai duality to certain twisted covariance algebras (an extension which is, in fact, implicit in [7]).

Given Proposition 1.3 we can now prove the "if" part of Theorem 1.2 and hence of Theorem 1.1. If $C^*(G, A, \tau_N)$ is $(G/N)\hat{}$ simple (which case we can certainly reduce to using Green's localisation plus Proposition 1.3) then there can only be one $(G/N)\hat{}$-quasiorbit in $\operatorname{Prim} C^*(G, A, \tau_N)$. By Dixmier ([3]) there exists a minimal primitive ideal I of $C^*(G, A, \tau_N)$. Now for any other primitive ideal J we can find a sequence $\{\gamma_n\} \in (G/N)\hat{}$ such that $\gamma_n \cdot J \to I$. As $(G/N)\hat{}$ is compact we can assume $\gamma_n \to \gamma$ for some $\gamma \in (G/N)\hat{}$. Thus I is in the closure of $\gamma \cdot J$. But I is minimal so $I = \gamma \cdot J$. So J is minimal. Thus all primitive ideals of $C^*(G, A, \tau_N)$ are minimal and hence also maximal. So $\operatorname{Prim} C^*(G, A, \tau_N)$ is T_1.

We isolate from this discussion:

DEFINITION 1.4. We say that $\operatorname{Prim} C^*(G, A, \tau_N)$ is *fibered over quasiorbits* in $\operatorname{Prim} A$ if whenever J_1 and J_2 are two primitive ideals of $C^*(G, A, \tau_N)$ which restrict on A to the same quasiorbit in $\operatorname{Prim} A$, there is a $\gamma \in (G/N)\hat{}$ with $\gamma \cdot J_1 = J_2$.

Thus the argument above shows that for A G-simple, with $(G/N)\hat{}$ compact, $\operatorname{Prim} C^*(G, A, \tau_N)$ is fibred over quasiorbits in $\operatorname{Prim} A$. This fibering property has proved useful in the study of nilpotent groups [2].

Fibering is unlikely to hold in any generality, although as is easily shown in Section 3, it is sufficient to guarantee that $\operatorname{Prim} C^*(G, A, \tau_N)$ is T_1.

We return in the final part of Section 3 to the motivating example analysed by Pukanzky [12], [11]. By using an abstraction of his setting due to Green [6] (see Pedersen [10]) we prove

THEOREM 1.5. *If G is a connected, simply connected Lie group then* $\operatorname{Prim} G$ *is T_1 whenever G-quasiorbits in $\hat{N} = [G, G]\hat{}$ are closed.*

2. Twisted covariance algebras

Let A be a separable C^* algebra with $g \to \alpha_g$ a strongly continuous homomorphism of G into $\operatorname{Aut} A$. Then G acts on $\operatorname{Prim} A$ and we suppose that there is a fixed normal subgroup N of G with G/N abelian such that the stabiliser of every element of Prim A contains N. If there is a map $\tau: N \to \mathfrak{M}(A)$ (the multiplier algebra of A) such that $\alpha_n = \operatorname{ad} \tau(n)$ for all $n \in N$ then we can form the twisted covariance algebra $C^*(G, A, \tau_N)$ as in [6]. Then there is a dense subalgebra $C_c(G, A, \tau_N)$ consisting of continuous functions f from G to A of compact support, satisfying $f(ns) = f(s)\tau_N(n)^{-1}$, $n \in N$, $s \in G$. (The subscript c will always denote functions of compact support.)

We define an action of $(G/N)\hat{}$ on $C^*(G, A, \tau_N)$:

$$(\hat{\alpha}_\gamma f)(g) = \gamma(g) f(g), \qquad \gamma \in (G/N)\hat{}, f \in C_c(G, A, \tau_N).$$

We can now form the cross-product of $C^*(G, A, \tau_N)$ by $(G/N)\hat{}$ which we will write as $C^*((G/N)\hat{}, C^*(G, A, \tau_N))$ (see [9], 7.8.3). On the other hand $(G/N)\hat{}$ acts trivially on A so we may form $C^*((G/N)\hat{}, A)$ (which is isomorphic to $C^*((G/N)\hat{}) \otimes A$) and define an action of G by

$$(g \cdot \phi)(\gamma) = \overline{\gamma(g)}\, \alpha_g(\phi(\gamma)), \quad \text{for } \phi \in C_0((G/N)\hat{}, A).$$

We can define a twisting map τ_N': $N \to \mathfrak{M}(C^*(G/N)\hat{} \otimes A)$ (the $\mathfrak{M}$ stands for multiplier algebra) by

$$(\tau_N'(n)\phi)(\gamma) = \tau_N(n)(\phi(\gamma)), \qquad \phi \in C_c((G/N)\hat{}, A).$$

The twisted covariance algebra $C^*(G, C^*((G/N)\hat{}, A, \tau_N'))$ may now also be constructed. However, rather than give details on the construction, we will identify these algebras with their faithful realisations described below.

We begin with a faithful representation π of A in a Hilbert space $\mathcal{H}$ and note that τ_N: $A \to \mathfrak{M}(a)$ is a continuous map into the unitaries in the multiplier algebra of A satisfying $\tau_N(n) a \tau_N(n) = \alpha_n(a)$ and $\tau_N(gng^{-1}) = \alpha_g(\tau_N(n))$ for all $n \in N$, $g \in G$, $a \in A$. Thus we can consider functions ψ: $G \to \mathcal{H}$ satisfying

$$\psi(ng) = \pi\left(\tau_N(n)^{-1}\right)\psi(g),$$

$$\|\psi\|^2 = \int_{G/N} \|\psi(g)\|^2\, dNg < \infty.$$

These functions define a Hilbert space $\mathcal{F}$ on which $C_c(G, A, \tau_N)$ acts via the representation ρ:

$$(\rho(f)\psi)(g') = \int \left(M(f(g)) V_g \psi\right)(g')\, dg, \qquad f \in C_c(G, A, \tau),$$

where V is the representation of G:

$$V(g)\psi(g') = \psi(g'g), \qquad g, g' \in G,$$

and M is the representation of A:

$$(M(a)\psi)(g') = \pi\left(\alpha_{g'}(a)\right)\psi(g'), \qquad a \in A, g' \in G.$$

All we are doing here is constructing ρ as the representation of $C_c(G, A, \tau_N)$ induced by π so ([6], Proposition 13) ρ extends to a faithful representation of $C^*(G, A, \tau_N)$ as G/N is abelian (and hence amenable). We note there is a representation U of $(G/N)\hat{}$ on $\mathcal{F}$ defined by

$$U_\gamma\psi(g) = \gamma(g)\psi(g), \qquad \psi \in \mathcal{F}, \gamma \in (G/N)\hat{},$$

from which the relation

$$U_\gamma\rho(f)U_\gamma^{-1} = \rho(\hat{\alpha}_\gamma(f)), \qquad f \in C_c(G, A, \tau_N),$$

follows. This shows that $\hat{\alpha}_\gamma$ extends to an automorphism of $C^*(G, A, \tau_N)$ and hence that (as in [9], 7.8.3) we can indeed form the cross-product $C^*((G/N)\hat{}, C^*(G, A, \tau_N))$. This cross-product can be faithfully represented via its regular representation on $L^2((G/N)\hat{}, \mathcal{F})$. We will realize this representation on functions Ψ: $(G/N)\hat{} \times G \to \mathcal{H}$ with

$$\int \|\Psi(\gamma, g)\|^2 \, d\gamma \, dNg < \infty,$$

$$\Psi(\gamma, ng) = \pi(\tau_N(n))^{-1}\Psi(\gamma, g),$$

and define for $u \in C_c((G/N)\hat{}, C_c(G, A, \tau_N))$ regarded as a function from $(G/N)\hat{} \times G$ into A:

$$(u \cdot \Psi)(\gamma', g') = \iint \pi(u(\gamma, g))\Psi(\gamma'\gamma, g'g) \, d\gamma \, dg.$$

Similarly, we produce a faithful representation of $C^*(G, C^*((G/N)\hat{}, A), \tau_N')$. Consider functions Φ: $G \times (G/N)\hat{} \to A$ satisfying

$$\int \|\Phi(g, \gamma)\|^2 \, dNg \, d\gamma < \infty,$$

$$\Phi(ng, \gamma) = \pi(\tau_N(n))^{-1}\Phi(g, \gamma)$$

and define for $v \in C_c(G, C^*((G/N)\hat{}, A), \tau_N')$

$$(v \cdot \Phi)(g', \gamma') = \int \pi(v(g, \gamma))\Phi(g'g, \gamma'\gamma) \, d\gamma \, dNg.$$

(We remark that this extends to a faithful representation of the corresponding twisted covariance algebra by ([6], Proposition 13), again using amenability of G/N and the fact that this is nothing more than the representation of $C^*(G, C^*((G/N)\hat{}, A), \tau_N')$ induced from the regular representation of $C^*((G/N)\hat{}, A)$.)

LEMMA 2.1. *$C^*((G/N)\hat{}, C^*(G, A, \tau_N))$ is isomorphic to $C^*(G, C^*((G/N)\hat{}, A), \tau_N')$. Moreover the isomorphism fixes the copies of $C^*(G, A, \tau_N)$ in the respective multiplier algebras.*

PROOF. This is now a routine argument using the two concrete realisations defined above and the method of ([9], 7.9.2). By regarding the multiplier algebras as subalgebras of the bounded operators on the appropriate Hilbert spaces, the second statement is also clear from the proof of 7.9.2 in [9].

LEMMA 2.2. $C^*(G, C^*((G/N)\hat{}, A), \tau') \triangleq A \otimes \mathcal{K}(L^2(G/N))$.

PROOF. This result is due to Green [7] using the isomorphism $C^*(N, A, \tau_N) \triangleq A$ and writing $\mathcal{K}(L^2(G/N))$ for the compact operators on $L^2(G/N)$.

With these preliminaries out of the way we move on to the main result of this section. There is an action of G on $C^*(G, A, \tau_N)$ which embeds G in the multiplier algebra of $C^*(G, A, \tau_N)$, namely:

$$(\delta_{g'} \cdot f)(g) = \alpha_{g'} f(g'^{-1}g), \qquad f \in C_c(G, A, \tau), g, g' \in G.$$

(This is the left multiplication by Dirac measure at $g' \in G$.) Then

$$(\delta_{g'} a \delta_{g'}^{-1} f)(g) = (a\alpha_{g'}^{-1}(f(g))) = \alpha_{g'}(a) f(g).$$

So $\delta_{g'} a \delta_{g'}^{-1} = \alpha_{g'}(a)$. If we assume $(G/N)\hat{}$ is compact then we can define a map I: $\mathfrak{M}(C^*(G, A, \tau_N)) \to \mathfrak{M}(C^*(G, A, \tau_N))$ by

$$I(x) = \int_{(G/N)\hat{}} \hat{\alpha}_\gamma(x)\, d\gamma.$$

Clearly I is continuous. Notice that if $f \in C_c(G, A, \tau_N)$ then

$$I(f)(g) = \int \hat{\alpha}_\gamma(f)(g)\, d\gamma = \int \overline{\gamma(g)}\, d\gamma \cdot f(g).$$

Thus $I(f)(g) = 0$ unless $g \in N$ in which case

$$I(f)(n) = f(n) = \tau_N(n)^{-1} f(e) \in A, n \in N.$$

Thus I is just the conditional expectation $f \to f|_N$ so that I extends from $C_c(G, A, \tau_N)$ to define a map from $C^*(G, A, \tau_N)$ into A.

LEMMA 2.3. *If J is a $(G/N)\hat{}$ invariant ideal of $C^*(G, A, \tau_N)$ then $I(J)$ is a non-zero G-invariant ideal of A.*

PROOF. Let $\{f_i\}_{i=1}^\infty$ be a sequence in $L^1(G/N)$ and $x \in C^*(G, A, \tau_N)$. Notice that the function $g \to f_i(Ng)I(x\delta_{g^{-1}})\delta_g$ is actually a function on G/N since $I(x\delta_{g^{-1}}) = \int \hat{\alpha}_\gamma(x)\overline{\gamma(g)}\, d\gamma$ using $\hat{\alpha}_\gamma(\delta_g) = \gamma(g)\delta_g$. So we can write

$$\int_{G/N} f_i(g) I(x\delta_{g^{-1}})\delta_g\, dNg = \int_{(G/N)\hat{}} \hat{\alpha}_\gamma(x)\hat{f}_i(\gamma^{-1})\, d\gamma,$$

where $\hat{f}_i(\gamma) = \int_{G/N} \gamma(g) f_i(Ng)\, dNg$. If $\hat{f}_i$ is an approximate unit for $L^1(G/N)$ then we have

$$(2.1) \qquad \int I(x\delta_{g^{-1}})\delta_g f_i(g)\, dg \to x \quad \text{as } i \to \infty.$$

Now if J is an ideal in $C^*(G, A, \tau_N)$ it is an ideal in $\mathfrak{M}(C^*(G, A, \tau_N))$ so $x\delta_{g^{-1}} \in J$. But if J is $(G/N)\hat{}$ invariant then $I(x\delta_{g^{-1}})$ also lies in J. Combining this with (2.1) yields the fact that $I(J)$ is a non-zero subset of $A \cap J$. Using $I(a) = a$, and $aI(x)b = I(axb)$, $a, b \in A$, it follows that $I(J)$ is a two-sided ideal in A. Finally G-invariance of $I(J)$ follows from the relation $\alpha_g(I(x)) = \delta_g I(x)\delta_{g^{-1}} = I(\delta_g x \delta_{g^{-1}})$. The above argument is taken from ([9], Section 7.9).

Now A is called G-simple if A has no non-trivial G-invariant ideals.

LEMMA 2.4. *If A is G-simple then $C^*(G, A, \tau_N)$ is $(G/N)\hat{}$ simple.*

PROOF. Again we follow ([9]). Suppose J is a non-trivial $(G/N)\hat{}$ invariant ideal in $C^*(G, A, \tau_N)$. Then Lemma 2.3 tells us that $I(J)$ is a non-zero G-invariant ideal of A. Thus the result follows provided $I(J)$ is not dense in A. But any state ϕ on $C^*(G, A, \tau_N)$ which annihilates J necessarily annihilates $I(J)$. However ϕ cannot annihilate A for if $\{a_\lambda\}$ is an approximate unit for A then $\phi(a_\lambda x) \to \phi(x)$ for $x \in C^*(G, A, \tau_N)$. So $I(J)$ is not dense.

Combining Lemma 2.4 with the argument in the introduction completes the "if" part of Theorem 1.2. The converse argument begins with the observation that if $(G/N)\hat{}$ is compact and $C^*(G, A, \tau_N)$ has T_1 primitive ideal space then a lemma of Moore and Rosenberg [8] implies that the $(G/N)\hat{}$ orbits in $C^*(G, A, \tau_N)$ are closed. Now Lemma 2.1 says that $C^*((G/N)\hat{}, C^*(G, A, \tau_N))$ is just the imprimitivity algebra for inducing from $C^*(N, A, \tau_N) \simeq A$ to $C^*(G, A, \tau_N)$. Consequently, inducing from A to $C^*(G, A, \tau_N)$ is the same as restricting from the imprimitivity algebra to $C^*(G, A, \tau_N)$. But any primitive ideal of $C^*((G/N)\hat{}$, $C^*(G, A, \tau_N))$ restricts to a $(G/N)\hat{}$ orbit in $\operatorname{Prim} C^*(G, A, \tau_N)$. Thus inducing a primitive ideal of A up to $C^*(G, A, \tau_N)$ gives a $(G/N)\hat{}$ orbit in $\operatorname{Prim} C^*(G, A, \tau_N)$. Since this orbit is closed and induction is continuous, its inverse image in Prim A is closed. Let $J \in \operatorname{Prim} C^*(G, A, \tau_N)$, then two ideals $\gamma_1 \cdot J$ and $\gamma_2 \cdot J$ in the $(G/N)\hat{}$ orbit containing J necessarily restrict to the same G-quasiorbit θ in Prim A. Thus a necessary condition for two primitive ideals of A to induce the orbit through J is that they lie in θ. But the set of all such primitive ideals is closed and so contains the closure of some G-orbit in θ and hence contains θ. Thus the inverse image of every $(G/N)\hat{}$-orbit in $\operatorname{Prim} C^*(G, A, \tau_n)$ is a G-quasiorbit in Prim A and so G-quasiorbits are closed.

It is worth mentioning here one consequence of our results. If G is a finitely generated discrete solvable group then Moore and Rosenberg [8] have shown that $\operatorname{Prim} G$ T_1 implies that G is a finite extension of a nilpotent group. So by our result if H is a discrete abelian group acting by automorphisms of Z^n (say a subgroup of $GL(n, Z)$) then the H-quasiorbits in T^n will be closed only if the semidirect product of H and Z^n is a finite extension of a nilpotent group. On the other hand this also suggests that it will be difficult to produce examples of solvable groups (satisfying the hypothesis of Theorem 1.1) which do not have a nilpotent subgroup of finite index.

3. Remarks on the general case.

Consider the situation where G/N is abelian, N is type I and G-quasiorbits in $\hat{N}$ are closed. If $\pi \in \hat{N}$ we let G_π denote the stabiliser in G of π and let σ be the 2-cocycle on G_π/N which is the Mackey obstruction to extending π to a representation of G_π. Introduce the group

$$K_\pi = \{g \in G_\pi \mid \sigma(\bar{g}, \bar{g}')/\sigma(\bar{g}', \bar{g}) = 1 \text{ for all } \bar{g} \in G_\pi/N\}$$

where we use the notation $\bar{g}$ for the coset Ng. Notice first that if $\operatorname{Prim} G$ is fibred over quasiorbits in $\hat{N}$ then $\operatorname{Prim} G$ is T_1 for, if $J_1 \subseteq J_2$ with $J_1 \in \operatorname{Prim} G$, $i = 1, 2$, then $\gamma \cdot J_1 = J_2$ for some $\gamma \in G/N$ so that $J_1 \subseteq \gamma \cdot J_1$. But if J_1 is minimal then $\gamma^{-1} \cdot J_1 \subseteq J_1$ implying that γ fixes J_1 and hence that $J_1 = J_2$. Thus fibering is obviously sufficiently to guarantee that $\operatorname{Prim} G$ is T_1 but it is also probably too restrictive an assumption to attempt to prove in general.

To handle the general problem one needs to reduce to the case where K_π and G_π are constant on quasiorbits. Since quasiorbits are closing, the cutting down argument of the introduction shows that we can reduce to the case where there is a G-simple type I C^*-algebra A. The results of Gootman and Olesen [4] suggest (see the remark at the end of the paper) that in this context (that is, G/N abelian) K_π and G_π are constant on $\hat{A}$, that is, on quasiorbits in $\hat{N}$.

However even given K_π and G_π constant on G-quasiorbits there are difficulties in proving the obvious conjecture that G-quasiorbits in $\operatorname{Prim} K_\pi$ are closed whenever G-quasiorbits in $\hat{N}$ are closed.

Pukanzky has considered a special case in which these difficulties may be overcome [11]. Following Green [6] and Pedersen [10] we formulate conditions which will enable us to show that $\operatorname{Prim} G$ is T_1 whenever G-quasiorbits in $\hat{N}$ are closed. The assumptions are that G can be embedded as a closed subgroup of a second countable topological group $\tilde{G}$ with G/N central in $\tilde{G}/N$, $[\tilde{G}, \tilde{G}] = [G, G]$ and N regularly embedded in $\tilde{G}$. If $\mathcal{O}$ is a quasiorbit in $\hat{N}$ and $\pi = \mathcal{O}$ then one can show easily that K_π and G_π are constant on $\mathcal{O}$. We write $K_{\mathcal{O}}$ and $G_{\mathcal{O}}$ for these groups.

Let $X(\mathcal{O}) = \{\rho \in \hat{K_{\mathcal{O}}} \mid \rho|_N \in \mathcal{O}\}$. If $\tilde{G}_\pi$ is the stabiliser in $\tilde{G}$ of π, let $G_1 = G \cdot \tilde{G}_\pi$. The following results is easily deduced following [10] and [12]:

LEMMA 3.1 ([9], Section 4). *$G_1 \times (G/N)\hat{}$ acts transitively on $X(\mathcal{O})$, the actions of G_1 and $(G/N)\hat{}$ commute and $X(\mathcal{O})$ is homeomorphic to the quotient $G_1 \times (G/N)\hat{}/M$ where M is the common stabiliser of the elements of $X(\mathcal{O})$.*

PROPOSITION 3.2. *With the above hypotheses on $N, G, \tilde{G}$,* Prim G *is T_1 whenever G-quasiorbits in $\hat{N}$ are closed.*

PROOF. Under these hypotheses on $N, G, \tilde{G}$ one can show as in [10] that if $J \in \operatorname{Prim} G$ and J restricts on N to $\mathcal{O}$ then there is an irreducible representation ρ_J of K which induces J. If now $J_2 \supseteq J_1$ are two primitive ideals lying over $\mathcal{O}$ then let ρ_{J_2} and ρ_{J_1} denote the corresponding representations of K. We aim to show that $J_1 = J_2$. Now as $J_2 \supseteq J_1$ we have $\rho_{J_1} \in \overline{G \cdot \rho_{J_2}}$ and since J_1 and J_2 both restrict to $\mathcal{O}$, which is closed, we must have ρ_{J_1} and ρ_{J_2} in $X(\mathcal{O})$. If we can show that ρ_{J_1} and ρ_{J_2} lie in the same G-quasiorbit in $X(\mathcal{O})$ we are through using continuity of inducing. But $X(\mathcal{O})$ is homeomorphic to the abelian group $(G/N)\hat{} \times G_1/M$. So we can identify ρ_{J_1} and ρ_{J_2} with cosets $h_1 M$ and $h_2 M$. Moreover as $\rho_{J_1} \in \overline{G \cdot \rho_{J_2}}$, there is a sequence $\{g_i\}$ in G such that $g_1 \cdot h_2 M \to h_1 M$. But then $g_i^{-1} h_1 M \to h_2 M$ which means that $\rho_{J_2} \in \overline{G \cdot \rho_{J_1}}$ and so ρ_{J_1} and ρ_{J_2} lie in the same G-quasiorbit.

The assumptions on the existence of $\tilde{G}$ with the required properties are satisfied whenever G is a connected, simply connected Lie group [12] however they are obviously too special to allow a general discussion of the question raised in the introduction.

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