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ON TRANSLATION-BOUNDED MEASURES

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Abstract

It is shown that a positive measure μ on the Borel subsets of $\mathbf{R}^k$ is translation-bounded if and only if the Fourier transform of the indicator function of every bounded Borel subset of $\mathbf{R}^k$ belongs to $L^2(\mu)$.

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1. Introduction

We shall be considering non-negative measures defined on the class $\mathfrak{B}$ of Borel subsets of $\mathbf{R}^k$, taking finite values on the subclass $\mathfrak{B}_0$ of bounded Borel sets; for convenience these will be called *Borel measures*. A Borel measure μ is called *translation-bounded* if, for every $A \in \mathfrak{B}_0$,

$$\sup\{\mu(A + x) : x \in \mathbf{R}^k\} < \infty.$$

It is clearly sufficient that this property hold for some A_0 with non-empty interior, for by a compactness argument any other $A \in \mathfrak{B}_0$ can be covered by a finite union of translates of A_0.

As in [3], we shall use the same notation for a set A and its indicator function; thus $A(x) = 1$ if $x \in A$ and $A(x) = 0$ otherwise. For each $A \in \mathfrak{B}_0$, its Fourier transform $\hat{A}$ is defined for all $\xi \in \mathbf{R}^k$ by

$$\hat{A}(\xi) = \int_{\mathbf{R}^k} A(x) e^{ix.\xi}\, dx,$$

where $x.\xi$ denotes the canonical inner product in $\mathbf{R}^k$.

It is shown in [3] that if μ is translation-bounded, then $\hat{A} \in L^2(\mu)$ for every $A \in \mathfrak{B}_0$. The purpose of this note is to prove the converse and thus establish the following result.

THEOREM. *A non-negative Borel measure μ on $\mathbf{R}^k$ is translation-bounded if and only if $\hat{A} \in L^2(\mu)$ for every bounded Borel subset A of $\mathbf{R}^k$.*

2. Proof of the theorem

We are given that μ is a non-negative Borel measure, finite on bounded sets, for which $\hat{A} \in L^2(\mu)$ for each $A \in \mathfrak{B}_0$.

Take any subset $I \in \mathfrak{B}_0$ with nonempty interior and denote by $B(I)$ the space of bounded Borel-measurable (complex-valued) functions on I, with the supremum norm. The indicator functions of Borel subsets of I form a subset $X(I)$, generating the dense vector subspace $S(I)$ of $B(I)$ consisting of the simple functions on I.

Every function $f \in B(I)$ has a Fourier transform $\hat{f} = T(f)$; the main part of the proof is to show the continuity of T.

LEMMA. *Under the hypothesis of the theorem, T is a continuous linear transformation from $B(I)$ to $L^2(\mu)$; that is, there is a constant c such that*

$$\int |\hat{f}(\xi)|^2 \, d\mu \leq c^2 \sup\{|f(x)|^2 : x \in I\} \quad \textit{for all } f \in B(I).$$

PROOF. Take any compact set K_1 in $\mathbf{R}^k$ and any $g \in L^2(\mu)$.
By hypothesis, $\hat{A} \in L^2(\mu)$ for each $A \in X(I)$ and so

$$\nu_1(A) = \int_{K_1} \hat{A} \, \bar{g} \, d\mu$$

is defined in $X(I)$. In fact, ν_1 is a (complex-valued) measure on the Borel subsets of I. For ν_1 is clearly finitely additive. Also, if (A_n) is a sequence decreasing to the empty set, then

$$|\nu_1(A_n)|^2 \leq \int_{K_1} |\hat{A}_n|^2 \, d\mu . \int_{K_1} |\bar{g}|^2 \, d\mu \leq \|g\|_2^2 \mu(K_1)(\lambda(A_n))^2,$$

where λ denotes Lebesgue measure in $\mathbf{R}^k$. So $\nu_1(A_n) \to 0$ as $n \to \infty$ and therefore ν_1 is countably additive.

Now take a sequence of compact sets K_n increasing to $\mathbf{R}^k$ and let ν_n be the corresponding measures. For each $A \in X(I)$,

$$\lim_{n\to\infty} \nu_n(A) = \lim_{n\to\infty} \int_{K_n} \hat{A}\bar{g}\, d\mu = \int_{\mathbf{R}^k} \hat{A}\bar{g}\, d\mu = \nu(A)$$

exists by hypothesis. Hence, by the theorem of Nikodým ([1] page 160, [2]) ν is a measure, which is therefore bounded on the Borel subsets of I. This shows that the set of $\hat{A}$ with $A \in X(I)$ is weakly bounded in $L^2(\mu)$ and hence, by the uniform boundedness theorem, it is norm-bounded:

$$\sup\{\|\hat{A}\|_2 : A \in X(I)\} < \infty.$$

Now, since every $f \in S(I)$ with $0 \leqslant f(x) \leqslant 1$ on I is a convex combination of elements of $X(I)$, $\{\hat{f} : f \in S(I)\}$ is also norm-bounded in $L^2(\mu)$. Hence there is a constant c such that

$$\|\hat{f}\|_2 \leqslant c\|f\|_\infty \quad \text{for all } f \in S(I).$$

Finally, any $f \in B(I)$ is the uniform limit of a sequence (f_n) of functions of $S(I)$. By the continuity of T on $S(I)$, $(\hat{f}_n)$ converges in $L^2(\mu)$, but also $(\hat{f}_n)$ converges to $\hat{f}$ pointwise on $\mathbf{R}^k$. So

$$\|\hat{f}\|_2 = \lim_{n\to\infty} \|\hat{f}_n\|_2 \leqslant \lim_{n\to\infty} c\|f_n\|_\infty = c\|f\|_\infty,$$

and the lemma is proved.

The proof of the theorem can now be completed. Since $\hat{I}$ is continuous and not identically zero, there is an open set D on which $\hat{I}$ is bounded away from zero; say $|\hat{I}(\xi)| \geqslant h > 0$ for $\xi \in D$. For any ζ, let $f(x) = I(x)e^{-ix.\zeta}$. Then

$$|\hat{f}(\xi)| = |\hat{I}(\xi - \zeta)| \geqslant h \quad \text{for } \xi \in D + \zeta.$$

So $h^2\mu(D+\zeta) = \int_{D+\zeta} h^2\, d\mu \leqslant \int |\hat{f}|^2\, d\mu \leqslant c^2 \sup_I |f|^2 = c^2$, whence

$$\sup\{\mu(D+\zeta) : \zeta \in \mathbf{R}^k\} \leqslant \frac{c^2}{h^2}$$

and μ is translation-bounded.

3. Comment

The paper [3] was concerned with a class of measures μ on $\mathbf{R}^k$ which satisfy the properties

(i) $\hat{A} \in L^2(\mu)$ for all $A \in \mathfrak{B}_0$, and

(ii) if (A_n) is a decreasing sequence of sets of $\mathfrak{B}_0$ with empty intersection, then $\hat{A}_n \to 0$ in $L^2(\mu)$, and it was there shown that translation-bounded measures have both these properties. Thus it follows as a corollary to the theorem that (i) implies (ii) for Borel measures μ.

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References

[1] N. Dunford and J. T. Schwartz, *Linear operators*, part I: *General theory* (Interscience, New York, 1958).

[2] A. P. Robertson, 'Unconditional convergence and the Vitali-Hahn-Saks theorem', *Colloque Anal. Function.* [1971, *Bordeaux*] *Bull. Soc. Math. France, Mémoire* 31–32 (1972), 335–341.

[3] M. L. Thornett, 'A class of second-order stationary random measures', *Stochastic Processes Appl.* **8** (1979), 323–334.

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