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CALCULUS IN *f*-ALGEBRAS

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Abstract

Let A be an Archimedean, uniformly complete, semiprime f-algebra and $F(X_1,\ldots,X_n) \in \mathbf{R}^+[X_1,\ldots,X_n]$ a homogeneous polynomial of degree p ($p \in \mathbf{N}$). It is shown that $(F(u_1,\ldots,u_n))^{1/p}$ exists in A^+ for all $u_1,\ldots,u_n \in A^+$.

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In an Archimedean, uniformly complete f-algebra A with unit element every positive element u has a (unique) square root $w = \sqrt{u}$ (that is, $w \in A^+$ and $w^2 = u$) (see, for example [1], Theorem 4.2). This property ceases to hold if the assumption of the unit element is dropped. However, if we assume instead the weaker condition that A is semiprime, then $\sqrt{uv}$ exists in A^+ for all $u, v \in A^+$ ([1], Theorem 4.2).

The main purpose of the present note is to generalize the latter theorem. In fact it will be shown that in any Archimedean, uniformly complete, semiprime f-algebra A for all $p = 1, 2, \ldots$ the pth root of a homogeneous polynomial of degree p, in the variables $u_1,\ldots,u_n \in A^+$ with positive coefficients exists in A^+.

We start with some preliminaries on Riesz spaces and f-algebras in Section 1 and we shall prove the main theorem in Section 2. For terminology and unproved properties of Riesz spaces and f-algebras we refer the reader to [3] and [2].

1. Some preliminaries

Let L be a Riesz space (vector lattice) with positive cone L^+. We assume throughout this note that all Riesz spaces (and hence all f-algebras) under consideration are Archimedean. Given the element $v \in L^+$, the sequence $\{f_n\}_{n=1}^{\infty}$ in L is said to converge v-uniformly to $f \in L$ whenever, for every $\varepsilon > 0$, there exists a natural number N_ε such that $|f - f_n| \leqslant \varepsilon v$ for all $n \geqslant N_\varepsilon$. This will be denoted by $f_n \to f(v)$ or by $f_n \to f$ (r.u.) if we do not want to specify the element v. In like manner the notion of uniform Cauchy sequence is defined. Uniform limits are unique if and only if L is Archimedean. The Archimedean Riesz space L is called uniformly complete whenever every uniform Cauchy sequence in L has a (unique) limit.

The Riesz space A is said to be a Riesz algebra (lattice ordered algebra) if there exists a multiplication in A with the usual algebra properties such that $uv \in A^+$ for all $u, v \in A^+$. Note that $0 \leqslant u \leqslant v$ in A implies that $u^p \leqslant v^p$ $(p = 1, 2, \ldots)$. The Riesz algebra A is called an f-algebra if A has the additional property that $u \wedge v = 0$ implies

$$(uw) \wedge v = (wu) \wedge v = 0$$

for all $w \in A^+$. As agreed upon, every f-algebra A we consider is Archimedean. Hence, A is automatically associative and commutative. If A has, in addition, a unit element, then A is semiprime (that is, the only nilpotent in A is zero). We mention another two properties of f-algebras which we shall use later on.

1) $uv = (u \vee v)(u \wedge v)$ for all $u, v \in A^+$;

2) $u(v \vee w) = (uv) \vee (uw)$

$u(v \wedge w) = (uv) \wedge (uw)$ for all $u, v, w \in A^+$.

Let A be an Archimedean semiprime f-algebra in the rest of this section. The element $u \in A^+$ is called a pth root $(p = 1, 2, \ldots)$ of the element $w \in A^+$ whenever $u^p = w$. We first show that such an element u, if existing, is necessarily unique. Once this is accomplished, the notation $u = \sqrt[p]{w} = w^{1/p}$ is justified.

PROPOSITION 1. *If $u, v \in A^+$, then*

$$(u \wedge v)^p = u^p \wedge v^p \quad \textit{and} \quad (u \vee v)^p = u^p \vee v^p.$$

PROOF. We show the validity of the infimum formula, the proof of the supremum formula being very similar. The method of proof is by induction on p. The case $p = 1$ being clear, suppose that $(u \wedge v)^q = u^q \wedge v^q$ for all $q \leqslant p$. From $uv = (u \vee v)(u \wedge v)$ it follows that

$$\begin{aligned}(uv^p) \wedge (u^p v) &= uv(u^{p-1} \wedge v^{p-1}) = (u \vee v)(u \wedge v)(u \wedge v)^{p-1} \\ &= (u \vee v)(u \wedge v)^p = (u \vee v)(u^p \wedge v^p) \\ &= (u^{p+1} \vee u^p v) \wedge (uv^p \vee v^{p+1}) \geqslant u^{p+1} \wedge v^{p+1}.\end{aligned}$$

Hence, $(u \wedge v)^{p+1} = (u \wedge v)^p (u \wedge v) = (u^p \wedge v^p)(u \wedge v) = u^{p+1} \wedge u^p v \wedge uv^p \wedge v^{p+1} = u^{p+1} \wedge v^{p+1}$, which finishes the induction step.

PROPOSITION 2. (i) $|u - v| \leqslant |u^p - v^p|$ *for all* $u, v \in A^+$.
(ii) *If* $u, v \in A^+$, *then* $u^p = v^p$ *if and only if* $u = v$.
(iii) *If* $u, v \in A^+$, *then* $u^p \leqslant v^p$ *if and only if* $u \leqslant v$.

PROOF. (i) Suppose first that $0 \leqslant v \leqslant u$ and put $w = u - v$. In this case $|u - v|^p = w^p \leqslant (w + v)^p - v^p = |u^p - v^p|$. The general case is reduced to this particular one. Indeed, if $u, v \in A^+$ are arbitrary, then

$$|u - v|^p = (u \vee v - u \wedge v)^p \leqslant (u \vee v)^p - (u \wedge v)^p$$
$$= u^p \vee v^p - u^p \wedge v^p = |u^p - v^p|,$$

where we use Proposition 1 and the identity $|f - g| = f \vee g - f \wedge g$ for all $f, g \in A$.
(ii) By (i), $u^p = v^p$ implies $|u - v|^p = 0$. Since A is semiprime, this yields $|u - v| = 0$, that is, $u = v$.
(iii) If $u^p \leqslant v^p$, then $u^p = u^p \wedge v^p = (u \wedge v)^p$. By (i), $u = u \wedge v$, that is, $u \leqslant v$. The converse is evident.

Obviously, the second part of the above proposition results in uniqueness of pth roots. For later purposes, we state and prove a corollary.

COROLLARY 3. (a) *If* $u_n \in A^+$ $(n = 1, 2, \ldots)$, $w \in A^+$ *and* $\{u_n^p\}_{n=1}^{\infty}$ *is a* w^p*-uniform Cauchy sequence, then* $\{u_n\}_{n=1}^{\infty}$ *is a* w*-uniform Cauchy sequence.*
(b) *If* $u_n^p \to v^p(w^p)$, *then* $u_n \to v(w)$.

PROOF. (a) Given $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbf{N}$ such that $|u_n^p - u_m^p| \leqslant \varepsilon^p w^p$ for all $n, m \geqslant N_\varepsilon$. By Proposition 2(i), this implies that $|u_n - u_m|^p \leqslant \varepsilon^p w^p$ for all $n, m \geqslant N_\varepsilon$. Using Proposition 2(ii) we derive $|u_n - u_m| \leqslant \varepsilon w$ for all $n, m \geqslant N_\varepsilon$.
(b) Similarly.

2. The main theorem

In the remainder of this paper A denotes an Archimedean, uniformly complete semiprime f-algebra. As stated before, $\sqrt{uv}$ exists in A^+ for all $u, v \in A^+$. Since

$$u^2 + v^2 = \left(u + v + \sqrt{2uv}\right)\left(u + v - \sqrt{2uv}\right)$$

is a positive product, it follows immediately that $\sqrt{u^2+v^2}$ exists in A^+ as well. Actually, the following extension is immediate: if $u, v \in A^+$; $\alpha, \beta, \gamma \in \mathbf{R}$ such that $\alpha \geqslant 0$, $\gamma \geqslant 0$ and $\beta^2 \leqslant \alpha\gamma$, the square root of the positive definite homogeneous polynomial $\alpha u^2 + 2\beta uv + \gamma v^2$ exists in A^+. We shall generalize this result to homogeneous polynomials of degree p in n variables. As a first step in this direction we prove

THEOREM 4. *If $u, v \in A^+$ and $u \leqslant v$, then $\sqrt[p]{u^{p-1}v}$ exists in A^+.*

PROOF. We recall that by [1], Proposition 4.1,

$$\inf_{\substack{\alpha=k/n\\ k=1,\ldots,n}} \frac{1}{\alpha}(u-\alpha v)^2 \leqslant n \cdot \frac{1}{n^2}v^2 = \frac{1}{n}v^2 \qquad (n=1,2,\ldots).$$

The following sequence $\{w_n\}_{n=1}^{\infty}$ will turn out to be the natural approximating Cauchy sequence for $\sqrt[p]{u^{p-1}v}$:

$$w_n = \inf_{\substack{\alpha=k/n\\ k=1,\ldots,n}} \left\{\frac{\alpha^{-1/p}}{p}((p-1)u+\alpha v)\right\} \qquad (n=1,2,\ldots).$$

The construction of the elements w_n is motivated by the fact that for all $u \in \mathbf{R}^+$ we have

$$\sqrt[p]{u^{p-1}} = \inf\left\{\frac{\alpha^{-1/p}}{p}((p-1)u+\alpha) : \alpha \in \mathbf{Q}^+\right\}$$

(note that the expression between brackets represents the tangent of $\sqrt[p]{x^{p-1}}$ at the point $x = \alpha$).

We claim that

$$0 \leqslant w_n^p - u^{p-1}v \leqslant \frac{C}{n}v^p \qquad (n=1,2,\ldots)$$

for some constant $C > 0$. Indeed, by Proposition 1,

$$w_n^p - u^{p-1}v = \frac{1}{p^p} \inf_{\substack{\alpha=k/n\\ k=1,\ldots,n}} \frac{1}{\alpha}\left\{[(p-1)u+\alpha v]^p - p^p u^{p-1}\alpha v\right\}.$$

Put $F(u, \alpha v) = [(p-1)u + \alpha v]^p - p^p u^{p-1}\alpha v$, which is a homogeneous polynomial of degree p in u and αv. Consider the corresponding inhomogeneous polynomial

$$F(X) = \{(p-1)X+1\}^p - p^p X^{p-1} \in \mathbf{R}[X].$$

Since $F(1) = F'(1) = 0$, we have $F(X) = (1 - X)^2 G(X)$ for some $G(X) \in \mathbf{R}[X]$ of degree $p - 2$. We assert that $G(X) \in \mathbf{R}^+[X]$, which will be deduced using formal power series. Indeed,

$$\begin{aligned} G(X) &= (1 - X)^{-2} F(X) \\ &= \left(1 + 2X + 3X^2 + \cdots\right)\left(1 + \alpha_1 X + \cdots + \alpha_{p-1} X^{p-1} + \alpha_p X^p\right) \end{aligned}$$

with $\alpha_i \geqslant 0$ ($i = 1, 2, \ldots, p - 2$). We do not compute the coefficients explicitly, since it is not relevant for the argument. In this formal product the constant is 1 and the coefficients of $X, X^2, \ldots, X^{p-2}$ are non-negative. However, the degree of $G(X)$ is $p - 2$, and so all coefficients of $G(X)$ are nonnegative, that is, $G(X) \in \mathbf{R}^+[X]$. Resuming the above, we find

$$F(u, \alpha v) = (u - \alpha v)^2 G(u, \alpha v) \geqslant 0,$$

in other words, $w_n^p - u^{p-1} v \geqslant 0$ ($n = 1, 2, \ldots$). Moreover, it follows from

$$G(u, \alpha v) = \beta_0 u^{p-2} + \beta_1 u^{p-3}(\alpha v) + \cdots + \beta_{p-2}(\alpha v)^{p-2}$$

($\beta_i \geqslant 0$, $i = 0, 1, \ldots, p - 2$), $0 < \alpha \leqslant 1$ and $0 \leqslant u \leqslant v$ that $G(u, \alpha v) \leqslant C' v^{p-2}$, with $C' > 0$ a constant not depending on α. Therefore,

$$0 \leqslant w_n^p - u^{p-1} v \leqslant C \inf_{\substack{\alpha = k/n \\ k = 1, \ldots, n}} \frac{1}{\alpha}(u - \alpha v)^2 \cdot v^{p-2}$$

(with $C = C'/p^p$). Hence, by the observation at the beginning of the proof,

$$0 \leqslant w_n^p - u^{p-1} v \leqslant C \frac{1}{n} v^2 \cdot v^{p-2} = \frac{C}{n} v^p \qquad (n = 1, 2, \ldots).$$

Therefore,

$$|w_n^p - w_m^p| \leqslant \frac{C}{n} v^p \quad \text{for all } m \geqslant n \ (n = 1, 2, \ldots).$$

By Corollary 3, the sequence $\{w_n\}_{n=1}^{\infty}$ is a v-uniform Cauchy sequence in A^+, so $w_n \to w$ (r.u.) for some $w \in A^+$. This implies that $w_n^p \to w^p$ (r.u.). On the other hand, $w_n^p \to u^{p-1} v$ (r.u.). Uniqueness of uniform limits yields $w^p = u^{p-1} v$, that is, $w = \sqrt[p]{u^{p-1} v}$. The proof is complete.

THEOREM 5. *Let A be an Archimedean, uniformly complete, semiprime f-algebra and let $F(X_1, \ldots, X_n) \in \mathbf{R}^+[X_1, \ldots, X_n]$ be a homogeneous polynomial of degree p ($p \in \mathbf{N}$). Then $(F(u_1, \ldots, u_n))^{1/p}$ exists in A^+ for all $u_1, \ldots, u_n \in A^+$.*

PROOF. The proof is divided in several steps and is reduced ultimately to the result of Theorem 4.

Step 1. Using the result of Theorem 4, we show by induction on p that $(u_1 \cdots u_p)^{1/p}$ exist in A^+ whenever $0 \leqslant u_1 \leqslant \cdots \leqslant u_p$. Indeed, the case $p = 1$ (and also $p = 2$) being clear, it follows from the induction hypothesis that $u_1 \cdots u_{p-1}u_p = v^{p-1}u_p$ (with $v = (u_1 \cdots u_{p-1})^{1/p-1}$). Since $v^{p-1} \leqslant u_{p-1}^{p-1} \leqslant u_p^{p-1}$, Proposition 2(iii) implies $v \leqslant u_p$. By Theorem 4, $(v^{p-1}u_p)^{1/p} = (u_1 \cdots u_p)^{1/p}$ exists in A^+.

Step 2. The pth root $(u^{p-1}v)^{1/p}$ exists in A^+ for all $u, v \in A^+$. This observation follows immediately from

$$uv = (u \wedge v)(u \vee v) \quad \text{and} \quad u^{p-1}v = (u \wedge v)u^{p-2}(u \vee v) \qquad (p \geqslant 3)$$

and step 1.

Step 3. The pth root of $u_1 \cdots u_p$ exists in A^+ for all $u_1, \ldots, u_p \in A^+$. Use step 2 and induction on p, just as in step 1.

Step 4. The pth root of $u_1^p + u_2^p$ exists in A^+ for all $u_1, u_2 \in A^+$. Indeed,

$$u_1^p + u_2^p = Q_1(u_1, u_2) \cdots Q_{p/2}(u_1, u_2) \qquad (p \text{ even})$$

or

$$u_1^p + u_2^p = (u_1 + u_2)Q_1(u_1, u_2) \cdots Q_{\frac{1}{2}(p-1)}(u_1, u_2) \qquad (p \text{ odd}),$$

where $Q_i(u_1, u_2)$ is a positive definite quadratic homogeneous polynomial in u_1 and u_2. By the remarks preceding Theorem 4, the square root of such $Q_i(u_1, u_2)$ exists in A^+. Therefore we have in either case that $u_1^p + u_2^p = w_1 w_2 \cdots w_p$ for appropriate $w_i \in A^+$ $(i = 1, \ldots, p)$. By step 3, the pth root of $u_1^p + u_2^p$ exists in A^+.

Step 5. The pth root of $u_1^p + \cdots + u_n^p$ exists in A^+ for all $u_1, \ldots, u_n \in A^+$. Immediate by induction on n.

Step 6. The pth root of $F(u_1, \ldots, u_n)$ exist in A^+. This follows from a combination of step 3 and step 5.

COROLLARY 6. *In an Archimedean, uniformly complete f-algebra A with unit element,* $\sqrt[p]{u}$ *exists for all* $u \in A^+$ *and all* $p \in \mathbf{N}$.

It should be noted that, independently and simultaneously, B. de Pagter has studied similar problems in a somewhat more general setting.

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