

## ON TRANSITIVE PERMUTATION GROUPS WITH A SUBGROUP SATISFYING A CERTAIN CONJUGACY CONDITION

CHERYL E. PRAEGER

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### Abstract

Let  $G$  be a transitive permutation group of degree  $n$  and let  $K$  be a nontrivial pronormal subgroup of  $G$  (that is, for all  $g$  in  $G$ ,  $K$  and  $K^g$  are conjugate in  $\langle K, K^g \rangle$ ). It is shown that  $K$  can fix at most  $\frac{1}{2}(n-1)$  points. Moreover if  $K$  fixes exactly  $\frac{1}{2}(n-1)$  points then  $G$  is either  $A_n$  or  $S_n$ , or  $GL(d, 2)$  in its natural representation where  $n = 2^d - 1 \geq 7$ . Connections with a result of Michael O’Nan are discussed, and an application to the Sylow subgroups of a one point stabilizer is given.

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This paper is concerned with finding an upper bound for the number of fixed points of certain subgroups of a transitive permutation group  $G$ . In [16] it was shown that the number of fixed points of a Sylow subgroup was strictly less than half the total number of points. Here we generalise that result to the class of nontrivial pronormal subgroups of  $G$ , that is, nontrivial subgroups  $K$  such that for all  $g$  in  $G$ ,  $K^g$  is conjugate to  $K$  by an element of  $\langle K^g, K \rangle$ . If  $p$  is a prime dividing the order of  $G$  a  $p$ -subgroup  $K$  of  $G$  is pronormal if and only if each Sylow  $p$ -subgroup of  $G$  contains exactly one conjugate of  $K$ , that is  $K$  weakly closed in any Sylow  $p$ -subgroup of  $G$  containing it. In particular Sylow subgroups are pronormal, and if  $G$  is soluble, then its Hall subgroups are pronormal. The main result of the paper is the following theorem. The proof is by induction and exploits results of Cameron [4, 5]. (Note that if  $K$  is a permutation group on a set  $\Omega$  then  $\text{fix}_\Omega K$  and  $\text{supp}_\Omega K$  denote its set of fixed points in  $\Omega$  and its support  $\Omega - \text{fix}_\Omega K$  in  $\Omega$  respectively.)

**THEOREM 1.** *Let  $G$  be a transitive permutation group on a set  $\Omega$  of  $n$  points, and let  $K$  be a nontrivial pronormal subgroup of  $G$ . Suppose that  $K$  fixes  $f$  points of  $\Omega$ . Then*

- (a)  $f \leq \frac{1}{2}(n - 1)$ , and
- (b) *if  $f = \frac{1}{2}(n - 1)$  then  $K$  is transitive on its support in  $\Omega$ , and either  $G \geq A_n$ , or  $G = \text{GL}(d, 2)$  acting on the  $n = 2^d - 1$  nonzero vectors, and  $K$  is the pointwise stabilizer of a hyperplane.*

This result may be compared, with a result of Michael O’Nan ([13], Theorem A) on subgroups of prime order of primitive permutation groups. We note that O’Nan’s result is true without the restriction that the subgroups have prime power order. As a simple consequence of Theorem 1 and O’Nan’s results we have Theorem 2.

**THEOREM 2.** *Let  $G$  be a primitive permutation group on a set  $\Omega$  of  $n$  points and let  $K$  be a nontrivial subgroup of  $G$  satisfying: if  $g \in G$  is such that  $\text{supp}_\Omega K \cap \text{supp}_\Omega K^g \neq \emptyset$ , then  $K$  is conjugate to  $K^g$  in  $\langle K, K^g \rangle$ .*

*Then one of the following is true.*

- (i)  $|\text{fix}_\Omega K| < \frac{1}{2}(n - 1)$ ,
- (ii)  $G \supseteq A_n$ ,
- (iii)  $G = \text{GL}(d, 2)$  on the  $n = 2^d - 1 \geq 7$  nonzero vectors,
- (iv)  $G = \text{AGL}(d, 2)$  on the  $n = 2^d \geq 8$  vectors.

(For if  $K$  is conjugate to  $K^g$  in  $\langle K, K^g \rangle$  for all  $g$  in  $G$  then  $K$  is pronormal and the result follows from Theorem 1; if not then the result follows from [16] Theorem A.) A less trivial consequence is the following; we note that O’Nan’s result is not strictly necessary for the proof.

**THEOREM 3.** *Let  $G$  be a primitive permutation group on a set  $\Omega$  of  $n$  points and let  $K$  be a nontrivial subgroup of  $G$  such that  $\text{fix}_\Omega K$  is nonempty. Assume that  $K$  satisfies*

*(\*) If  $g \in G$  is such that  $\text{fix}_\Omega K \cap \text{fix}_\Omega K^g \neq \emptyset$  then  $K$  is conjugate to  $K^g$  in  $\langle K, K^g \rangle$ .*

*Then  $f = |\text{fix}_\Omega K| \leq \frac{1}{2}n$ , and if  $f = \frac{1}{2}n$  either  $G \supseteq A_n$  or  $G = \text{AGL}(m, 2)$  with  $n = 2^m \geq 8$ .*

This has the following corollary.

**COROLLARY TO THEOREM 3.** *Let  $G$  be a primitive permutation group on a set  $\Omega$  of  $n$  points, let  $p$  be a prime dividing  $|G|/n$ , and let  $K$  be a Sylow  $p$ -subgroup of the stabilizer  $G_\alpha$  of the point  $\alpha$  of  $\Omega$ . Then  $f = |\text{fix}_\Omega K| \leq \frac{1}{2}n$  and if  $f = \frac{1}{2}n$  then  $n = 2p$  and  $G \supseteq A_n$ .*

Theorem 1 is proved in Section 2, and Section 1 consists of preliminary results among which is the following application of a result of Wielandt.

**PROPOSITION 4.** *Let  $G$  be a primitive permutation group on  $\Omega$ . For  $\alpha \in \Omega$  suppose that  $K$  is a subgroup of  $G_\alpha$  satisfying:*

*(\*\*) If  $K^g \leq G_\alpha$  for  $g \in G$ , then  $K^{g^h} = K$  for some  $h$  in  $G_\alpha$ .*

*Then  $K$  acts nontrivially on each orbit of  $G_\alpha$  in  $\Omega - \{\alpha\}$ .*

In Section 3 we prove a generalization of Theorem 3 and its corollary for transitive groups. Our notation is fairly standard and follows the conventions of [20]. However when it is convenient we shall use the notation of D. G. Higman [7] for suborbits of a permutation group. If  $G$  acts as a permutation group on a set  $\Omega$ , we shall call an orbit of  $G$  in  $\text{supp}_\Omega G$  a nontrivial orbit.

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## 1. Preliminary results

In this section we first prove Proposition 4. Then we examine some properties of the groups of Theorem 1(b) and some properties of pronormal subgroups which will be useful in the inductive proof of Theorem 1. Finally we state for convenience some known results about primitive groups with a small subdegree.

**PROOF OF PROPOSITION 4.** Let  $G, K$  satisfy the hypotheses of Proposition 4, and let  $\Gamma$  be an orbit of  $G_\alpha$  in  $\Omega - \{\alpha\}$ . By [20] 18.1, for some  $g$  in  $G$ ,  $K^g \leq G_\alpha$  and acts nontrivially on  $\Gamma$ . Thus for some  $x$  in  $K^g$ ,  $\beta$  in  $\Gamma$ , we have  $\beta^x \neq \beta$ . By condition (\*\*),  $K^{g^h} = K$  for some  $h$  in  $G_\alpha$ . Then  $x^h \in K$ ,  $\beta^h \in \Gamma^h = \Gamma$ , and  $(\beta^h)^{h^{-1}xh} = \beta^{xh} \neq \beta^h$ . Thus  $K$  acts nontrivially on  $\Gamma$ .

Note we consider the groups  $A_n$  and  $\text{GL}(d, 2)$ .

**LEMMA 1.1.** *Suppose that  $G, K$  satisfy the hypotheses of Theorem 1 and  $G \geq A_n$ . Then  $K$  has only one nontrivial orbit. If this orbit has length  $a$  then  $f = a - i_a(n)$ , where  $i_a(n)$  is the integer satisfying  $1 \leq i_a(n) \leq a$ ,  $n + i_a(n) \equiv 0 \pmod{a}$ . Thus if  $f = \frac{1}{2}(n - 1)$  then  $n = 2a - 1$ .*

**PROOF.** Let  $a$  be the length of the shortest nontrivial orbit of  $K$ . Suppose that  $f \geq a$  and let  $\Gamma$  be a  $K$ -orbit of length  $a$ . Then there is an element  $g$  in  $A_n$  such that  $\Gamma^g \subseteq \text{fix}_\Omega K$ . Then  $\Gamma^g$  is an orbit both of  $K^g$  and of  $\langle K, K^g \rangle$ . Since  $K$  is

pronormal  $K^{gh} = K$  for some  $h$  in  $\langle K, K^g \rangle$ , but then  $\Gamma^{gh} = \Gamma^g$  is an orbit for  $K$ , contradiction. Hence  $f < a$  and so  $f = a - i_a(n)$  and the rest follows immediately.

Clearly  $G \geq A_n$  has nontrivial pronormal subgroups. As a corollary to the proof we have

**COROLLARY 1.2.** *If  $G, K$  satisfy the hypotheses of Theorem 1 and  $G$  is  $a$ -transitive then  $f = a - i_a(n)$  (where  $a$  is the length of the shortest nontrivial orbit of  $K$ ).*

**LEMMA 1.3.** *Suppose that  $G = \text{GL}(d, 2)$ ,  $d \geq 3$ , acting on the set  $\Omega$  of  $n = 2^d - 1$  nonzero vectors. Let  $K$  be the pointwise stabilizer of a hyperplane  $\Delta$ . Then  $K$  is elementary abelian of order  $2^{d-1}$ , is regular on  $\Omega - \Delta$ , and is pronormal. Moreover no subgroup of  $K$  of index 2 is pronormal in  $G$ .*

**PROOF.** It is well known that  $K$  is elementary abelian of order  $2^{d-1}$  and acts regularly on  $\Omega - \Delta$ . Moreover  $K$  is pronormal since each Sylow 2-subgroup of  $\text{GL}(d, 2)$  contains exactly one conjugate of  $K$ . Now  $K$  has  $2^{d-1}$  subgroups of index 2 and each of these groups has two orbits of equal length in  $\Omega - \Delta$ . Thus there are  $2^d - 2$  orbits in  $\Omega - \Delta$  of these subgroups, and each of these orbits is the intersection of  $\Omega - \Delta$  with one of the  $2^d - 2$  hyperplane distinct from  $\Delta$ . Since  $G$  is 2-transitive on the set of hyperplanes it follows that all subgroups of index 2 in  $K$  are conjugate in  $G$ . If  $L, L'$  are distinct subgroups of  $K$  of index 2, then both  $L$  and  $L'$  are normal in  $\langle L, L' \rangle = K$  and so neither is pronormal in  $G$ .

In the proof of Theorem 1 we shall use the following results repeatedly.

**LEMMA 1.4.** *Suppose that  $K$  is a nontrivial pronormal subgroup of the group  $G$ .*

(a) *If  $K \leq H \leq G$  then  $K$  is a pronormal subgroup of  $H$ .*

(b) *If  $X$  is a normal subgroup of  $G$  not containing  $K$ , then  $KX/X$  is a nontrivial pronormal subgroup of  $G/X$ .*

**LEMMA 1.5.** *Suppose that  $K$  is a nontrivial pronormal subgroup of a primitive permutation group  $G$  on  $\Omega$ . If  $\alpha \in \text{fix}_\Omega K$  then  $K$  acts nontrivially on each orbit of  $G_\alpha$  in  $\Omega - \{\alpha\}$ .*

The proof of Lemma 1.4 is straightforward and that of Lemma 1.5 is simply an application of Proposition 4. Finally we quote some results about primitive permutation groups with small subdegrees.

**LEMMA 1.6.** *Let  $G$  be a simply transitive primitive permutation group on  $\Omega$  of degree  $n$ . Let  $\alpha \in \Omega$  and suppose that  $G_\alpha$  has an orbit of length  $k$  in  $\Omega - \{\alpha\}$ .*

- (a) ([20] 18.7, 18.8) *If  $k = 2$  then  $n$  is prime and  $G$  is a Frobenius group of order  $2n$ .*
- (b) ([21]) *If  $k = 3$  then  $G_\alpha$  is soluble with order dividing 48.*
- (c) ([18] Theorem 2.1 (10)) *If  $k$  is 3 or 4 then either  $n$  is divisible by a prime greater than 3 or  $n = 2^c$  or  $n = 3^c$  for some  $c < k$ .*

## 2. Proof of Theorem 1

Suppose that Theorem 1 is false, let  $n$  be the least integer for which a counterexample of degree  $n$  exists, and let  $G$  be a counterexample of degree  $n$ . Thus  $G$  has a nontrivial pronormal subgroup  $K$  with  $f \geq \frac{1}{2}(n-1)$  fixed points in  $\Omega$ ,  $G \not\geq A_n$ , and  $G$  is not  $\text{GL}(d, 2)$  in its natural representation for any  $d$ .

LEMMA 2.1.  *$G$  is primitive on  $\Omega$ .*

PROOF. Suppose that  $G$  has a set  $\Sigma = \{B_1, \dots, B_t\}$  of blocks of imprimitivity in  $\Omega$  with  $|\Sigma| = t > 1$ ,  $|B_i| = b > 1$ , and  $n = tb$ . Since any block containing a point of  $\text{fix}_\Omega K$  is fixed setwise by  $K$  and since  $f > 0$  it follows that  $\text{fix}_\Sigma K$  is nonempty. By [20] 3.5 applied to  $K^\Sigma$  as a subgroup of  $G^\Sigma$ ,  $N_G(K)$  is transitive on  $\text{fix}_\Sigma K$ . It follows that  $f_B = |\text{fix}_B K|$  is independent of the choice of  $B$  in  $\text{fix}_\Sigma K$ . Thus if we set  $f_\Sigma = |\text{fix}_\Sigma K|$  then  $f = f_\Sigma f_B$ . If for  $B$  in  $\text{fix}_\Sigma K$ ,  $K^B \neq 1$  then the hypotheses of Theorem 1 hold for the setwise stabilizer of  $B$  acting on  $B$ . Hence by minimality,  $f_B \leq \frac{1}{2}(b-1)$ , and so  $f \leq tf_B < \frac{1}{2}(n-1)$ . If on the other hand  $K^B = 1$  for  $B$  in  $\text{fix}_\Sigma K$ , then  $f = bf_\Sigma$  and so  $K^\Sigma \neq 1$ . The hypotheses of Theorem 1 then hold for  $G^\Sigma$  and again by minimality,  $f_\Sigma \leq \frac{1}{2}(t-1)$ , and  $f < \frac{1}{2}(n-1)$ . Thus  $G$  is primitive on  $\Omega$ , since  $f \geq \frac{1}{2}(n-1)$ .

LEMMA 2.2.  *$G$  is 2-transitive on  $\Omega$  and  $f = \frac{1}{2}(n-1)$ .*

PROOF. Let  $\alpha \in \text{fix}_\Omega K$ . Then by Lemma 1.5,  $K$  acts nontrivially on each orbit  $\Gamma$  of  $G_\alpha$  in  $\Omega - \{\alpha\}$ . By Lemma 1.4, the hypotheses of the theorem hold for  $G_\alpha^\Gamma$  with nontrivial pronormal subgroup  $K^\Gamma$ . Let  $\{\Gamma_j; 1 \leq j \leq s\}$  be the  $G_\alpha$ -orbits in  $\Omega - \{\alpha\}$ ,  $s \geq 1$ , and let  $|\Gamma_j| = n_j$  and  $|\text{fix } K \cap \Gamma_j| = f_j$ ,  $1 \leq j \leq s$ . Then by minimality  $f = 1 + \sum f_j \leq 1 + \frac{1}{2} \sum (n_j - 1) = \frac{1}{2}(n + 1 - s)$ . Thus either  $f = \frac{1}{2}(n-1)$  and  $s \leq 2$  or  $s = 1$  and  $f = \frac{1}{2}n$ . Assume that  $G_\alpha$  is transitive on  $\Omega - \{\alpha\}$  and  $|\text{fix } K - \{\alpha\}| = \frac{1}{2}((n-1) - 1)$ . Then by minimality, either  $G_\alpha \geq A_{n-1}$ , or  $G_\alpha = \text{GL}(d, 2)$  with  $n-1 = 2^d - 1$  and  $K$  the pointwise stabilizer of a hyperplane. Since  $G \not\geq A_n$ ,  $G_\alpha = \text{GL}(d, 2)$  is the collineation group of  $(d-1)$ -dimensional projective geometry over a field of order 2 and  $d \geq 3$ . It follows from [6] 2.4.34 that  $G$  is a collineation group of a  $d$ -dimensional affine geometry over a field of

order 2. Since  $\text{fix } K - \{\alpha\}$  is a hyperplane of the projective geometry, then both  $\text{fix } K$  and  $\text{supp } K$  are hyperplanes of the affine geometry. Now  $G$  is transitive on hyperplanes (since  $G_\alpha$  is transitive on the hyperplanes containing  $\alpha$ ), and so  $(\text{supp } K)^g = \text{fix } K$  for some  $g$  in  $G$ . Then  $K$  and  $K^g$  centralise each other, a contradiction since  $K$  is pronormal.

Thus  $f = \frac{1}{2}(n - 1)$  and  $s \leq 2$ . Assume that  $s = 2$ . Then  $f_j = \frac{1}{2}(n_j - 1)$  for  $j = 1, 2$ , and by minimality  $G_\alpha$  is doubly transitive on both  $\Gamma_1$  and  $\Gamma_2$ , contradicting [20] 17.7. Thus  $G$  is 2-transitive on  $\Omega$ .

Let  $G$  be  $d$ -transitive but not  $(d + 1)$ -transitive. Then since  $G \not\cong A_n$  it follows from [20] 15.1 that  $2 \leq d \leq 5$ . It is easy to check that  $n > 11$  and hence  $f > d$ . Let  $\Delta$  be a subset of  $\Omega$  of size  $m = n - d + 1$  such that  $\Omega - \Delta \subseteq \text{fix}_\Omega K$ , and let  $H$  be the pointwise stabilizer in  $G$  of  $\Omega - \Delta$ . Then  $H$  is transitive but not 2-transitive on  $\Delta$ ,  $K \subseteq H$ , and  $|\text{fix}_\Delta H| = f - d + 1 = \frac{1}{2}(m - d) = e > 0$ , say. Information about an imprimitive  $H$  is given by the next lemma.

**LEMMA 2.3.** *Let  $X$  be a transitive imprimitive permutation group on a set  $\Delta$  of  $m < n$  points having a nontrivial pronormal subgroup  $K$  with  $e = \frac{1}{2}(m - z) > 0$  fixed points where  $2 \leq z \leq 5$ . Then one of the following is true.*

(i)  $m = yz$ ,  $X$  has a set of  $z$  or  $y$  blocks of imprimitivity in  $\Delta$  of length  $y$  or  $z$  respectively, and in either case the action of degree  $y$  is  $A_y$  or  $S_y$  (with  $y \geq 3$  and  $y$  odd), or  $\text{GL}(r, 2)$  (with  $y = 2^r - 1 \geq 7$ ).

(ii)  $m = 4y$ ,  $z = 4$ ,  $X$  has  $2y$  blocks of imprimitivity of length 2 and  $X$  acts on the set of  $2y$  blocks as an imprimitive group with 2 blocks of imprimitivity of length  $y$ . The representation of degree  $y$  is as in (1).

(iii)  $m = 2x$ ,  $z = 4$ ,  $X$  has 2 or  $x$  blocks of imprimitivity of length  $x$  or 2, the representation of degree  $x$  is primitive and the fixed point set of  $K$  in this representation has size  $\frac{1}{2}(x - 2)$ .

**PROOF.** Assume that  $X$  is imprimitive on  $\Delta$  with a set  $\Sigma = \{B_1, \dots, B_t\}$  of  $t > 1$  blocks of imprimitivity of length  $b > 1$ , where  $m = tb$ . Assume that the blocks are maximal proper blocks so that  $X$  acts primitively on  $\Sigma$ . As in Lemma 2.1  $\text{fix}_\Sigma K$  is nonempty and  $N_X(K)$  is transitive on  $\text{fix}_\Sigma K$ . Thus if  $e_\Sigma = |\text{fix}_\Sigma K|$  and  $e_B = |\text{fix}_B K|$  for  $B$  in  $\text{fix}_\Sigma K$ , then  $e = e_\Sigma e_B$ . Suppose first that for  $B$  in  $\text{fix}_\Sigma K$ ,  $K^B \neq 1$ . Then by minimality  $e_B = \frac{1}{2}(b - u)$  for some positive integer  $u$ . Also either  $e_\Sigma = t$  or  $K^\Sigma \neq 1$  so that (again by minimality)  $e_\Sigma < \frac{1}{2}t$ . In the latter case  $\frac{1}{2}(m - z) = e < \frac{1}{2}te_B = \frac{1}{4}(m - tu)$ , that is  $m < 2z - tu$  so that  $e = \frac{1}{2}(m - z) < \frac{1}{2}(z - tu) \leq \frac{3}{2}$ . Hence  $e = 1$ ,  $z = 5$ , and  $m = 7$  which is a contradiction. Thus  $\frac{1}{2}(m - z) = e = te_B = \frac{1}{2}(m - tu)$ , so that either  $t = z$ ,  $u = 1$  and (i) follows by minimality, or  $z = 4$  and  $t = u = 2$ ; here consideration of the action on  $B_1$  and  $B_2$  of the subgroup of  $X$  fixing  $B_1$  and  $B_2$  setwise shows that one of (i), (ii), (iii), is

true. Thus assume that  $K$  fixes pointwise each block in  $\text{fix}_\Sigma K$ . Then  $K^\Sigma \neq 1$  and by minimality  $e_\Sigma = \frac{1}{2}(t - u)$  for some positive integer  $u$ , so that  $\frac{1}{2}(m - z) = e = be_\Sigma = \frac{1}{2}(m - bu)$ . As above either  $b = z, u = 1$  and (i) is true, or  $z = 4, b = u = 2$ , and as  $X$  is primitive on  $\Sigma$ , (iii) is true.

The next lemma gives information about a primitive group  $H$ , and we make this explicit in the corollary.

**LEMMA 2.4.** *Let  $X$  be a primitive permutation group on a set  $\Delta$  of  $m \leq n$  points having a nontrivial pronormal subgroup  $K$  with  $e = \frac{1}{2}(m - z) \geq 0$  fixed points where  $1 \leq z \leq 5$ . Then*

- (i) if  $z = 1$ ,  $X$  is 4-primitive or  $X = \text{GL}(d, 2)$ , or  $X \supseteq A_m$ ,
- (ii) if  $z = 2$ ,  $X$  is 3-primitive, or  $m = 6$ ,  $X = \text{PGL}(2, 5)$ , or  $m = 4$ ,  $X = A_4$ , or  $m = 2$ ,
- (iii) if  $z = 3$ ,  $X$  is 2-primitive, or  $m = 9$  and  $X$  is  $\text{ASL}(2, 3)$  or  $\text{AGL}(2, 3)$ , or  $m = 5$  and  $|X|$  is 10 or 20, or  $m = 3$ ,
- (iv) if  $4 \leq z \leq 5$ ,  $X$  has rank at most  $z$ ; and if  $z = 4$  and  $X$  is 2-transitive then  $X$  is 2-primitive.

**COROLLARY TO LEMMA 2.4.**  *$H$  is primitive and  $d$  is 4 or 5.*

This corollary follows immediately from Lemma 2.2 and Lemma 2.4 parts (i), (ii), and (iv).

**PROOF OF LEMMA 2.4.** Suppose that  $X$  satisfies the hypotheses of the lemma with degree  $m \leq n$ , and  $e = |\text{fix}_\Delta K| = \frac{1}{2}(m - z) \geq 0$ ,  $1 \leq z \leq 5$ . Suppose also that the lemma is true for groups of smaller degree. Clearly we may assume that  $m \geq 7$ , and  $e > 0$ . The proof is given in two steps.

*Step 1.* First assume that  $X$  is not 2-transitive; then  $z \geq 2$  by minimality and Lemma 2.2. Let  $\delta \in \text{fix}_\Delta K$  and let  $\Gamma_1, \dots, \Gamma_s$  be the orbits of  $X_\delta$  in  $\Delta - \{\delta\}$ , where  $|\Gamma_i| = m_i$  for each  $i \leq s$ , and  $s \geq 2$ . By Lemma 1.5,  $K$  acts nontrivially on each  $\Gamma_i$ , and so by minimality  $e_i = |\text{fix } K \cap \Gamma_i| = \frac{1}{2}(m_i - z_i)$  for some positive integer  $z_i$ , and  $\sum z_i = z + 1$ . If  $z_j = 1$  for some  $j \leq s$  then by minimality  $X_\delta$  acts on  $\Gamma_j$  as  $A_{m_j}$ ,  $S_{m_j}$ , or  $\text{GL}(r, 2)$  with  $m_j = 2^r - 1$ . By [20] 17.7,  $X_\delta$  cannot act 2-transitively on all of the  $\Gamma_j$  so we conclude that at least one of the  $z_j$  is greater than 1. It is convenient to order the  $\Gamma_j$  so that  $z_1 \leq z_2 \leq \dots \leq z_s$ ; then  $z_s \geq 2$ . Suppose next that  $z_j = 1$  for all  $j \leq s - 1$ . Then by [5],  $s \leq 3$  and if  $s = 3$  then  $m = 4t^2(t + 2)^2$ , and  $m_1 = m_2 = t(2t^2 + 4t + 1)$  for some odd positive integer  $t$ . Moreover by [4, 17],  $m_3 = m_1(m_1 - 1)/k$  where  $k$  is 1, 2, or 3. It follows that  $t = 1$ ,  $m_1 = 7$ ,  $k = 2$ , and (by [5]),  $X_\delta^{\Gamma_1} \supseteq A_7$ . Here  $X_\delta$  acts on  $\Gamma_3$  as on unordered pairs of points of  $\Gamma_1$ , and as  $K$  fixes 3 points of  $\Gamma_1$  and has one orbit of length 4 (in order to be

pronormal) it follows that  $K$  fixes exactly 3 points of  $\Gamma_3$ . Thus  $z_3 = 15 > z$  which is a contradiction. Thus  $s = 2$ ,  $z_1 = 1$ , and  $2 \leq z_2 = z \leq 5$ . Assume here that  $z$  is 2 or 3. Since  $X_8$  on  $\Gamma_1$  is alternating or symmetric or  $\text{GL}(r, 2)$ , by [4, 17],  $m_2 = m_1(m_1 - 1)/k$  where  $k$  is 1, 2, or 3, and if  $k = 3$  then  $r \geq 4$ . Also, as  $X_8$  cannot be 2-transitive on  $\Gamma_2$  it follows that  $X_8$  is imprimitive on  $\Gamma_2$  or  $z_2 = 3$  and  $m_2$  is 3 or 5. In the latter case,  $m_2 \neq 5$  as  $m_2 = m_1(m_1 - 1)/k$ , and  $m_2 \neq 3$  follows from [20] 18.4. In the imprimitive case it follows from [20] 18.2 and Lemma 2.3(i) that  $m_2 = z_2 m_1$  so that  $(m_1, m_2)$  is (3, 6), (5, 10), or (7, 21). If  $m_1 = 3$  then (see [18]),  $X$  is  $A_5$  or  $S_5$  on unordered pairs; here  $e = 4$  and so  $K$  is generated by a transposition. However such a group is not pronormal. If  $m_1 = 5$  then  $e = 7$  divides  $|X|$  which is impossible by [18]. If  $m_1 = 7$  then  $m = 39$ , and  $e = 13$  divides  $|X|$  which is impossible by [20] 13.10. Thus if  $X$  is not 2-transitive then  $z \geq 3$ ; if  $3 \leq z \leq 6$  then  $X$  has rank at most  $z$ ; and if  $z = 3$  and  $X$  has rank 3 then  $z_1 = z_2 = 2$ .

Assume then that  $z = 3$ ,  $z_1 = z_2 = 2$ . Then as  $X_8$  cannot be 2-transitive on both  $\Gamma_1$  and  $\Gamma_2$  we may assume that it is imprimitive on  $\Gamma_1$  and its action satisfies Lemma 2.3(i); in particular  $m_1 \equiv 2 \pmod{4}$  and  $m_1 \geq 6$ . Suppose that  $X_8$  is primitive on  $\Gamma_2$ . Then either  $X_8$  is 3-transitive on  $\Gamma_2$ , or  $m_2 = 4$  and  $X_8^{\Gamma_2} = A_4$ , or  $m_2 = 2$ . By Lemma 1.6,  $m_2 \neq 2$  (since  $m_1 \geq 6$ ). By [4],  $m_1 = m_2(m_2 - 1)/k$  where  $k$  is 1 or 2, or  $m = (x + 1)^2(x + 4)^2$ ,  $m_2 = (x + 1)(x^2 + 5x + 5)$ ,  $k = (x + 1)(x + 2)$  for some integer  $x \geq 1$ . In the latter case  $x$  is odd since  $m_2$  is even, and hence  $m_1$  is odd, contradiction. Hence  $k$  is 1 or 2. By [20] 17.6, the only nonabelian composition factor of  $X_8$  is  $A_x$ ,  $x = \frac{1}{2}m_1$ , or  $\text{GL}(r, 2)$ ,  $\frac{1}{2}m_1 = 2^r - 1$ ,  $r \geq 3$ . Also if  $m_2$  is a power of 2 then, since  $m_1 \equiv 2 \pmod{4}$  and  $m_1 \geq 6$ , we have  $m_1 = 6$ ,  $m_2 = 4$ ,  $m = 11$ , a contradiction to [20] 11.6 and 11.7. Thus we may assume (by [20] 11.3 and 12.1 and [3] page 202), that  $X_8^{\Gamma_2}$  has a simple normal subgroup  $S$  which is 2-transitive of even degree  $m_2$ , where  $S$  is  $A_x$ , or  $\text{GL}(r, 2)$ , and  $\frac{1}{2}m_1$  is  $x$ , or  $2^r - 1$  respectively. By [2, 10],  $(\frac{1}{2}m_1, m_2)$  is (5, 6), (7, 8), or (15, 8) all of which contradict  $m_1 = m_2(m_2 - 1)/k$ ,  $k \leq 2$ .

Thus  $X_8$  is imprimitive on both  $\Gamma_1$  and  $\Gamma_2$ , with the actions given by Lemma 2.3. By [20] 18.2 it follows that  $m_1 = m_2 \equiv 2 \pmod{4}$  and  $m_1 \geq 6$ . If  $m_1 = 6$  then  $m = 13$  contradicting [20] 11.6 and 11.7. If  $m_1 = 10$  then  $m = 21$ ,  $e = 9$ , and  $|K|$  is divisible by 3. Hence  $|N(K)|$  is divisible by 27. However by [20] 13.10,  $|X|$  is not divisible by 25 and so  $X_8$  has only one composition factor  $A_5$ . It follows that  $|X_8|$  is not divisible by 9, a contradiction. Thus  $m_1 \geq 14$ . Suppose that  $A_x$ ,  $x = \frac{1}{2}m_1$  is a composition factor of  $X_8$ . Then  $X_8$  contains a 5-element of degree at most 20, a contradiction to [20] 13.10. Thus  $x = \frac{1}{2}m_1 = 2^r - 1$ , and  $X_8$  has a composition factor  $\text{GL}(r, 2)$ ,  $r \geq 3$ . Let  $Y$  be the smallest normal subgroup of  $X_8$  such that  $X_8/Y$  is a (possibly trivial) 2-group. Then  $KY$  is represented as  $\text{GL}(r, 2)$  (acting on points or hyperplanes), on either a set of  $x$  blocks of imprimitivity in  $\Gamma_j$  or on each of two blocks of length  $x$  in  $\Gamma_j$ , for  $j = 1$  and  $j = 2$ . Also the kernel of



all these representations of  $KY$  is a possibly trivial 2-group. Let  $\theta$  be one of the sets of size  $x$  on which  $KY$  is represented. Then  $K^\theta$  is the pointwise stabilizer of a hyperplane and its normalizer in  $(KY)^\theta$  is the setwise stabilizer of the hyperplane and has index  $x$  in  $(KY)^\theta$ ; that is to say, if  $Y_1$  is the kernel of  $KY$  on  $\theta$  then  $N_{KY}(KY_1)$  has index  $x$  in  $KY$ . If  $g \in N_{KY}(KY_1)$  then  $K^g \leq KY_1$  and so  $K$  is conjugate to  $K^g$  in  $\langle K, K^g \rangle \leq KY_1$ . Thus since  $N_{KY}(KY_1)$  contains  $N_{KY}(K)$ , then  $|KY: N_{KY}(K)| = x |N_{KY}(KY_1): N_{KY}(K)| = x |KY_1: N(K) \cap KY_1|$ . Now  $(KY)/Y_1 \cong \text{GL}(r, 2)$  and  $KY$  has  $j$  composition factors isomorphic to  $\text{GL}(r, 2)$  for some  $1 \leq j \leq 4$ . If  $j > 1$  then  $KY_1$  is represented as  $\text{GL}(r, 2)$  on one of the sets of length  $x$  described above. We define  $Y_k$  inductively as the kernel of a representation of  $KY_{k-1}$  as  $\text{GL}(r, 2)$  as above, for  $1 < k \leq j$ . Then as above,  $|KY_{k-1}: N(KY_k) \cap KY_{k-1}| = x$  and the number of conjugates of  $K$  in  $N(KY_k) \cap KY_{k-1}$  is equal to the number of conjugates of  $K$  in  $KY_k$ . Thus  $|KY_{k-1}: N(K) \cap KY_{k-1}| = x |N(KY_k) \cap KY_{k-1}: (N(K) \cap KY_{k-1})| = x |KY_k: N(K) \cap KY_k|$  for  $1 < k \leq j$ . Hence  $|KY: N_{KY}(K)| = x^j |KY_j: N(K) \cap KY_j|$ , whether or not  $j = 1$ . As we remarked above  $Y_j$  is a possibly trivial 2-group, and  $|X_\delta: KY|$  is a power of 2. Thus  $|X_\delta: N(K) \cap X_\delta| = x^j 2^c$  for some  $c \geq 0$ ,  $1 \leq j \leq 4$ . Now  $e = 4(2^{r-1} - 1) + 1 = 2x - 1$  divides  $|X: N(K) \cap X_\delta|$  which divides  $mx^4 2^c$ , and this is clearly impossible. Thus if  $X$  is not 2-transitive then  $z \geq 4$ .

*Step 2.* In this second part of the proof we assume that  $X$  is 2-transitive but not 2-primitive on  $\Delta$  of degree  $m \leq n$ , and  $e = |\text{fix } K| = \frac{1}{2}(m - z)$  where  $1 \leq z \leq 4$ . Then if  $\delta \in \text{fix}_\Delta K$ ,  $X_\delta$  satisfies one of (i), (ii), (iii) of Lemma 2.3 where  $e - 1 = |\text{fix}_\Delta K - \{\delta\}| = \frac{1}{2}((m - 1) - (z + 1))$ ,  $2 \leq z + 1 \leq 5$ . As  $m \geq 7$ ,  $e - 1 > 0$ . Set  $u = z + 1$ . If in (i) or (iii),  $X_\delta$  has a set of  $u$  or 2 blocks respectively then the kernel of the action on blocks is 2-transitive on each of the blocks. By [12] Theorem D it follows that  $X \geq \text{PSL}(3, 3)$ , and  $e = 5$  divides  $|X|$  which is impossible. If in (i)  $X_\delta$  acts as  $A_y$  or  $S_y$  on a set of  $y$  blocks of length  $u$  then as  $K$  fixes a block pointwise and  $y$  is odd, it follows from [14] that  $X$  contains  $\text{PSL}(3, u)$ ,  $u \equiv 2$  or  $4$ , or  $X$  is an extension of an elementary abelian group of order 16 by  $A_5$  or  $S_5$ ,  $u = 3$ . If  $u = 2$  then (i) is true. If  $u = 4$  then  $|X|$  is divisible by  $e|K|$  which is divisible by 27, a contradiction. If  $u = 3$  then  $e = 7$  divides  $|X|$ , also a contradiction. Thus in case (i)  $X_\delta$  acts as  $\text{GL}(r, 2)$  on a set of  $2^r - 1$  blocks of length  $u$ . If  $r = 3$  then  $m = 1 + 7u$  and  $e = 1 + 3u$ ;  $e$  does not divide  $|X|$  if  $u$  is 3 or 4 so  $u$  is 2 or 5. If  $u = 2$  then the 7-element in  $N_X(K)$  must be a 7-cycle on  $\Delta$ , contradiction. If  $u = 5$  then by [15] Corollary B1, the translates of  $B \cup \{\delta\}$  form the blocks of a design on  $\Delta$  with  $\lambda = 1$ , where  $B$  is one of the blocks of  $X_\delta$  of length  $u$ ; further  $\text{fix}_\Delta K$  is the union of three blocks of this design containing  $\delta$  which forces  $N_X(K)$  to fix  $\delta$  whereas  $N_X(K)$  is transitive on  $\text{fix}_\Delta K$ . Thus  $r \geq 4$ . Let  $Y$  be the setwise stabilizer of one of the blocks  $B$  of the set  $\Sigma$  of blocks of  $X_\delta$  in  $\Delta - \{\delta\}$ . Then  $Y^\Sigma$  is an elementary abelian group  $N$  of order  $2^{r-1}$  extended by  $\text{GL}(r - 1, 2)$  acting irreducibly on  $N$ ; thus  $Y^\Sigma$  has no transitive representations of

degree  $u$ ,  $2 \leq u \leq 5$ , and so the kernel  $Z$  of  $X_\delta$  on  $\Sigma$  is nontrivial. It follows from [14] Lemma 1.1 that either  $u = 2$ ,  $X = \text{GL}(r+1, 2)$  (and so (i) is true), or  $Z$  is semiregular on  $\Delta - \{\delta\}$ . Assume the latter. Since  $K$  is the kernel of  $KZ$  acting on  $B$ , where  $B \in \text{fix}_\Sigma K$ , it follows that  $K$  and  $Z$  centralise each other. Now  $|X_\delta: N(NZ) \cap X_\delta| = 2^r - 1$ ; any conjugate  $K^g$  of  $K$  by an element  $g$  in  $N(KZ) \cap X_\delta$  is conjugate to  $K$  in  $\langle K, K^g \rangle \subseteq KZ$  and hence is equal to  $K$ . Thus  $N(KZ) \cap X_\delta \subseteq N(K) \cap X_\delta$  and it follows that  $|X_\delta: N(K) \cap X_\delta| = 2^r - 1$ . Thus  $e = 1 + u(2^{r-1} - 1) = |N(K): N(K) \cap X_\delta|$  divides  $|X: N(K) \cap X_\delta| = (1 + u(2^r - 1))(2^r - 1)$ . It follows that  $u = 2$ , and by [9] 6C(2),  $X$  has a regular normal subgroup of order  $m = 3^c = 2^{r+1} - 1$ ; but there is no solution  $c$  for any  $r \geq 4$ . Thus we may assume that case (i) of Lemma 2.3 does not hold, and so  $u = 4$ .

If in case (iii) of Lemma 2.3  $X_\delta$  has a set  $\Sigma$  of blocks of length 2, by minimality  $X_\delta$  is 3-primitive on  $\Sigma$  or  $x$  is 6 and  $X_\delta^\Sigma = \text{PGL}(2, 5)$  or  $x$  is 4 and  $X_\delta^\Sigma = A_4$  (since  $m \geq 7$ ). In the first and third cases  $X$  is  $\text{AGL}(2, 3)$  or  $\text{ASL}(2, 3)$  by [15] Theorem C and Theorem B respectively, while the case  $x = 6$  cannot arise (since  $e = 11$  cannot divide  $|X|$ ). Suppose that case (ii) of Lemma 2.3 holds, with  $m - 1 = 4y$ . If  $A_y$  is involved then consideration of a 5-element in  $A_y$  and [20] 13.10 shows that  $y$  is 3 or 5. If  $y = 3$  then  $X \geq \text{PSL}(3, 3)$  (see [18]) and  $X_\delta$  does not have blocks of size 2. If  $y = 5$  then 25 does not divide  $|X|$ , by [20] 13.10, while 27 divides  $e|K|$  which divides  $|X|$ ; these two assertions are incompatible with the structure of  $X_\delta$  as its only composition factors are  $Z_2$  and  $A_5$ . Thus  $\text{GL}(r, 2)$  is involved in  $X_\delta$  where  $y = 2^r - 1 \geq 7$ . By [11] the kernel  $Z$  of  $X_\delta$  on its set of  $2y$  blocks in  $\Delta - \{\delta\}$  has order at most 2 and so  $K$  and  $Z$  centralise each other. It follows as above that  $e = 1 + 4(2^{r-1} - 1)$  divides  $|X: N(K) \cap X_\delta| = (1 + 4(2^r - 1))(2^r - 1)$ , which gives a contradiction.

Steps (1) and (2) complete the proof of Lemma 2.4 after noting that none of  $\text{ASL}(2, 3)$ ,  $\text{AGL}(2, 3)$  and  $\text{PGL}(2, 5)$  have transitive extensions.

**LEMMA 2.5.** *Let  $X$  be a primitive permutation group on a set  $\Delta$  of  $m < n$  points having a nontrivial pronormal subgroup  $K$  with  $e = \frac{1}{2}(m - 4) \geq 0$  fixed points. Then  $X$  is 2-transitive.*

**PROOF.** Suppose that  $X$  is not 2-transitive, and that  $m$  is the least degree for which such a group  $X$  exists. Then (see [18])  $m \geq 16$ . Let  $\delta \in \text{fix}_\Delta K$  and let  $\Gamma_1, \dots, \Gamma_s$ ,  $s \geq 2$ , be the orbits of  $X_\delta$  in  $\Delta - \{\delta\}$ , where  $|\Gamma_i| = m_i$ ,  $|\text{fix } K \cap \Gamma_i| = e_i = \frac{1}{2}(m_i - z_i)$ , and  $1 \leq z_1 \leq z_2 \leq \dots \leq z_s$ . By Lemma 2.4,  $2 \leq s \leq 3$  and in the proof of that result we showed that if  $s = 3$  then  $z_2 > 1$ , that is  $z_1 = 1, z_2 = z_3 = 2$ . In this case by [4, 17] and minimality,  $m_2$ , say, is  $m_1(m_1 - 1)/k$  where  $k \leq 3$  and  $m_1 \geq 15$  if  $k = 3$ . Further by Lemma 2.4, [20] 17.7, and [5],  $X_\delta$  is imprimitive on both  $\Gamma_2$  and  $\Gamma_3$ , and by Lemma 2.3, and [20] 18.2–18.4,  $m_2 = m_3 = 2m_1$ . Thus  $m_1 = 2k + 1$  is 3 or 5. If  $m_1 = 3$  then  $m = 16$ , and we have a contradiction to

Lemma 1.6. If  $m_1 = 5$  then  $m = 26$ , and  $e = 11$  divides  $|X|$  contradicting [20] 13.10.

Thus  $s = 2$  and  $(z_1, z_2)$  is  $(1, 4)$  or  $(2, 3)$ . Consider the case  $(1, 4)$  first. By minimality and [4, 17]  $m_2 = m_1(m_1 - 1)/k$  where  $k \leq 3$  and  $m_1$  is odd. By the minimality of  $m$  and [20] 17.7,  $X_\delta$  is imprimitive on  $\Gamma_2$  and so by Lemma 2.3,  $m_2 \equiv 0 \pmod{4}$  so that  $m_1 \equiv 1 \pmod{4}$ . The case  $k = 1$  is impossible by [1, 7]. Thus if  $X_\delta^{\Gamma_1}$  is alternating or symmetric then by [4],  $k = 2$  and  $m_2 = 10 \not\equiv 0 \pmod{4}$ , contradiction. Hence  $m_1 = 2^d - 1$ ,  $d \geq 3$ ; but again  $m_1 \not\equiv 1 \pmod{4}$ .

Thus  $z_1 = 2$ ,  $z_2 = 3$ . By [20] 17.7,  $X_\delta$  is not primitive on both  $\Gamma_1$  and  $\Gamma_2$ . Suppose that  $X_\delta$  is primitive on  $\Gamma_1$ , so that  $X_\delta$  is imprimitive on  $\Gamma_2$ . By the minimality of  $m$  either  $X_\delta$  is 4-transitive on  $\Gamma_1$  or  $m_1$  is 2, 4 or 6 and  $X_\delta$  is 2-transitive on  $\Gamma_1$ . The case  $m_1 = 2$  is impossible by Lemma 1.6 since  $e > 1$ . Also by [4],  $m_2 = m_1(m_1 - 1)/k$  where  $k$  is 1 or 2 (for even if  $m_1$  is 4 or 6 then  $k \leq \frac{1}{2}(m_1 - 1)$  so  $k \leq 2$ ). Since  $m_1$  is even and  $m_2$  is odd,  $k = 2$  and  $m \equiv 2 \pmod{4}$ . By [20] 17.6,  $X_\delta$  is faithful on  $\Gamma_2$  and so the only nonabelian composition factors of  $X_\delta$  are  $A_x$ , where  $x = m_2/3$  is odd, or  $\text{GL}(r, 2)$  where  $m_2/3 = 2^r - 1 \geq 7$ . Since  $m_1 \equiv 2 \pmod{4}$ , and by [3] page 202 and [20] 11.3 and 12.1, either  $X_\delta^{\Gamma_1}$  has a simple normal subgroup  $S$  which is 3-transitive on  $\Gamma_1$ , or  $m_1 = 6$ . Thus if  $m_1 > 6$ ,  $S$  is  $A_x$ ,  $x$  odd, or  $\text{GL}(r, 2)$ . By [2, 10] and since  $m_1 \equiv 2 \pmod{4}$ , it follows that  $m_1 = 6$ , and  $X_\delta^{\Gamma_1} \simeq \text{PGL}(2, 5)$ . Thus  $m_2 = 15$ ,  $m = 22$ , and  $e \mid |K^{\Gamma_2}|$  which is 27 or 81 divides  $|X|$ ; further  $K$  contains a 3-element of degree at most 9 and this contradicts [13] Theorem E.

Thus  $X_\delta$  is imprimitive on  $\Gamma_1$  and its action is given by Lemma 2.3, in particular  $m_1 \equiv 2 \pmod{4}$ . Suppose that  $X_\delta$  is primitive on  $\Gamma_2$ . Then by the minimality of  $m$ , either  $X_\delta$  is 3-transitive on  $\Gamma_2$  with  $m_2 \geq 5$ , or  $m_2$  is 3, 5, or 9 and  $X_\delta^{\Gamma_2}$  is soluble. In the latter case  $X_\delta$  is soluble and  $m_1 = 6$ ,  $m_2 = 3$  or 9, by [20] 18.3, 18.4. If  $m_2 = 3$  then  $m = 10$  and this is impossible by [18], as  $S_5$  on pairs has no subgroup fixing  $e = 3$  pairs. If  $m_2 = 9$  then  $K$  contains a 3-element of degree 6, a contradiction to [13] Corollary 4. Thus  $X_\delta$  is 3-transitive on  $\Gamma_2$  of odd degree  $m_2 \geq 5$ . Then by [3] page 202, and [20] 11.3, 12.1,  $X_\delta^{\Gamma_2}$  has a simple normal subgroup  $S$  which is 2-transitive on  $\Gamma_2$ . By [20] 17.6,  $X_\delta$  is faithful on  $\Gamma_1$  and it follows that  $S$  is either  $A_x$  where  $x = \frac{1}{2}m_1$  is odd or  $\text{GL}(r, 2)$  where  $r \geq 3$ ,  $m_1 = 2(2^r - 1)$ . Hence by [2, 10] either  $m_1 = 2m_2$  or  $(m_1, m_2)$  is  $(14, 15)$ . It follows from [4] that  $m_1 = 2m_2 = 10$ ,  $e \mid |K^{\Gamma_1}| = 18$  divides  $|X|$ , and  $K$  contains a 3-element of degree 6, contradicting [13] Corollary 4.

Thus  $X_\delta$  is imprimitive on both  $\Gamma_1$  and  $\Gamma_2$ . It follows from Lemma 2.3 and [20] 18.2 that  $m_1/2 = m_2/3 = x$  for some odd  $x \geq 3$ . If  $X_\delta$  involves  $A_x$  with  $x \geq 9$  then  $X_\delta$  contains a 7-element of degree at most 35, contradicting [20] 13.10. Thus if  $X_\delta$  involves  $A_x$  then  $x$  is 3, 5 or 7. If  $x = 7$ , a Sylow 5-subgroup of  $X$  fixes 11 points and so  $|X|$  is divisible by 11, contradiction. If  $x = 5$  then  $e = 11$  divides  $|X|$ , contradiction. If  $x = 3$  then  $m = 16$ ,  $m_2 = 9$  divides  $|X|$ , and  $N(K)$

contains a 3-element  $g$  which fixes  $\Gamma_1$  pointwise and so has degree at most 9. It follows from [13] Theorem E that a Sylow 3-subgroup of  $X$  has order 9, and it clearly fixes only one point and has an orbit length 9. Since  $|\text{fix}_\Delta g| = 7$  does not divide  $|X|$  it follows by [20] 3.5 that  $\langle g \rangle$  is not weakly closed in a Sylow 3-subgroup of  $X$ , and this clearly has the wrong orbit lengths. Thus  $x = 2^r - 1$ ,  $r \geq 3$ , and the only insoluble composition factor of  $X_\delta$  is  $\text{GL}(r, 2)$ . By a similar argument to that in the proof of Lemma 2.4 we can show that  $e = 5 \cdot 2^{r-1} - 4$  divides  $mx^5 6^c = (5x + 1)x^{5 \cdot 6^c}$  for some  $c \geq 0$ . It follows that  $r$  is 3 or 4. If  $r = 4$  then  $m_1 = 30$ , and  $m_2 = 45$ . If  $\gamma \in \Gamma_1$  then  $X_{\delta_\gamma}$  has orbits in  $\Gamma_1 - \{\gamma\}$  of lengths 1, 14, 14 or 14, 15. If  $\Gamma(\gamma)$  is the orbit of  $X_\gamma$  of length 30 then  $\lambda = |\Gamma_1 \cap \Gamma(\gamma)|$  is 0, 1, 14, 15, 28 or 29; since  $X$  is primitive  $\lambda \neq 29$  by [7] Corollary 3, and by [7] Lemma 5,  $2(19 - \lambda)/3$  is an integer. Hence  $\lambda = 14$ , which contradicts [7] Lemma 7. If  $r = 3$  a similar argument shows that  $\lambda = 7$ ,  $\mu = 4$ . However if  $\eta \in \Gamma_2$  then the  $X_{\delta_\eta}$  orbit lengths in  $\Gamma_1$  are sums of 1, 1, 6, 6, and no sum of these is equal to  $\mu = 4$ . This completes the proof of Lemma 2.5.

It follows from Lemmas 2.4 and 2.5 that  $d = 5$  and  $H$  is primitive but not 2-transitive of rank at most 5. To complete the proof of Theorem 1 we show that this situation is impossible. This follows from the next lemma since we are assuming that  $H$  has degree  $m = n - 4 > 7$ .

**LEMMA 2.6.** *Let  $X$  be a primitive permutation group on a set  $\Delta$  of  $m < n$  points having a nontrivial pronormal subgroup  $K$  with  $e = \frac{1}{2}(m - 5) \geq 0$  fixed points. Then either  $m$  is 5 or 7, or  $X$  is 2-transitive.*

**PROOF.** By [18] the result is true for  $m \leq 13$ , so assume that  $m \geq 15$  is minimal such that  $X$  is not 2-transitive. Let  $\delta \in \text{fix}_\Delta K$  and let  $\Gamma_1, \dots, \Gamma_s$ ,  $s \geq 2$  be the orbits of  $X_\delta$  in  $\Delta - \{\delta\}$ , where  $|\Gamma_i| = m_i$ ,  $|\text{fix } K \cap \Gamma_i| = e_i = \frac{1}{2}(m_i - z_i)$  and  $1 \leq z_1 \leq \dots \leq z_s$ . By Lemma 2.4,  $\sum z_i = 6$  and  $2 \leq s \leq 4$ , and from the proof of that result, if  $s \geq 3$  then  $z_{s-1} \geq 2$ . First let  $s = 4$ ; then  $z_1 = z_2 = 1$ ,  $z_3 = z_4 = 2$ . By Lemma 2.4 and [5]  $X_\delta$  is imprimitive on  $\Gamma_3$  and  $\Gamma_4$  and these actions are given by Lemma 2.3. By [20] 18.2, and Lemma 2.3,  $2m_1 = 2m_2 = m_3 = m_4$ . Now by [4] one of the subdegrees is equal to  $m_1(m_1 - 1)/k \geq 2m_1$ . Thus  $k = \frac{1}{2}(m_1 - 1)$  and by [4],  $m_1$  is 3 or 5. Thus  $m = 1 + 6m_1$  is 19 or 31, a contradiction to [20] 11.6 and 11.7. Thus  $s$  is 2 or 3.

Suppose that  $s$  is 3. Then  $(z_1, z_2, z_3)$  is  $(1, 2, 3)$  or  $(2, 2, 2)$ . Consider the case  $(1, 2, 3)$ ;  $X_\delta$  is 2-transitive on  $\Gamma_1$  and (by Lemma 2.4, [5], Lemma 1.6, and [20] 17.7 and 18.3), imprimitive on  $\Gamma_2$  and either imprimitive on  $\Gamma_3$  or  $m_3$  is 3 or 5 and  $X_\delta$  is soluble. From Lemma 2.3, and [20] 18.2, 18.3,  $(m_1, m_2, m_3)$  is  $(x, 2x, 3x)$  for some odd  $x \geq 3$  or is  $(3, 6, 3)$ . The latter is impossible by [20] 11.6 and 11.7, so  $m_1 = \frac{1}{2}m_2 = m_3/3 = x \geq 3$ . Now by [4, 17] one of  $m_2, m_3$  is  $x(x - 1)/k$  where

$k \leq 3$ . If  $m_2 = x(x-1)/k$  then  $k = (x-1)/2$  is 1 or 2 by [4] and  $m$  is 19 or 31, a contradiction as before. Similarly if  $m_3 = x(x-1)/k$  then  $x = 3k+1 = 7$  as  $m_1$  is odd. Then  $m = 43$ , again a contradiction.

Thus if  $s = 3$  then  $z_1 = z_2 = z_3 = 2$ . If  $X_\delta$  is imprimitive on all three suborbits then by Lemma 2.3 and [20] 18.2,  $m_1 = m_2 = m_3 = 2x \equiv 2 \pmod{4}$ . If  $X_\delta$  has  $A_x$  as a composition factor with  $x \geq 9$  then  $X$  contains a 7-element of degree at most 42, a contradiction to [19]. Thus if  $X_\delta$  has  $A_x$  as a factor then  $x$  is 3, 5, or 7 and  $m$  is 19, 31, or 43 respectively, a contradiction to [20] 11.6 and 11.7. So  $x = 2^r - 1 \geq 7$ . By a similar argument to that in the proof of Lemma 2.4, we can show that  $e = 3x - 2$  divides  $mx^6 2^c = (6x + 1)x^6 2^c$  for some  $c \geq 0$ , a contradiction. Thus we may assume that  $X_\delta$  is primitive on at least one suborbit and we may suppose that  $m_1$  is maximal among the  $m_i$  such that  $X_\delta$  is primitive on  $\Gamma_i$ . By the minimality of  $m$ ,  $X_\delta$  is 5-transitive on  $\Gamma_1$ , or  $m_1 \leq 6$  and  $X_\delta$  is 2-transitive on  $\Gamma_1$ . Then by [4],  $m_2$  say is  $m_1(m_1 - 1)/k$  where  $k$  is 1 or 2 (even if  $m_1 \leq 6$ ). By the maximality of  $m_1$ ,  $X_\delta$  is imprimitive on  $\Gamma_2$ , and by Lemma 2.3,  $m_2 \geq 6$  so  $m_1 \geq 4$ ; also  $m_2 \equiv 2 \pmod{4}$ . By [20] 17.5,  $X_\delta$  acts faithfully on the union of suborbits on which it is imprimitive. Hence by Lemma 2.3 and [20] 18.2 the only insoluble composition factor of  $X_\delta$  is  $A_x$ , where  $x = m_2/2 \geq 3$  is odd, or  $\text{GL}(r, 2)$  where  $m_2 = 2(2^r - 1)$ ,  $r \geq 3$ . By [20] 11.3, 12.1 and [3] page 202, if  $m_1 > 6$  then  $X_\delta$  has a simple normal subgroup  $S$  which is 4-transitive on  $\Gamma_1$  of even degree  $m_1$ . Since  $S$  must be  $A_x$  or  $\text{GL}(r, 2)$  this is impossible. Hence  $m_1$  is 4 or 6. If  $m_1$  is 4 then by [20] 18.3,  $X_\delta$  is soluble so that  $m_2 = 6$ , and  $m_3$  is 4 (if  $X_\delta$  is primitive on  $\Gamma_3$ ) or 6 (if  $X_\delta$  is imprimitive on  $\Gamma_3$ ). If  $m_3$  is 4 we have a contradiction to [5] while if  $m_3$  is 6 then  $m = 17$ , contradicting [20] 11.6 and 11.7. If  $m_1 = 6$  then since  $m_2$  is even  $m_2 = 30$ , a contradiction by Lemma 2.3 and [20] 18.2.

Thus  $s = 2$  and  $(z_1, z_2)$  is  $(1, 5)$ ,  $(2, 4)$ , or  $(3, 3)$ . Consider the case  $(1, 5)$ . By [4, 17],  $m_2 = m_1(m_1 - 1)/k$  where  $k \leq 3$ . It follows from the minimality of  $m$  and [20] 17.7 that  $X_\delta$  is imprimitive on  $\Gamma_2$ , and by Lemma 2.3 and [20] 18.2,  $m_2 = 5m_1$ ; so  $m_1 = 5k + 1$ . Since  $m_1$  is odd  $k = 2$  and  $m = 67$ , a contradiction to [20] 11.6 and 11.7.

Next consider the case  $z_1 = 2, z_2 = 4$ . Suppose first that  $X_\delta$  is primitive on  $\Gamma_1$ . Then by Lemma 2.5 and [20] 17.7,  $X_\delta$  is 4-transitive on  $\Gamma_1$  or  $m_1 \leq 6$  and  $X_\delta$  is 2-transitive on  $\Gamma_1$ , and  $X_\delta$  is imprimitive on  $\Gamma_2$ . By [4],  $m_2 = m_1(m_1 - 1)/k$  where  $k$  is 1 or 2. Suppose that  $m_1 < m_2/4$  and that  $\gamma \in \Gamma_1$ . By Lemma 2.3,  $X_\delta^{\Gamma_2}$  involves a 2-transitive representation of degree  $m_2/2$  or  $m_2/4$ , and so by [8] Hilfsatz 1, all orbits of  $X_{\delta\gamma}$  in  $\Gamma_2$  have length a multiple of  $m_2/4$ . Now if  $\Gamma(\gamma)$  is the orbit of  $X_\gamma$  of length  $m_1$  then  $X_{\delta\gamma}$  is transitive on  $\Gamma_1 - \{\gamma\}$  and  $\Gamma(\gamma) - \{\delta\}$  and it follows from [7] Corollary 3 that  $\Gamma(\gamma) - \{\delta\} \subseteq \Gamma_2$ . Hence  $m_1 - 1 \geq m_2/4$ , contradiction. Therefore  $m_1 \geq m_2/4 = m_1(m_1 - 1)/4k$ , where  $k = 1$  or 2, and so  $(m_1, m_2)$  is  $(8, 28)$ ,  $(6, 15)$ ,  $(4, 12)$ ,  $(4, 6)$  or  $(2, 2)$ . Now  $m_2$  is divisible by 4, so  $m_1$  is 4 or 8 and  $X_\delta^{\Gamma_1}$  is alternating or symmetric. By [4],  $k = 1$ , so  $m = 17$ , a

contradiction to [20] 11.6, 11.7. Hence  $X_\delta$  is imprimitive on  $\Gamma_1$ . If  $m_2 < \frac{1}{2}m_1$  and if  $\gamma \in \Gamma_2$ , then all orbits of  $X_{\delta_\gamma}$  in  $\Gamma_1$  have length a multiple of  $\frac{1}{2}m_1$  (by Lemma 2.3 and [8] Hilfsatz 1). If  $\Gamma_2(\gamma)$  is the orbit of  $X_\gamma$  of length  $m_2$ , then if  $m_2 < \frac{1}{2}m_1$  we must have  $\Gamma_2(\gamma) - \{\delta\} \subseteq \Gamma_2$ . Hence  $\Gamma_2 \cup \{\delta\}$  is fixed setwise by  $\langle X_\delta, X_\gamma \rangle = X$ , contradiction. Thus  $m_2 \geq \frac{1}{2}m_1$ . If  $X_\delta$  is primitive on  $\Gamma_2$  then by Lemma 2.5 it is 2-transitive and hence by [4],  $m_1 = m_2(m_2 - 1)/k \geq m_1(m_2 - 1)/2k$ , that is  $k \geq (m_2 - 1)/2$ . By [4],  $m_2$  is 3 or 5, a contradiction since  $m_2$  is even. Hence  $X_\delta$  is imprimitive on both  $\Gamma_1$  and  $\Gamma_2$  and  $m_1 \leq 2m_2$ . Suppose first that  $X_\delta^{\Gamma_2}$  satisfies Lemma 2.3(i) or (ii). Then by [20] 18.2,  $m_1 = 2x$ ,  $m_2 = 4x$  for some odd  $x \geq 3$ , and  $m = 1 + 6x$ . Since  $m$  is not prime  $x \geq 9$ . As above we can show that  $A_x$  is not involved; hence  $x = 2^r - 1 \geq 15$ , and we show as above that  $e = 3x - 2$  divides  $(6x + 1)x^{6^c}$ , for some  $c \geq 0$ , a contradiction. Thus  $X_\delta^{\Gamma_2}$  satisfies Lemma 2.3(iii), and by the minimality of  $m$  either the representation of degree  $y = \frac{1}{2}m_2$  is 5-transitive or  $y \leq 6$ . If  $y \leq 6$  then by [20] 18.4,  $(m_1, m_2)$  is  $(6, 8)$  or  $(10, 12)$ . The first case is impossible by [18] since  $S_6$  on pairs has no subgroup fixing  $e = 5$  pairs; in the other case it is also impossible since  $m = 23$  is prime. Thus  $y \geq 8$  and so by [20] 11.3, 12.1,  $X_\delta^{\Gamma_2}$  has a composition factor  $S$  which is 4-transitive of degree  $y$ . If  $S$  is not a composition factor of  $X_\delta^{\Gamma_1}$  then the kernel  $Y$  of  $X_\delta$  on  $\Gamma_1$  has two orbits of length  $y$  in  $\Gamma_2$  (by [20] 13.1), and is 4-transitive on each. If  $\gamma \in \Gamma_1$  and  $\Gamma_1(\gamma)$ ,  $\Gamma_2(\gamma)$  are the orbits of  $X_\gamma$  of length  $m_1$ ,  $m_2$  respectively, then  $\mu = |\Gamma_2 \cap \Gamma_2(\gamma)|$  is 0,  $y$ , or  $2y$ . By [7] Corollary 3,  $\mu = y$ . Thus  $Y$  has 1 orbit of length  $y$  in  $\Gamma_1(\gamma)$  and fixes the remaining points of  $\Gamma_1(\gamma)$ . Since the lengths of the orbits of  $X_{\delta_\gamma}$  in  $\Gamma_1(\gamma)$  are either 1, 1,  $\frac{1}{2}m_1 - 1$ ,  $\frac{1}{2}m_1 - 1$ , or 1,  $\frac{1}{2}m_1 - 1$ ,  $\frac{1}{2}m_1$ , and since  $Y$  is normal in  $X_{\delta_\gamma}$  and  $y$  is even, it follows that  $y$  is  $\frac{1}{2}m_1 - 1$ . Then as  $Y$  is 4-transitive on this orbit of length  $y$  it follows that  $X_\delta^{\Gamma_1}$  involves the alternating group of degree  $\frac{1}{2}m_1 = y + 1$ , a contradiction to [20] 18.2. Thus  $S$  is a composition factor of  $X_\delta^{\Gamma_1}$  hence is either  $A_x$  where  $x = \frac{1}{2}m_1$  is odd or  $\text{GL}(r, 2)$  where  $m_1 = 2(2^r - 1) \geq 14$ . Since  $S$  is 4-transitive of even degree  $y$  we have a contradiction.

The final case is  $s = 2$ ,  $z_1 = z_2 = 3$ . By Lemma 2.4, [20] 17.7, and since  $m \geq 15$ ,  $X_\delta$  is not primitive on both suborbits. We may therefore assume that  $X_\delta$  is imprimitive on  $\Gamma_1$ . If  $X_\delta$  is also imprimitive on  $\Gamma_2$  then by Lemma 2.3, [20] 18.2 and 18.4,  $m_1 = m_2$ . If  $A_x$ ,  $x = m_1/3$  odd, is involved then by considering a 7-element as before,  $x \leq 7$ , but then  $m$  is prime. Hence  $m_1/3 = 2^r - 1 \geq 7$ . If  $r$  is 3 then  $m$  is prime; if  $r$  is 4 then  $e = \frac{1}{2}(m - 5) = 43$  divides  $|X|$ , a contradiction to [23] 13.10. If  $r \geq 5$  then arguing as before we can show that  $e = 3x - 2$  divides  $(6x + 1)x^{6^c}$  for some  $c \geq 0$ , a contradiction. Thus  $X_\delta$  is primitive on  $\Gamma_2$  and by Lemmas 2.4 and 2.5 is either 3-transitive, or  $m_2$  is 9, 5 or 3 and  $X_\delta$  is soluble, by [20] 18.3. In the latter case  $m_1 = 9$ , and so only the primes 2 and 3 divide  $|X_\delta|$ ; thus  $m_2$  is 3 or 9. Since  $m \geq 15$ ,  $m_2$  is 9 and then  $m = 19$  is prime. Thus  $X_\delta$  is 3-transitive on  $\Gamma_2$  and  $m_2 \geq 5$ . If  $m_2 < m_1/3$  and if  $\gamma \in \Gamma_2$  then all orbits of  $X_{\delta_\gamma}$

in  $\Gamma_1$  have length a multiple of  $m_1/3 \geq m_2$  by [8] Hilfsatz 1; hence  $\Gamma_1$  is also an orbit for  $X_\gamma$ , a contradiction as before. So  $m_2 \geq m_1/3$  and both  $m_1$  and  $m_2$  are odd. By [4] it follows that  $m_1 = 21$ ,  $m_2 = 7$ . In this case  $X_\delta^{\Gamma_2} \geq A_7$  and we have a contradiction to [4]. This completes the proof of Lemma 2.6.

Thus the proof of Theorem 1 is complete.

### 3. Proof of Theorem 3 and its corollary

In this section we prove the following generalizations of Theorem 3 and its corollary for transitive groups.

**THEOREM 3'.** *Let  $G$  be a transitive permutation group on a set  $\Omega$  of  $n$  points and let  $K$  be a nontrivial subgroup of  $G$  such that  $\text{fix}_\Omega K$  is nonempty. Assume that  $K$  satisfies*

(\*) *If  $g \in G$  is such that  $\text{fix}_\Omega K \cap \text{fix}_\Omega K^g \neq \emptyset$  then  $K$  is conjugate to  $K^g$  in  $\langle K, K^g \rangle$ .*

*Then  $f = |\text{fix}_\Omega K| \leq \frac{1}{2}n$ , and if  $f = \frac{1}{2}n$  either*

(i)  *$\text{fix}_\Omega K$  is a block of imprimitivity for  $G$ , or*

(ii)  *$G$  has a set  $\Sigma$  of  $m$  blocks of imprimitivity in  $\Omega$  such that  $G^\Sigma$  is  $A_m$  or  $S_m$ , or  $\text{AGL}(d, 2)$  in its natural representation where  $m = 2^d \geq 8$ . Moreover  $K$  fixes half the blocks pointwise and is transitive on the remaining blocks.*

**COROLLARY TO THEOREM 3'.** *Let  $G$  be a transitive permutation group on a set  $\Omega$  of  $n$  points, let  $p$  be a prime dividing  $|G|/n$ , and let  $K$  be a Sylow  $p$ -subgroup of the stabilizer  $G_\alpha$  of the point  $\Omega$ . Then  $f = |\text{fix}_\Omega K| \leq \frac{1}{2}n$  if  $f = \frac{1}{2}n$  then  $K$  is semiregular on  $\Omega$  and either*

(i)  *$\text{fix}_\Omega K$  is a block of imprimitivity for  $G$ , or*

(ii)  *$G$  has a set  $\Sigma$  of  $2p$  blocks of imprimitivity in  $\Omega$  such that  $G^\Sigma \supseteq A_{2p}$ .*

**PROOF OF THEOREM 3'.** Let  $G, K$  be as in Theorem 3'. Suppose first that, for all  $g$  in  $G$ ,  $\text{fix}_\Omega K \cap \text{fix}_\Omega K^g$  is nonempty. Then by assumption  $K$  and  $K^g$  are conjugate in  $\langle K, K^g \rangle$ , that is  $K$  is pronormal in  $G$ . Thus by Theorem 1,  $f < \frac{1}{2}n$ . So suppose that  $K$  has a conjugate  $K^g$  such that  $\text{fix}_\Omega K$  and  $\text{fix}_\Omega K^g$  are disjoint. Then  $n \geq |\text{fix}_\Omega K \cup \text{fix}_\Omega K^g| = 2f$  so that  $f \leq \frac{1}{2}n$ . If  $f = \frac{1}{2}n$  then clearly  $\text{supp}_\Omega K^g = \text{fix}_\Omega K$ .

To complete the proof we must examine the case  $f = \frac{1}{2}n$  more closely. Let  $\alpha \in \text{fix}_\Omega K$  and define  $H = \langle K^g | K^g \leq G_\alpha, g \in G \rangle$ . Let  $\mathcal{C}$  denote the conjugacy class of  $K$  in  $G$  and if  $L$  is a subgroup of  $G$  let  $\mathcal{C} \cap L$  denote the set of conjugates of  $K$  contained in  $L$ . Then  $\mathcal{C} \cap G_\alpha$  is a generating set of  $H$ . Let  $B = \text{fix}_\Omega H$ . Then clearly if  $\beta \in B$ ,  $\mathcal{C} \cap G_\alpha = \mathcal{C} \cap G_\beta$ . Suppose that  $g \in G$  is such that  $B \cap B^g \neq \emptyset$ ,



say  $\beta^g = \gamma$  for some  $\beta, \gamma \in B$ . Then  $(\mathcal{C} \cap G_\alpha)^g = (\mathcal{C} \cap G_\beta)^g = \mathcal{C} \cap G_\gamma = \mathcal{C} \cap G_\alpha$ , that is  $g$  fixes setwise a set of generators of  $H$ . Thus  $g \in N_G(H)$  and so  $B^g = B$ . We have therefore shown that  $B$  is a block of imprimitivity for  $G$  in  $\Omega$ . Now  $B$  is a subset of  $\text{fix}_\Omega K$  and if  $B = \text{fix}_\Omega K$  then part (i) is true. So assume that  $B$  is a proper subset of  $\text{fix}_\Omega K$ . Then there is a conjugate  $K^h$  of  $K$  such that  $K^h \leq G_\alpha$  and  $\text{fix}_\Omega K \neq \text{fix}_\Omega K^h$ , that is  $\text{fix}_\Omega K^h$  contains points of both  $\text{fix}_\Omega K$  and  $\text{supp}_\Omega K$ .

Let  $\Sigma = \{B^g \mid g \in G\}$  and consider the action of  $G$  on  $\Sigma$ . The setwise stabilizer  $X$  of  $B$  in  $G$  is  $N_G(H)$ , for clearly  $N_G(H) \subseteq X$ , and if  $x \in X$ , say  $\alpha^x = \beta \in B$ , then  $(\mathcal{C} \cap G_\alpha)^x = \mathcal{C} \cap G_\beta = \mathcal{C} \cap G_\alpha$  so that  $x \in N_G(H)$ . Let  $K' \in \mathcal{C} \cap X$ . If  $\text{fix}_\Omega K' \cap \text{fix}_\Omega K \neq \emptyset$  then  $K'$  and  $K$  are conjugate in  $\langle K', K \rangle \leq X$ . If not then  $\text{fix}_\Omega K' = \text{supp}_\Omega K$ , and the subgroup  $K^h$  defined above is such that  $\text{fix}_\Omega K^h$  contains points of  $\text{fix}_\Omega K$  and  $\text{fix}_\Omega K'$ . It follows that  $K'$  and  $K$  are conjugate in  $\langle K', K^h, K \rangle \leq X$ . Thus all conjugates of  $K$  contained in  $X$  are conjugate to  $K$  in  $X$ . By [20] 3.5,  $N_G(K)$  is transitive on  $\text{fix}_\Sigma K$ , and so  $K$  fixes pointwise all members of  $\text{fix}_\Sigma K$ .

Let  $\Delta$  be an orbit of  $X$  in  $\Sigma - \{B\}$ . Suppose that  $K$  acts trivially on  $\Delta$  and let  $C \in \Delta$ . By our remark above  $C \subseteq \text{fix}_\Omega K$ . Let  $K' \in \mathcal{C} \cap G_\alpha \subseteq \mathcal{C} \cap X$ . Then  $K' = K^x$  for some  $x \in X$  and so  $(K')^\Delta = (K^x)^\Delta = (K^\Delta)^x = 1$ . Thus  $C \in \text{fix}_\Sigma K'$  and so  $C \subseteq \text{fix}_\Omega K'$ . Hence  $C$  is fixed pointwise by all members of a generating set for  $H$ , and so  $C \subseteq \text{fix}_\Omega H = B$ , a contradiction. Thus  $K^\Delta \neq 1$  and in particular  $X^\Delta \neq 1$ . So  $X^\Delta$  is a transitive group with nontrivial pronormal subgroup  $K^\Delta$  (by Lemma 1.4) and so by Theorem 1,  $f_\Delta = |\text{fix}_\Delta K| \leq \frac{1}{2}(|\Delta| - 1)$ . Thus  $\frac{1}{2}|\Sigma| = |\text{fix}_\Omega K|/|B| = |\text{fix}_\Sigma K| = 1 + \Sigma f_\Delta \leq 1 + \Sigma \frac{1}{2}(|\Delta| - 1) = \frac{1}{2}(|\Sigma| + 1 - r) \leq \frac{1}{2}|\Sigma|$ , where  $r$  is the number of orbits of  $X$  in  $\Sigma - \{B\}$ . It follows from Theorem 1 that  $X$  is transitive on  $\Sigma - \{B\}$  and  $X^{\Sigma - \{B\}}$  is alternating or symmetric, or is  $\text{GL}(d, 2)$  for some  $d \geq 3$ . In the former case  $G^\Sigma$  is alternating or symmetric. In the case of  $\text{GL}(d, 2)$ ,  $K^\Sigma$  is a 2-group and by O'Nan's result [13] Theorem A,  $G^\Sigma = \text{AGL}(d, 2)$ . This completes the proof of Theorem 3'.

**PROOF OF COROLLARY TO THEOREM 3'.** Let  $G$  be a transitive permutation group on  $\Omega$  of degree  $n$ , let  $\alpha \in \Omega$ , let  $p$  be a prime dividing  $|G_\alpha|$ , and let  $K$  be a Sylow  $p$ -subgroup of  $G_\alpha$ . It is easy to check that  $K$  satisfies condition \* of Theorem 3' and so  $f = |\text{fix}_\Omega K| \leq \frac{1}{2}n$ . Suppose that  $f = \frac{1}{2}n$ . We showed in the proof of Theorem 3' that in this case  $K$  has a conjugate  $K'$  such that  $\text{fix}_\Omega K = \text{supp}_\Omega K'$ . Then  $\langle K, K' \rangle = K \times K'$  is a  $p$ -subgroup of  $G$  containing  $K$  and for all  $\beta \in \text{fix}_\Omega K$ ,  $K \times K'_\beta \leq G_\beta$ . Since  $K$  is a Sylow  $p$ -subgroup of  $G_\beta$  we must have  $K'_\beta = 1$ . Thus  $K'$  and hence  $K$  are semiregular on the points they permute.

Finally we must consider the action of  $G$  on the set  $\Sigma$  of blocks of imprimitivity in case (ii) of Theorem 3'. Let  $H, B, \Sigma$  and  $X$  be as in the proof of Theorem 3'. Let  $Y$  be the pointwise stabilizer of  $B$ . Then  $K \leq Y$  and  $Y \trianglelefteq X$ . Now  $X^{\Sigma - \{B\}}$  is  $A_{m-1}$



or  $S_{m-1}$ , or  $GL(d, 2)$  where  $|\Sigma| = m$ , and  $|\Sigma| = 2^d \geq 8$  respectively. Since  $K$  acts nontrivially on  $\Sigma$  it follows that  $Y^{\Sigma-(B)}$  contains  $A_{m-1}$  or  $GL(d, 2)$  respectively. Since  $K$  is a Sylow  $p$ -subgroup of  $Y$  and fixes half the blocks of  $\Sigma$ , the groups  $GL(d, 2)$  do not arise, and in the case of  $A_{m-1}$  and  $S_{m-1}$ ,  $n$  must be  $2p$ . This completes the proof.

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Department of Mathematics  
University of Western Australia  
Nedlands, W. A. 6009  
Australia