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SOME NON-CRITICAL IDEMPOTENTS IN THE CLOSURE OF THE CHARACTERS IN THE MAXIMAL IDEAL SPACE OF $M(\mathbf{D}_2)$

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Abstract

This paper shows that the idempotent generalized characters associated with a Raikov System generated by a K_2 set in $\mathbf{D}_2 = \prod_{i=1}^{\infty}(\mathbf{Z}_2)_i$ is contained in the closure of the characters $\overline{\hat{\mathbf{D}_2}}$ in $\Delta M(\mathbf{D}_2)$.

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1. Introduction

We will be working with the compact totally disconnected abelian group $\mathbf{D}_2 = \prod_{i=1}^{\infty}(\mathbf{Z}_2)_i$ which has dual group $\hat{\mathbf{D}_2} = \oplus_{i=1}^{\infty}(\mathbf{Z}_2^{\cdot})_i$ where $\mathbf{Z}_2^{\cdot}$ is the multiplicative group of order 2, $\mathbf{Z}_2^{\cdot} = \{1, -1; \cdot\}$. The dual group $\hat{\mathbf{D}_2}$ is canonically embedded in $\Delta M(\mathbf{D}_2)$, the maximal ideal space of $M(\mathbf{D}_2)$.

A compact perfect subset K of $\mathbf{D}_2$ is called a K_2 subset of $\mathbf{D}_2$ if for any continuous function f: $K \to \mathbf{Z}_2^{\cdot}$ there is a character $\phi \in \hat{\mathbf{D}_2}$ such that ϕ restricted to K is equal to f.

It will be shown that the idempotent associated with any Raikov System generated by a K_2 set is contained in the closure of the characters $\overline{\hat{\mathbf{D}_2}}$ in $\Delta M(\mathbf{D}_2)$. Dunkl and Ramirez (1972) have shown that the idempotent associated with the Raikov System generated by a closed subgroup is in the closure of the characters.

As the maximal ideal space $\Delta M(\mathbf{D}_2)$ has the weak topology from the Fourier-Stieltjes transforms of the measures in $M(\mathbf{D}_2)$, an idempotent associated with a Raikov System is in the closure of the characters if and only if the Fourier-Stieltjes transforms of the measures satisfy the following condition:

For all measures μ concentrated on the Raikov System and for all measures ν which annihilate all the sets in the Raikov System

$$\|\hat{\mu}\|_\infty \leqslant \|(\mu+\nu)\hat{}\|_\infty$$

where the sup norm is taken over $\hat{\mathbf{D}_2}$.

We will prove that Raikov Systems generated by K_2 subsets of $\mathbf{D}_2$ satisfy this bound by constructing a series of positive definite functions such that, for each measure μ and ν as above, there is a positive definite function P_γ such that

$$\left|\int_{\mathbf{D}_2} P_\gamma \, d\mu - \hat{\mu}(1)\right| < \varepsilon \quad \text{and} \quad \left|\int_{\mathbf{D}_2} P_\gamma \, d\nu\right| < \varepsilon.$$

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2. Raikov systems and idempotents

Let G be a locally abelian group and let A be a subset of G. We define the Raikov System of sets of G generated by A, $\mathfrak{R}_A$, to be the collection of all measurable subsets of some countable union of translates of sums of A.

That is to say

$$\mathfrak{R}_A = \left\{ B \subseteq G : \begin{array}{c} 1.\ B \text{ is measurable;} \\ 2.\ \exists x_i \in G, m_i \in \mathbf{Z}^+, \text{ for } i \in \mathbf{Z}^+, \\ \text{such that } B \subseteq \bigcup_{i=1}^{\infty} x_i + (m_i)A \end{array} \right\}$$

where for $m \in \mathbf{Z}^+$,

$$(m)A = \underbrace{A + A + \cdots + A}_{m \text{ times}} = \left\{ \sum_{i=1}^{m} x_i : x_i \in A, i = 1, \ldots, m \right\}.$$

We notice that $\mathfrak{R}_A$ is closed under translation, intersection, countable unions and addition of sets.

The Raikov System $\mathfrak{R}_A$ is now used to define a direct sum splitting of $M(G)$ into the L-algebra $\mathfrak{A}_A$ of measures concentrated on the sets in the Raikov System, and the L-ideal $\mathfrak{I}_A$ of measures which annihilate all the sets in the Raikov System.

That is to say,

$$\mathcal{Q}_A = \{\mu \in M(G) : \exists B \in \mathfrak{R}_A \text{ such that } \mu \text{ is concentrated on } B\}, \quad \text{and}$$
$$\mathcal{I}_A = \{\nu \in M(G) : |\nu|(B) = 0 \text{ for all } B \in \mathfrak{R}_A\}.$$

The idempotent I_A associated with this Raikov system is the projection from $M(G)$ onto $\mathcal{Q}_A$ and hence is a homomorphism and an element of the maximal ideal space of $M(G)$.

The group of characters on G, $\hat{G}$, is canonically embedded in $\Delta M(G)$ and the idempotent I_A is contained in the closure of the characters $\overline{\hat{G}}$ in $\Delta M(G)$ if and only if the projection

$$I_A : M(G) \to \mathcal{Q}_A$$

is bounded in the Fourier-Stieltjes transform norm, that is to say, for any measure $\mu \in M(G)$

$$\|(I_A\mu)\hat{}\|_\infty \leqslant \|\hat{\mu}\|_\infty$$

where the sup norm is taken over $\hat{G}$. If we have two direct sum splittings of $M(G)$ into an L-subalgebra and L-ideal associated with idempotents in the closure of the characters, then the direct sum splitting

$$\mathcal{Q}_1 \cap \mathcal{Q}_2 \oplus \mathcal{I}_1 + \mathcal{I}_2$$

is also a splitting of $M(G)$ associated with an idempotent in the closure of the characters of G.

3. Properties of K_2 sets

Let $K \subseteq \mathbf{D}_2$ be a compact perfect K_2 subset of $\mathbf{D}_2$. $\overline{\mathrm{Gp}\,K}$ is a closed subgroup of $\mathbf{D}_2$ and so $\mathbf{D}_2 = \overline{\mathrm{Gp}\,K} \oplus H$ where H is also a closed subgroup of $\mathbf{D}_2$. $\overline{\mathrm{Gp}\,K}$ is isomorphic to $\mathbf{D}_2$ and the idempotent associated with the Raikov System generated by $\overline{\mathrm{Gp}\,K}$ is in the closure of the characters (Dunkl and Ramirez (1972)).

If we have a Raikov splitting of $M(\mathbf{D}_2)$ generated by a set $A \subseteq \overline{\mathrm{Gp}\,K}$ then the idempotent associated with this splitting is in the closure of the characters if and only if the condition

$$\|(I_A\mu)\hat{}\|_\infty \leqslant \|\hat{\mu}\|_\infty$$

holds for all measures ν in $M(\overline{\mathrm{Gp}\,K})$. For convenience therefore we assume that $\overline{\mathrm{Gp}\,K} = \mathbf{D}_2$.

For any continuous function ϕ: $K \to \hat{\mathbf{Z}}_2$ there is a character $\chi \in \hat{\mathbf{D}}_2$ such that $\chi|_K = \phi$. For each character χ in $\hat{\mathbf{D}}_2$ we let $P_\chi = \{x \in K: \chi(x) = -1\}$. We say that a set of characters $\{\chi_1 \cdots \chi_n\}$ determines a partition of K if

1. $$P_{\chi_i} \cap P_{\chi_j} = \phi \qquad \forall\, i \neq j,\ 1 \leqslant i, j \leqslant n;$$

2. $$K = \bigcup_{i=1}^{n} P_{\chi_i}.$$

Given two partitions $\mathcal{P} = \{P_{\chi_1}, P_{\chi_2}, \ldots, P_{\chi_n}\}$ and $\mathcal{P}' = \{P_{\phi_1}, P_{\phi_2}, \ldots, P_{\phi_m}\}$ of K determined by the characters $\{\chi_1, \chi_2, \ldots, \chi_n\}$ and $\{\phi_1, \ldots, \phi_m\}$ respectively we say $\mathcal{P}$ is an everywhere finer partition of K than $\mathcal{P}'$ if for each $1 \leqslant i \leqslant m$ there exists an $I_i \subsetneq \{1, \ldots, n\}$ with $\# I_i \geqslant 2$ such that

$$P_{\phi_i} = \bigcup_{j \in I_i} P_{\chi_j}.$$

Since $\overline{\mathrm{Gp}\, K} = \mathbf{D}_2$ this implies that

$$\phi_i = \prod_{j \in I_i} \chi_j.$$

As K is a K_2 subset of $\mathbf{D}_2$, for any continuous function f from K into $\dot{\mathbf{Z}}_2$ there is a partition (in fact, trivial) $\mathcal{P} = \{P_{\chi_1}, P_{\chi_2}, \ldots, P_{\chi_m}\}$ such that

$$f = \prod_{j \in I} \chi_j \Big|_K \quad \text{for some } I \subsetneq \{1, 2, \ldots, m\}.$$

We say that the function f can be generated by the partition $\mathcal{P}$.

Given two partitions of K, $\mathcal{P} = \{P_{\phi_1}, P_{\phi_2}, \ldots, P_{\phi_n}\}$ and $\mathcal{P}' = \{P_{\chi_1}, P_{\chi_2}, \ldots, P_{\chi_m}\}$, there exists a partition $\mathcal{P}'''$ which is everywhere finer than $\mathcal{P}$ and $\mathcal{P}'$ since, for each $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, if we have that $P_{\chi_i} \cap P_{\phi_j} \neq \varnothing$ then there exists a character ω_{ij} such that

$$\omega_{ij} = \begin{cases} -1 & \text{on } P_{\chi_i} \cap P_{\phi_j} \\ 1 & \text{elsewhere on } K \end{cases}$$

and so the partition determined by the characters

$$\left\{\omega_{ij} \colon P_{\chi_i} \cap P_{\phi_j} \neq \varnothing,\ 1 \leqslant i \leqslant m,\ i \leqslant j \leqslant n\right\}$$

is finer than $\mathcal{P}$ and $\mathcal{P}'$. Since K is totally disconnected there exists a partition $\mathcal{P}'''$ which is everywhere finer than $\{\omega_{ij} \colon P_{\chi_i} \cap P_{\phi_j} \neq \varnothing,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$ and so is everywhere finer than both $\mathcal{P}$ and $\mathcal{P}'$.

We define a sequence of partitions of K, $\{\mathcal{P}_i\}_{i \in \mathbf{Z}^+}$ where

$$\mathcal{P}_i = \left\{P_{\phi_1^i}, P_{\phi_2^i}, \ldots, P_{\phi_{j(i)}^i}\right\}$$

determined by a set of characters $\{\phi_1^i, \phi_2^i, \ldots, \phi_{j(i)}^i\}$, to be a "separating sequence of partitions of K" if

1. $\forall\, n \in \mathbf{Z}^+$ $\mathcal{P}_{n+1}$ is an everywhere finer partition of K than $\mathcal{P}_n$;
2. for each continuous function $f \colon K \to \dot{\mathbf{Z}}_2$ there is an $N \in \mathbf{Z}^+$ such that f can

be generated by the partition $\mathcal{P}_N$ and hence f can be generated by each partition $\mathcal{P}_n$ for $n \geqslant N$.

LEMMA 1. *Let K be a K_2 subset of $\mathbf{D}_2$. Then there is a separating sequence of partitions of K.*

PROOF. The set of continuous functions from K into $\mathbf{Z}_2^{\cdot}$ is countable: denote it by $\{f_i : i \in \mathbf{Z}^+\}$. We will define the sequence of partitions inductively. Let $\mathcal{P}_1$ be a partition of K which generates f_1. Let $\mathcal{P}_2$ be a partition of K everywhere finer than $\mathcal{P}_1$ which also generates f_2. Inductively, let $\mathcal{P}_{n+1}$ be an everywhere finer partition than $\mathcal{P}_n$ that generates f_{n+1} and hence also generates $f_1, f_2, \ldots, f_n$.

We can now characterize elements of K, $(m)K$ and $\overline{\mathrm{Gp}\,K}$ using a separating sequence of partitions of K. Let $\{\mathcal{P}_n\}_{n \in \mathbf{Z}^+}$, where $\mathcal{P}_n = \{P_{\phi_1^n}, P_{\phi_2^n}, \ldots, P_{\phi_{j(n)}^n}\}$, be a separating sequence of partitions of K. We define a sequence of characters $\{\phi_{k(m)}^m\}_{m \in \mathbf{Z}^+}$ where $1 \leqslant k(m) \leqslant j(m)$ to be a "chain" of characters from the separating sequence $\{\mathcal{P}_n\}_{n \in \mathbf{Z}^+}$ if

$$P_{\phi_{k(1)}^1} \supset P_{\phi_{k(2)}^2} \supset P_{\phi_{k(3)}^3} \supset \cdots \supset P_{\phi_{k(n)}^n} \supset P_{\phi_{k(n+1)}^{n+1}} \supset \cdots.$$

Obviously if we have two chains $\{\phi_{k(i)}^i\}_{i \in \mathbf{Z}^+}$ and $\{\phi_{m(i)}^i\}_{i \in \mathbf{Z}^+}$ such that for some $N \in \mathbf{Z}^+$, $\phi_{k(N)}^N \neq \phi_{m(N)}^N$, then $\phi_{k(n)}^n \neq \phi_{m(n)}^n$ for all $n \geqslant N$. We also have the following lemma.

LEMMA 2. *Given K a K_2 subset of $\mathbf{D}_2$ with $\overline{\mathrm{Gp}\,K} = \mathbf{D}_2$, and $\{\mathcal{P}_i\}_{i \in \mathbf{Z}^+}$ a separating sequence of partitions of K, then K is equal to the set H where*

$$H = \left\{ x \in \mathbf{C}_2 : \begin{array}{l} \exists \text{ chain } \left\{\phi_{k(i)}^i\right\}_{i \in \mathbf{Z}^+} \text{ of characters from the separating} \\ \text{sequence such that} \\ 1.\ \phi_{k(i)}^i(x) = -1 \qquad \forall\, i \in \mathbf{Z}^+; \\ 2.\ \phi_m^i(x) = 1 \qquad \forall\, m \neq k(i),\ 1 \leqslant m \leqslant j(i). \end{array} \right\}$$

PROOF. Obviously $K \subseteq H$. Let $\{\phi_{k(i)}\}_{i \in \mathbf{Z}^+}$ be a chain of characters from the separating sequence of K. By definition $P_{\phi_{k(i)}^i} \subseteq K$, and so we have $\bigcap_{i \in \mathbf{Z}^+} P_{\phi_{k(i)}^i}$ is non empty, as $\{\phi_{k(i)}^i\}_{i \in \mathbf{Z}^+}$ is a chain, and so $\bigcap_{i \in \mathbf{Z}^+} P_{\phi_{k(i)}^i} = \{x\}$ for some $x \in K$ as $\{\mathcal{P}_i\}_{i \in \mathbf{Z}^+}$ is a separating sequence of partitions of K. So we have

$$\phi_{k(i)}^i(x) = -1 \qquad \forall\, i \in \mathbf{Z}^+$$

and

$$\phi_m^i(x) = 1, \qquad m \neq k(i), \forall\, 1 \leqslant m \leqslant j(i).$$

To show that $K = H$, we need to show that for each chain $\{\phi^i_{k(i)}\}_{i \in \mathbf{Z}^+}$ from the separating sequence there is a unique $x \in \mathbf{D}_2$ with

$$\phi^i_{k(i)}(x) = -1 \qquad \forall\, i \in \mathbf{Z}^+$$

and

$$\phi^i_m(x) = 1 \qquad \forall\, m \neq k(i),\ 1 \leqslant m \leqslant j(i).$$

Assume x, $y \in \mathbf{D}_2$ with

$$\phi^i_{k(i)}(x) = \phi^i_{k(i)}(y) = -1 \qquad \forall\, i \in \mathbf{Z}^+$$

and

$$\phi^i_m(x) = \phi^i_m(y) = 1 \qquad \forall\, m \neq k(i),\ 1 \leqslant m \leqslant j(i),$$

so if x and y are distinct there must exist a character $\chi \in \hat{\mathbf{D}_2}$ such that $\chi(x) \neq \chi(y)$, but $\overline{\operatorname{Gp} K} = \mathbf{D}_2$ so $\chi\,|_K$ is not identically equal to 1. As $\{\mathcal{P}_i\}_{i \in \mathbf{Z}^+}$ is a separating sequence of partitions of K there must be an $i \in \mathbf{Z}^+$ such that χ is generated by the functions $\{\phi^i_1, \phi^i_2, \ldots, \phi^i_{j(i)}\}$ on K. Thus

$$\phi^i_m(x) \neq \phi^i_m(y) \quad \text{for some } 1 \leqslant m \leqslant j(i)$$

and so $x = y$.

As $\overline{\operatorname{Gp} K} = \mathbf{D}_2$, every character on $\mathbf{D}_2$ is uniquely determined by its restrictions to K, so given $\{\mathcal{P}_i\}_{i \in \mathbf{Z}^+}$, a separating sequence of partitions of K, we have that every element of $\mathbf{D}_2$ is uniquely determined by the values of the $\phi^i_k(x)$ where ϕ^i_k are the characters from the separating sequence $\{\mathcal{P}_i\}_{i \in \mathbf{Z}^+}$.

For $x \in \overline{\operatorname{Gp} K} = \mathbf{D}_2$ we define the length of x on the nth partition of K to be

$$l(n, x) = \sum_{i=1}^{u(n)} \frac{1}{2}\left(1 - \phi^n_i(x)\right) = \#\{\phi^n_i\colon \phi^n_i(x) = -1,\ 1 \leqslant i \leqslant j(n)\}.$$

We now have a lemma.

LEMMA 3. *Let K be a K_2 subset of $\mathbf{D}_2$ with $\overline{\operatorname{Gp} K} = \mathbf{D}_2$ and let $\{\mathcal{P}_n\}_{n \in \mathbf{Z}^+}$ be a separating sequence of partitions of K with associated length function $l(n,.)$. Then*:

1. *For $x \in K$, $l(n, x) = 1\ \forall\, n \in \mathbf{Z}^+$.*
2. *For $x \in (m)K \setminus \bigcup_{i=1}^{m-1}(i)K$, $\lim_{n \to \infty} l(n, x) = m$.*
3. *For $x \in \overline{\operatorname{Gp} K}$, $n \in \mathbf{Z}^+$*
   a) *if $l(n, x) = 0$ then $l(m, x) = 0\ \forall\, m \leqslant n$,*
   b) *$l(n+1, x) \geqslant l(n, x)$.*
4. *For $x \in \overline{\operatorname{Gp} K} \setminus \bigcup_{i=1}^{\infty}(i)K$, $\lim_{n \to \infty} l(n, x) = \infty$.*

PROOF. 1. We can see from Lemma 2 that each $x \in K$ is uniquely associated with a chain of characters, say $\{\phi^i_{k(i)}\}_{i\in\mathbf{Z}^+}$, from the separating sequence with

$$\phi^i_{k(i)}(x) = -1 \qquad \forall\, i \in \mathbf{Z}^+$$

and

$$\phi^i_m(x) = 1 \qquad \forall\, 1 \leqslant m \leqslant j(i), m \neq k(i),$$

and so $l(n, x) = 1$.

2. $x = x_1 + x_2 + \cdots + x_m$, $x_i \in K$, all distinct. Each x_i is uniquely associated with a chain $\{\phi^n_{k(n,i)}\}_{n\in\mathbf{Z}^+}$ from the separating sequence with

$$\phi^n_{k(n,i)}(x_i) = -1 \qquad \forall\, n \in \mathbf{Z}^+$$

and

$$\phi^n_j(x_i) = 1, \qquad 1 \leqslant j \leqslant j(n),\ j \neq k(n,i).$$

As the x_i are distinct there exists an $N \in \mathbf{Z}^+$ with

$$\phi^N_{k(N,i)} \neq \phi^N_{k(N,j)} \qquad \forall\, i \neq j,\ 1 \leqslant i, j \leqslant m$$

and so

$$\phi^n_{k(n,i)} \neq \phi^n_{k(n,j)} \qquad \forall\, i \neq j,\ 1 \leqslant i, j \leqslant m \text{ and } n \geqslant N$$

so $l(n, x) = m$ for all $n \geqslant N$.

3. a) As $\overline{\mathrm{Gp}\,K} = \mathbf{D}_2$ and $\{\mathcal{P}_i\}_{i\in\mathbf{Z}^+}$ is a separating sequence of partitions of K, letting

$$H_n = \{x \in \mathbf{D}_2 \colon l(i, x) = 0, i = 1, \ldots, n\}$$

we have that $\{H_n \colon n \in \mathbf{Z}^+\}$ forms a base of open neighbourhoods of zero in $\mathbf{D}_2$. Let $x \in \overline{\mathrm{Gp}\,K} = \mathbf{D}_2$ be such that $l(n, x) = 0$. We can write

$$x = x_1 + x_2 + \cdots + x_r + h$$

where $h \in H_{n+1}$ and $x_i \in K$, $i = 1, \ldots, r$ are distinct, so each x_i is uniquely associated with a chain $\{\phi^m_{k(m,i)}\}_{m\in\mathbf{Z}^+}$ where $\phi^m_{k(m,i)}(x_i) = -1$ and

$$\phi^m_j(x_i) = 1, \qquad 1 \leqslant j \leqslant j(m),\ j \neq k(m,i).$$

Now $l(n, x) = l(n, x_1 + x_2 + \cdots + x_r) = 0$ so we must be able to group the x_i in pairs x_i, x_j with $\phi^n_{k(n,i)} = \phi^n_{k(n,j)}$, and so

$$\phi^m_{k(m,i)} = \phi^m_{k(m,j)} \qquad \text{for all } m \leqslant n.$$

Thus

$$l(m, x_1 + x_2 + \cdots + x_r) = 0 \qquad \forall\, m \leqslant n$$

so $l(m, x) = 0$ for all $m \leqslant n$.

3. b) For $x \in \overline{\mathrm{Gp}\,K} = \mathbf{D}_2$ let $l(n + 1, x) = m$. Then we can write $x = x_1 + x_2 + \cdots + x_m + h$ where $l(n + 1, h) = 0$, so

$$l(i, h) = 0, \qquad 1 \leqslant i \leqslant n + 1,$$

and so

$$l(n, x) = l(n, x_1 + x_2 + \cdots + x_m) \leqslant m \leqslant l(n+1, x).$$

4. Let $x \in \overline{\mathrm{Gp}\, K}$ and suppose that $\lim_{n\to\infty} l(n, x) = m$, so for each $n \in \mathbf{Z}^+$ we can find an $h \in H_n$ and $x_1 \cdots x_m \in K$ so that

$$x = x_1 + x_2 + \cdots + x_m + h.$$

As $\{H_n : n \in \mathbf{Z}^+\}$ forms a base of open neighbourhoods of zero we have that

$$x \in \overline{(m)K} = (m)K.$$

4. Positive definite functions

We will now use the separating sequence of partitions of K, $\{\mathcal{P}_m\}_{m\in\mathbf{Z}^+}$, with generating characters $\{\phi_1^m, \phi_2^m, \ldots, \phi_{j(m)}^m\}$, for $m \in \mathbf{Z}^+$, to construct a sequence of positive definite functions on $\mathbf{D}_2$.

LEMMA. *Let $r \in (0, 1)$ and $n \in \mathbf{Z}^+$. Then the function*

$$F_r^n : \overline{\mathrm{Gp}\, K} \to \mathbf{R}$$
$$: x \rightsquigarrow r^{l(n,x)}$$

is a positive definite function on $\mathbf{D}_2 = \overline{\mathrm{Gp}\, K}$.

PROOF. Consider the nth partition of K, $\mathcal{P}_n = \{P_{\phi_1^n}, P_{\phi_2^n}, \ldots, P_{\phi_{j(n)}^n}\}$, generated by the characters $\{\phi_1^n, \phi_2^n, \ldots, \phi_{j(n)}^n\}$. The measure $\mu_{n,r}$ on $\mathbf{D}_2$

$$\mu_{n,r} = \mathop{*}_{i=1}^{j(n)} \left(\left(\frac{1+r}{2}\right)\delta(1) + \left(\frac{1-r}{2}\right)\delta(\phi_i^n)\right)$$

is a positive measure for $r \in (0, 1)$ and has Fourier transform

$$\hat{\mu}_{n,r}(x) = r^{l(n,x)}$$

and so $F_r^n : \mathbf{D}_2 \to \mathbf{R} : x \rightsquigarrow r^{l(n,x)}$ is a positive definite function on $\mathbf{D}_2 = \overline{\mathrm{Gp}\, K}$.

We now have the main theorem of this section.

THEOREM 1. *Let $K \subseteq \mathbf{D}_2$ be a K_2 subset of $\mathbf{D}_2$ such that $\overline{\mathrm{Gp}\, K} = \mathbf{D}_2$. Then, for each $m \in \mathbf{Z}^+$ and $\varepsilon, \delta > 0$ we can choose an $h \in \mathbf{Z}^+$ such that, for any open neighbourhood H of zero, there exists a positive definite function F with $F(0) = 1$*

and

$$1.\quad F(x) > 1-\varepsilon \quad \text{for } x \in \bigcup_{i=1}^{m} (i)K,$$

$$2.\quad |F(x)| < \delta \quad \text{for } x \in \mathbf{D}_2 \backslash \left\{ \left(\bigcup_{j=1}^{m+h} (i)K \right) + H \right\}.$$

PROOF. Choose an $r \in (0,1)$ such that $(1-r^m) < \varepsilon$ and choose an $h \in \mathbf{Z}^+$ so that $r^{m+h} < \delta$. Now let H' be an open neighbourhood of zero of the form

$$H' = \left\{ x \in \mathbf{D}_2 \colon \phi_j^i(x) = 1 \ \forall\, 1 \leqslant j \leqslant j(i),\ 1 \leqslant i \leqslant I \right\} = H_I$$

for some $I \in \mathbf{Z}^+$. So we have

$$\left\{ \left(\bigcup_{i=1}^{m+h} (i)K \right) + H' \right\} = \{ x \in \mathbf{D}_2 \colon l(p,x) \leqslant m+h \text{ for } 1 \leqslant p \leqslant I \}$$
$$= \{ x \in \mathbf{D}_2 \colon l(I,x) \leqslant m+h \}.$$

Now observe that $F_r^I(x) = r^{l(I,x)}$; so, for $x \in \bigcup_{i=1}^{m}(i)K$,

$$|1 - F_r^I(x)| \leqslant |1 - r^m| < \varepsilon.$$

For $x \in \mathbf{D}_2 \backslash \{ (\bigcup_{i=1}^{m+h}(i)K) + H' \}$ we have $l(I,x) > m+h$, so $F_r^I(x) \leqslant r^{m+h} < \delta$.

5.

We now prove a general theorem about Raikov idempotent generalized characters in the closure of the characters of $\mathbf{D}_2$, given the existence of positive definite functions with certain properties. The main result is then a corollary of Theorem 1 and the following theorem.

THEOREM 2. *Let $A \subsetneq \mathbf{D}_2$ be a compact perfect subset of $\mathbf{D}_2$ such that, for every $m \in \mathbf{Z}^+$, $\varepsilon, \delta > 0$ and open neighbourhood H of zero, there exists an integer $h \in \mathbf{Z}^+$ independent of the neighbourhood H and a positive definite function F on $\mathbf{D}_2$ with*

$$1.\quad F(0) = 0;$$

$$2.\quad |F(x) - 1| < \varepsilon \quad \forall\, x \in \bigcup_{i=1}^{m} (i)A;$$

$$3.\quad |F(x)| < \delta \quad \forall\, x \in \mathbf{D}_2 \backslash \left\{ \left(\bigcup_{i=1}^{m+h} (i)A \right) + H \right\}.$$

Then the idempotent generalized character I_A associated with the Raikov System generated by A on $\mathbf{D}_2$ is in the closure of the characters $\overline{\hat{\mathbf{D}_2}}$ in $\Delta M(\mathbf{D}_2)$.

PROOF. $\Delta M(\mathbf{D}_2)$ has the weak topology induced from the Fourier-Stieltjes transforms of the measures in $M(\mathbf{D}_2)$, so the idempotent I_A is in $\overline{\hat{\mathbf{D}_2}}$ if and only if

$$\|(I_A\mu)\hat{}\|_\infty \leqslant \|\hat{\mu}\|_\infty \qquad \forall \mu \in M(\mathbf{D}_2)$$

where the sup norms are taken over $\hat{\mathbf{D}_2}$.

Let $\mu \in \mathscr{Q}_A$ and $\varepsilon > 0$. We can find an $l \in \mathbf{Z}^+$ so that

$$\mu = \sum_{i=1}^{n} \delta_{x_i} * \mu_i + \mu'$$

where $\mu_i \in M(\bigcup_{j=1}^{l}(j)A)$, $\|\hat{\mu'}\| < \varepsilon$ and $x_i \in \mathbf{D}_2$. We will consider the measure $\sum_{i=1}^{n} \delta_{x_i} * \mu_i$ which is concentrated on

$$\bigcup_{j=1}^{n}\left(x_j + \bigcup_{i=1}^{l}(i)A\right).$$

We can assume (without loss of generality) that $S = \{x_1, x_2, \ldots, x_n\}$ is a finite subgroup of $\mathbf{D}_2$. We can find a subgroup S_0 of S such that

$$S + \operatorname{Gp} A = S_0 + \operatorname{Gp} A \quad \text{and} \quad S_0 \cap \operatorname{Gp} A = \{0\}$$

and can find an $m \in \mathbf{Z}^+$ so that

$$\bigcup_{x\in S}\left(x + \bigcup_{i=1}^{l}(i)A\right) \subset \bigcup_{y\in S_0}\left(y + \bigcup_{i=1}^{m}(i)A\right).$$

Now we have for each $q \in \mathbf{Z}^+$ and $x \neq y \in S_0$ that

$$\left\{x + \bigcup_{i=1}^{q}(i)A\right\} \cap \left\{y + \bigcup_{i=1}^{q}(i)A\right\} = \varnothing$$

so we can choose an open neighbourhood $H(q)$ of zero such that for all $x, y \in S_0$, $x \neq y$,

$$\left\{\left(x + \bigcup_{i=1}^{q}(i)A\right) + H(q)\right\} \cap \left\{\left(y + \bigcup_{i=1}^{q}(i)A\right) + H(q)\right\} = \varnothing.$$

Now choose an $h \in \mathbf{Z}^+$ such that for every open neighbourhood H of zero there exists a positive definite function F on $\mathbf{D}_2$ with

1. $F(0) = 1$;
2. $|F(x) - 1| < \varepsilon \quad \forall x \in \bigcup_{i=1}^{m}(i)A$;
3. $|F(x)| < \dfrac{\varepsilon}{|S_0|} \quad \forall x \in \mathbf{D}_2 \backslash \left\{\left(\bigcup_{i=1}^{m+h}(i)A\right) + H\right\}$.

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Let H be an open neighbourhood of zero contained in $H(m+h)$. Then we can find a positive definite function F satisfying

1. $F(0) = 1$;
2. $|F(x) - 1| < \varepsilon \quad \forall x \in \bigcup_{i=1}^{m} (i)A$;
3. $|F(x)| < \dfrac{\varepsilon}{|S_0|} \quad \forall x \in \mathbf{D}_2 \setminus \left\{ \left(\bigcup_{i=1}^{m+h} (i)A \right) + H \right\}$.

Now the measure $|S_0| \cdot m_{S_0}$ where m_{S_0} is haar measure on S_0 has positive Fourier transform, so $\mathcal{F} = |S_0| \cdot m_{S_0} * F$ is a positive definite function on $\mathbf{D}_2$, and

$$\mathcal{F} = \sum_{x \in S_0} \delta_x * F. \tag{1}$$

$\mathcal{F}$ has the following properties:

1. $\mathcal{F}(0) \leqslant 1 + \varepsilon$.
2. $|\mathcal{F}(x) - 1| < 2\varepsilon, \quad x \in \bigcup_{y \in S_0} \left(y + \bigcup_{i=1}^{m} (i)A \right)$.
3. $|\mathcal{F}(x)| < \varepsilon, \quad x \in \mathbf{D}_2 \setminus \left\{ \left(\bigcup_{y \in S_0} y + \bigcup_{i=1}^{m+h} (i)A \right) + H \right\}$.

Let ν be a measure in I_A. Then

$$|\nu| \left(\bigcup_{y \in S_0} \left(y + \bigcup_{i=1}^{m+h} (i)A \right) \right) = 0.$$

So we can choose an open neighbourhood H of zero contained in $H(m+h)$ such that

$$|\nu| \left\{ \left(\bigcup_{y \in S_0} \left(y + \bigcup_{i=1}^{m+h} (i)A \right) \right) + H \right\} < \varepsilon$$

and let $\mathcal{F}$ be the associated positive definite function as in (1).

We then have, for $\gamma \in \hat{\mathbf{D}_2}$,

$$|\hat{\mu}(\gamma)| \leqslant \left| \left(\sum_{i=1}^{n} \delta x_i * \mu_i \right)\hat{} (\gamma) \right| + \varepsilon$$

$$\leqslant \left| \int_{\mathbf{D}_2} \gamma \mathfrak{F} d\left(\sum_{i=1}^{n} \delta x_i * \mu_i \right) \right| + 2\varepsilon \|\mu\| + \varepsilon$$

$$\leqslant \left| \int_{\mathbf{D}_2} \gamma \mathfrak{F} d\left(\sum_{i=1}^{n} \delta x_i * \mu_i \right) + \int_{\mathbf{D}_2} \gamma \mathfrak{F}\, d\nu \right| + \varepsilon + 2\varepsilon\|\mu\| + \varepsilon(\|\nu\| + 2)$$

$$\leqslant \left| \int_{\mathbf{D}_2} \gamma \mathfrak{F}\, d\left(\sum_{=1}^{n} \delta x_i * \mu_i + \nu \right) \right| + \varepsilon + 2\varepsilon\|\mu\| + \varepsilon(\|\nu\| + 2)$$

$$\leqslant \left\| \left(\sum_{i=1}^{n} \delta x_i * \mu_i + \nu \right)\hat{} \right\|_\infty \mathfrak{F}(0) + \varepsilon + 2\varepsilon\|\mu\| + \varepsilon(\|\nu\| + 2)$$

(where the sup norm is taken over $\hat{\mathbf{D}_2}$)

$$\leqslant (1 + \varepsilon)\|(\mu + \nu)\hat{}\|_\infty + (1 + \varepsilon)(\varepsilon) + \varepsilon + 2\varepsilon\|\mu\| + \varepsilon(\|\nu\| + 2)$$

and so

$$\|\hat{\mu}\|_\infty \leqslant \|\mu + \nu\|_\infty$$

where the supremum norm is taken over $\hat{\mathbf{D}}$.

From this we have the corollary.

COROLLARY 1. *Let $K \subseteq \mathbf{D}_2$ be a compact perfect K_2 subset of $\mathbf{D}_2$ such that $\overline{\mathrm{Gp}\, K} = \mathbf{D}_2$. Then the idempotent associated with the Raikov System generated by K is contained in the closure of the characters $\overline{\hat{\mathbf{D}_2}}$ in $\Delta M(\mathbf{D}_2)$.*

COROLLARY 2. *Let $K \subseteq \mathbf{D}_2$ be a compact perfect K_2 subset of $\mathbf{D}_2$. Then the idempotent associated with the Raikov System generated by K is contained in the closure of the characters $\overline{\hat{\mathbf{D}_2}}$ in $\Delta M(\mathbf{D}_2)$.*

PROOF. $\mathbf{D}_2 = \overline{\mathrm{Gp}\, K} \oplus H$ for some closed subgroup H of $\mathbf{D}_2$. We can give $\mathbf{D}_2$ a finer l.c.a. topology $\mathfrak{F}$ where

$$(\mathbf{D}_2)_{\mathfrak{F}} = \overline{\mathrm{Gp}\, K} \oplus H_d$$

where H_d is the group H with the discrete topology. The positive definite function F with

1. $F(x) = 1 \quad \forall x \in \overline{\mathrm{Gp}\, K}$
2. $F(x) = 0 \quad$ elsewhere

is continuous on $(\mathbf{D}_2)_{\mathscr{F}}$ and so there exist continuous positive definite functions on $(\mathbf{D}_2)_{\mathscr{F}}$ as required in Theorem 2. Hence the idempotent $I_K \in \overline{((\mathbf{D}_2)_{\mathscr{F}})\hat{}}$ but $\overline{((\mathbf{D}_2)_{\mathscr{F}})\hat{}} \subseteq \overline{\hat{\mathbf{D}}_2}$ so $I_K \in \overline{\hat{\mathbf{D}}_2}$.

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