

ON ALMOST LOCALLY CONNECTED SPACES

TAKASHI NOIRI

(Received March 25, 1982)

Communicated by J. H. Rubinstein

Abstract

In this paper it is shown that almost local connectedness is hereditary for the subspace that is the union of regular open sets and is preserved under almost-open (in the sense of Singal) θ -continuous surjections.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 54C10, 54D05.

1. Introduction

Recently, V. J. Mancuso [3] has introduced and investigated the concept of almost locally connected spaces. In [3], among others, the following theorems have been established:

THEOREM A. *Let $f: X \rightarrow Y$ be an almost-open, almost-continuous and connected surjection. If X is almost locally connected and Y is almost-regular, then Y is almost locally connected.*

THEOREM B. *Let $f: X \rightarrow Y$ be an open, almost-continuous and connected surjection. If X is almost locally connected, then so is Y .*

The main purpose of the present paper is to improve the previous theorems. In Section 4 it will be shown that almost local connectedness is preserved under almost-open and almost-continuous surjections. By making use of this result we

shall show in Section 5 that the word “regular open” in Theorem 3.8 of [3] can be replaced by “the union of regular open sets”.

2. Preliminaries

Throughout this paper spaces mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let S be a subset of a space X . The closure of S and the interior of S in X are denoted by $\text{Cl}_X(S)$ and $\text{Int}_X(S)$ (or simply $\text{Cl}(S)$ and $\text{Int}(S)$), respectively. A subset S is said to be *regular open* (resp. *regular closed*) if $\text{Int}(\text{Cl}(S)) = S$ (resp. $\text{Cl}(\text{Int}(S)) = S$). The family of all regular open sets of a space X is denoted by $\text{RO}(X)$. A function $f: X \rightarrow Y$ is said to be *almost-continuous* [12] (resp. *θ -continuous* [1], *weakly-continuous* [2]) if for each point $x \in X$ and each open set V of Y containing $f(x)$ there exists an open set U of X containing x such that $f(U) \subset \text{Int}_Y(\text{Cl}_Y(V))$ (resp. $f(\text{Cl}_X(U)) \subset \text{Cl}_Y(V)$, $f(U) \subset \text{Cl}_Y(V)$).

REMARK 2.1. It is known that *continuity* \Rightarrow *almost-continuity* \Rightarrow *θ -continuity* \Rightarrow *weak-continuity* and none of these implications is reversible ([6], [12]).

3. Almost locally connected spaces

DEFINITION 3.1. A space X is said to be *almost locally connected* (simply a.l.c.) [3] if for each $x \in X$ and each $G \in \text{RO}(X)$ containing x there exists an open connected set V such that $x \in V \subset G$.

We shall begin by giving a characterization of a.l.c. spaces which will be used in the subsequence.

THEOREM 3.2. *The following statements are equivalent for a space X :*

- (1) X is a.l.c.
- (2) *The components of regular open sets in X are regular open in X .*
- (3) *For each $x \in X$ and each $G \in \text{RO}(X)$ containing x , there exists a regular open connected set V such that $x \in V \subset G$.*

PROOF. (1) \Rightarrow (2): Let $G \in \text{RO}(X)$ and C be a component of G . By Theorem 3.5 of [3], C is open in X and $C \subset \text{Int}_X(\text{Cl}_X(C))$. On the other hand, since C is connected, so is $\text{Int}_X(\text{Cl}_X(C))$. Since C is a component of G , $\text{Int}_X(\text{Cl}_X(C)) \subset C$. Therefore, we have $C = \text{Int}_X(\text{Cl}_X(C))$ which shows that $C \in \text{RO}(X)$.

(2) \Rightarrow (3) and (3) \Rightarrow (1) are easy and the proofs are thus omitted.

DEFINITION 3.3. A space X is said to be *nearly-compact* [11] if every regular open cover of X has a finite subcover.

COROLLARY 3.4. A nearly-compact a.l.c. space has a finite number of components.

PROOF. Let X be a nearly-compact a.l.c. space. Since $X \in \text{RO}(X)$, by Theorem 3.2 the family of components of X is a regular open cover of X . Therefore, X has a finite number of components.

A space X is said to be *weakly-Hausdorff* [13] if every point of X is the intersection of regular closed sets.

THEOREM 3.5. A nearly-compact weakly-Hausdorff space X is a.l.c. if and only if every regular open cover of X is refined by a cover consisting of a finite number of regular open connected sets.

PROOF. Let X be a nearly-compact a.l.c. space and $\mathcal{V} = \{V_\alpha \mid \alpha \in \nabla\}$ a regular open cover of X . By Theorem 3.2, for each $\alpha \in \nabla$ the components $C_{\alpha(j)}$ of V_α are regular open in X , where $\alpha(j) \in \nabla(\alpha)$. Since X is nearly-compact, there exist a finite subset ∇_0 of ∇ and a finite subset $\nabla_0(\alpha)$ of $\nabla(\alpha)$ for each $\alpha \in \nabla_0$ such that

$$X = \bigcup \{C_{\alpha(j)} \mid \alpha(j) \in \nabla_0(\alpha), \alpha \in \nabla_0\}.$$

Thus, the family $\{C_{\alpha(j)} \mid \alpha(j) \in \nabla_0(\alpha), \alpha \in \nabla_0\}$ is a desirable refinement of \mathcal{V} .

Conversely, under the condition that X is a weakly-Hausdorff space and the hypothesis holds, we shall show that X is a.l.c. Let $x \in X$ and $x \in G \in \text{RO}(X)$. Since X is weakly-Hausdorff, for each $y \in X - G$ there exists $U_y \in \text{RO}(X)$ such that $y \in U_y$ and $x \notin U_y$. Then $G \cup \{U_y \mid y \in X - G\}$ is a regular open cover of X . By the hypothesis, it has a refinement $\{V_\alpha \mid \alpha \in \nabla\}$ consisting of a finite number of regular open connected sets. There exists an $\alpha_0 \in \nabla$ such that $x \in V_{\alpha_0}$. If $V_{\alpha_0} \subset U_y$ for some $y \in X - G$, then $x \in U_y$. This is a contradiction. Therefore, we obtain $x \in V_{\alpha_0} \subset G$. This shows that X is a.l.c.

4. Preservation theorems

A function $f: X \rightarrow Y$ is said to be *almost-open* (simply a.o.R.) [10] if $f(U) \subset \text{Int}_Y(\text{Cl}_Y(f(U)))$ for every open set U of X . A function $f: X \rightarrow Y$ is said to be *connected* [8] if for each connected set C of X , $f(C)$ is connected in Y . The following theorem is an improvement of Theorem B.

THEOREM 4.1. *Let $f: X \rightarrow Y$ be an a.o.R., weakly-continuous and connected surjection. If X is a.l.c., then so is Y .*

PROOF. Let $y \in Y$ and $y \in G \in \text{RO}(Y)$. It follows from Theorem 3.4 of [6] that $f^{-1}(G) \in \text{RO}(X)$. Since X is a.l.c., for $x \in f^{-1}(y)$ there exists an open connected set U of X such that $x \in U \subset f^{-1}(G)$. Since f is a.o.R., we have $f(U) \subset \text{Int}(\text{Cl}(f(U)))$. Since f is connected, $f(U)$ is connected and so is $\text{Int}(\text{Cl}(f(U)))$. Moreover, we obtain

$$y \in f(U) \subset \text{Int}(\text{Cl}(f(U))) \subset G.$$

This shows that Y is a.l.c.

We shall show the main theorem of this paper which is an improvement of Theorem A and Theorem B. For this purpose we need some lemmas.

DEFINITION 4.2. A function $f: X \rightarrow Y$ is said to be *almost-open* (simply a.o.S.) [12] if for each $U \in \text{RO}(X)$ $f(U)$ is open in Y .

REMARK 4.3. It is known in [7] that “a.o.S.” neither implies “a.o.R.”, nor does “a.o.R.” imply “a.o.S.”.

LEMMA 4.4. *Let $f: X \rightarrow Y$ be a weakly-continuous surjection and X_0 be a subset of X . If $f(X_0)$ is open in Y , then the function $f_0: X_0 \rightarrow f(X_0)$, defined by $f_0(x) = f(x)$ for each $x \in X_0$, is weakly-continuous.*

PROOF. Put $Y_0 = f(X_0)$. Let $x \in X_0$ and V_0 be an open set of Y_0 containing $f_0(x)$. Since Y_0 is open in Y , V_0 is open in Y . By weak-continuity of f , there exists an open set U of X containing x such that $f(U) \subset \text{Cl}_Y(V_0)$. Put $U_0 = U \cap X_0$, then U_0 is an open set of X_0 containing x and $f_0(U_0) \subset \text{Cl}_Y(V_0) \cap Y_0 = \text{Cl}_{Y_0}(V_0)$. This shows that f_0 is weakly-continuous.

LEMMA 4.5. *Let $f: X \rightarrow Y$ be an a.o.S. weakly-continuous surjection. If U is a regular open connected set of X , then $f(U)$ is open connected in Y .*

PROOF. Since f is a.o.S. and $U \in \text{RO}(X)$, $f(U)$ is open in Y . It follows from Lemma 4.4 that the function $f_0: U \rightarrow f(U)$ is weakly-continuous. Since U is a connected set of X , by Theorem 3 of [4] $f_0(U) = f(U)$ is connected.

THEOREM 4.6. *Let $f: X \rightarrow Y$ be an a.o.S. θ -continuous surjection. If X is a.l.c., then so is Y .*

PROOF. Let $y \in Y$ and $y \in G \in \text{RO}(Y)$. Every a.o.S. function is weakly-open [7, Lemma 1.4]. Hence, it follows from Theorem 4.4 of [6] that $f^{-1}(G) \in \text{RO}(X)$. Since X is a.l.c., for $x \in f^{-1}(y)$ by Theorem 3.2 there exists a regular open connected set U of X such that $x \in U \subset f^{-1}(G)$. Every θ -continuous function is weakly-continuous. Therefore, by Lemma 4.5 $f(U)$ is open connected in Y and $y \in f(U) \subset G$. This shows that Y is a.l.c.

COROLLARY 4.7. *Almost local connectedness is preserved under a.o.S. almost-continuous surjections.*

PROOF. This is an immediate consequence of Theorem 4.6.

REMARK 4.8. The previous corollary shows that the hypothesis “connected” on f in Theorems A and B and also “almost-regular” on Y in Theorem A can be removed.

In this paper, for simplicity, we call the set X with the topology having $\text{RO}(X)$ as a basis the *semi-regularization*, denoted by X_s , of a space X .

COROLLARY 4.9. *A space X is a.l.c. if and only if the semi-regularization X_s is locally connected.*

PROOF. *Necessity.* Let X be a.l.c. The identity function $i_X: X \rightarrow X_s$ is a.o.S. and continuous. Thus, by Corollary 4.7 X_s is a.l.c. and it is locally connected by Proposition 3.3 of [3].

Sufficiency. Let X_s be locally connected. The identity function $j_X: X_s \rightarrow X$ is open and almost-continuous. It follows from Corollary 4.7 that X is a.l.c.

THEOREM 4.10. *Let $f: X \rightarrow Y$ be an almost-continuous surjection. If X is compact a.l.c. and Y is Hausdorff, then Y is a.l.c.*

PROOF. Since f is almost-continuous and X is compact, $Y = f(X)$ is nearly-compact [11, Theorem 3.2]. Moreover, since Y is Hausdorff, it is almost-regular [11, Theorem 2.4]. It follows from Theorems 4.9 and 2.5 of [5] that f_s is a continuous function of a compact space X_s onto a Hausdorff space Y_s . Therefore, f_s is closed. Since X is a.l.c., by Corollary 4.9 X_s is locally connected and hence $f_s(X_s) = Y_s$ is locally connected. It follows from Corollary 4.9 that Y is a.l.c.

5. Subspaces of a.l.c. spaces

Theorem 3.8 of [3] states that if A is a regular open set of an a.l.c. space X then the subspace A is a.l.c. The following theorem shows that the hypothesis “regular open” on A in this result can be replaced by “the union of regular open sets”.

THEOREM 5.1. *If X is an a.l.c. space and A is the union of arbitrarily many regular open sets, then the subspace A is a.l.c.*

PROOF. It follows from Corollary 4.9 that X_s is locally connected. Since the identity function $i_X: X \rightarrow X_s$ is a.o.S., $i_X(A)$ is open in X_s and hence the subspace $i_X(A)$ is locally connected. The identity function $j_X: X_s \rightarrow X$ is open and almost-continuous. Since A is open in X , by Lemma 4.4 the induced identity function $(j_X)_0: i_X(A) \rightarrow A$ is open weakly-continuous. Moreover, every open weakly-continuous function is almost-continuous [12, Theorem 2.3]. Therefore, it follows from Corollary 4.7 that A is a.l.c.

A subset K of a Hausdorff space X is said to be *H-closed relative to X* [9] if for every cover $\{\nabla_\alpha \mid \alpha \in \nabla\}$ of K by open sets of X there exists a finite subset ∇_0 of ∇ such that $K \subset \bigcup \{\text{Cl}_X(V_\alpha) \mid \alpha \in \nabla_0\}$.

COROLLARY 5.2. *If a space X is a.l.c. Hausdorff and K is H-closed relative to X , then $X - K$ is a.l.c.*

PROOF. Let $x \in X - K$. Since X is Hausdorff, for each $y \in K$ there exist regular open sets $V(y)$ and $W(y)$ containing x and y , respectively, such that $V(y) \cap \text{Cl}(W(y)) = \emptyset$. Since $\{W(y) \mid y \in K\}$ is a cover of K by open sets of X , there exists a finite subset K_0 of K such that

$$K \subset \bigcup \{\text{Cl}_X(W(y)) \mid y \in K_0\}.$$

Put $V_x = \bigcap \{V(y) \mid y \in K_0\}$, then $x \in V_x \in \text{RO}(X)$ and $V_x \subset X - K$. It follows from Theorem 5.1 that $X - K$ is a.l.c.

References

- [1] S. Fomin, ‘Extensions of topological spaces’, *Ann. of Math.* **44** (1943), 471–480.
- [2] N. Levine, ‘A decomposition of continuity in topological spaces’, *Amer. Math. Monthly* **68** (1961), 44–46.
- [3] V. J. Mancuso, ‘Almost locally connected spaces’, *J. Austral. Math. Soc. Ser. A* **31** (1981), 421–428.

- [4] T. Noiri, 'On weakly continuous mappings', *Proc. Amer. Math. Soc.* **46** (1974), 120–124.
- [5] T. Noiri, 'On δ -continuous functions', *J. Korean Math. Soc.* **16** (1980), 161–166.
- [6] T. Noiri, 'A note on η -continuous functions', *J. Korean Math. Soc.* **18** (1981), 37–42.
- [7] T. Noiri, 'Semi-continuity and weak-continuity', *Czechoslovak. Math. J.* **31** (1981), 314–321.
- [8] W. J. Pervin and N. Levine, 'Connected mappings of Hausdorff spaces', *Proc. Amer. Math. Soc.* **9** (1958), 488–495.
- [9] J. Porter and J. Thomas, 'On H -closed and minimal Hausdorff spaces', *Trans. Amer. Math. Soc.* **138** (1969), 159–170.
- [10] D. A. Rose, 'Weak continuity and almost continuity', preprint.
- [11] M. K. Singal and Asha Mathur, 'On nearly-compact spaces', *Boll. Un. Mat. Ital.* (4) **2** (1969), 702–710.
- [12] M. K. Singal and A. R. Singal, 'Almost-continuous mappings', *Yokohama Math. J.* **16** (1968), 63–73.
- [13] T. Soundararajan, 'Weakly Hausdorff spaces and the cardinality of topological spaces', *General topology and its relations to modern analysis and algebra* III, Proc. Conf. Kampur, 1968, pp. 301–306 (Academia, Prague, 1971).

Department of Mathematics
Yatsushiro College of Technology
Yatsushiro-shi, Kumamoto-ken
866 Japan