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WHEN IS THE ALGEBRA OF REGULAR SETS FOR A FINITELY ADDITIVE BOREL MEASURE A σ-ALGEBRA?

THOMAS E. ARMSTRONG

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Abstract

It is shown that the algebra of regular sets for a finitely additive Borel measure μ on a compact Hausdorff space is a σ-algebra only if it includes the Baire algebra and μ is countably additive on the σ-algebra of regular sets. Any infinite compact Hausdorff space admits a finitely additive Borel measure whose algebra of regular sets is not a σ-algebra. Although a finitely additive measure with a σ-algebra of regular sets is countably additive on the Baire σ-algebra there are examples of finitely additive extensions of countably additive Baire measures whose regular algebra is not a σ-algebra. We examine the particular case of extensions of Dirac measures. In this context it is shown that all extensions of a $\{0, 1\}$-valued countably additive measure from a σ-algebra to a larger σ-algebra are countably additive if and only if the convex set of these extensions is a finite dimensional simplex.

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Introduction and synopsis

In [16], Kupka noted that if a vector-valued Borel measure on a compact Hausdorff space X is countably additive then its algebra of regular sets is in fact a σ-algebra. In Question 3.3.1 of [16], he asked whether countable additivity is necessary for this result. We essentially answer this question in the negative but do show that a good deal of countable additivity is implicit in the assumption that the algebra of regular sets of a finitely additive Borel measure is a σ-algebra. More specifically we show that, on any σ-algebra contained in the algebra of regular sets of a finitely additive Borel measure μ, μ is countably additive (Lemma 1). If in fact the algebra of regular sets for μ is a σ-algebra then it includes the μ-completion of the Baire algebra and μ agrees with the canonical regular extension of μ to the Borel algebra from the Baire algebra at least on the σ-algebra of regular sets (Propositions 3 and 5). In fact, the latter statement holds even if μ is only assumed to be countably additive on the Baire algebra but with the algebra of regular sets not necessarily a σ-algebra. Corollary 3.1 answers Kupka's question affirmatively for completion regular compact Hausdorff spaces. Here a finitely additive Borel measure is countably additive if and only if its algebra of regular sets is σ-algebra. Corollary 3.2 shows that on any infinite compact Hausdorff space there is a finitely additive Borel measure which does not have a σ-algebra of regular sets. This follows from Proposition 4 which asserts that a Boolean algebra admits a non-countably additive measure if and only if it is not Cantor separable if and only if its Stone space is not an almost P-space, a result of independent interest.

The latter part of the paper examines the regular algebras of finitely additive Borel measures μ whose restriction to the Baire algebra is countably additive when μ is $\{0,1\}$-valued on the Baire algebra. Proposition 6 deals with the convex compact set of all extensions of a countably additive $\{0,1\}$-valued measure δ on a σ-algebra Σ_1 to a larger σ-algebra Σ_2. It is shown that this convex compact set is finite dimensional if and only if all extensions of δ are countably additive. Otherwise, there exist 2^c mutually singular non-atomic purely finitely additive extensions or c $\{0,1\}$-valued extensions where $c = 2^{\aleph_0}$, (Corollary 6.1). This is applied to the case where Σ_1 is the Baire algebra, Σ_2 is the Borel algebra and δ is δ_x for some non-G_δ-point $x \in X$. If the extensions of δ_x to the Borel algebra are all countably additive there is a countably additive extension μ whose regular algebra is just the δ_x-completion of the Baire algebra. However, for this to be true X must be topologically pathological near x.

We conclude with an example which yields finitely additive Borel measures whose regular algebras are not σ-algebras yet contain the Baire algebra. If real valued measurable cardinals exist an example is given of a countably additive Borel measure whose regular σ-algebra is properly contained in the Borel algebra and properly contains the completed Baire algebra.

1. When is the algebra of regular sets for a finitely additive Borel measure a σ-algebra?

$\mathfrak{B}_0$ and $\mathfrak{B}$ denote, respectively, the Baire and Borel σ-algebras on X. $\mathcal{C}(X)$ denotes the real continuous functions on X and $\mathfrak{M}(X)$ the dual of $\mathcal{C}(X)$. $\mathfrak{M}(X)$ is identified, as usual, with both $CA(\mathfrak{B}_0)$ the countable additive Baire measures and with $CA_t(\mathfrak{B})$ the regular countably additive Borel measures. For any Boolean algebra $\mathcal{Q}$, $BA(\mathcal{Q})$ denotes the finitely additive real measures of bounded variation on $\mathcal{Q}$ with $CA(\mathcal{Q})$ the band of countably additive elements of $BA(\mathcal{Q})$. If $\mu \in BA^+(\mathfrak{B})$ we denote by $\mathrm{Reg}(\mu)$ all $A \in \mathfrak{B}$ so that $\inf\{\mu(\theta \setminus K)$: K compact $\subset A \subset \theta$ open$\} = 0$. Note that $\mathrm{Reg}(\mu)$ is an algebra which is μ-complete in $\mathfrak{B}$ in that whenever $\{A_n\}$ is an increasing sequence and $\{B_n\}$ is a decreasing sequence in $\mathrm{Reg}(\mu)$ with $A_n \subset B_n$ for all n and with $\lim_{n\to\infty} \mu(B_n \setminus A_n) = 0$ then $A \in \mathrm{Reg}(\mu)$ provided $A \in \mathfrak{B}$ and $A_n \subset A \subset B_n$ for all n. For any algebra $\mathcal{Q} \subset \mathfrak{B}$, $\hat{\mathcal{Q}}^\mu$ will denote its completion in $\mathfrak{B}$ with respect to the finitely additive Borel measure μ. Thus, $\mathrm{Reg}(\mu) = (\mathrm{Reg}(\mu)\hat{\ })^\mu$. This lemma was pointed out by Douglas Dokken. It is a generalization of Problem 7 on page 11 of [6].

LEMMA 1. *If Σ is a σ-algebra contained in* $\mathrm{Reg}(\mu)$ *for $\mu \in BA^+(\mathfrak{B})$ then μ is countably additive on Σ.*

PROOF. It must be shown that if $\{D_n\} \subset \Sigma$ is a disjoint sequence with union D then $\mu(D) = \sum_{n=1}^\infty \mu(D_n)$. That $\mu(D) \geqslant \sum_{n=1}^\infty \mu(D_n)$ is immediate. If we show that $\mu(D) \leqslant \sum_{n=1}^\infty \mu(D_n) + \varepsilon$ for any $\varepsilon > 0$ the assertion will be established. Pick K compact $\subset D$ with $\mu(D) \leqslant \mu(K) + \varepsilon/2$. Pick θ_n open with $D_n \subset \theta_n$ and with $\mu(\theta_n \setminus D_n) \leqslant \varepsilon 2^{-n-1}$. Since $K \subset D \subset \bigcup_{n=1}^\infty \theta_n$ there is an integer m so that $K \subset \theta_1 \cup \cdots \cup \theta_m$. For this m it is true that $\mu(K) \leqslant \sum_{n=1}^\infty \mu(\theta_m) \leqslant \sum_{n=1}^\infty \mu(\theta_n) < \sum_{n=1}^\infty \mu(D_n) + \varepsilon/2$. Thus, $\mu(D) < \sum_{n=1}^\infty \mu(D_n) + \varepsilon$.

REMARK. Lemma 1 is a consequence of Proposition 1.6 in Chapter V of [4] and of Lemma 1 of [25].

COROLLARY 1.1. a) *If $\mu \in BA^+(\mathfrak{B})$ and* $\mathrm{Reg}(\mu)$ *is a σ-algebra then μ is countably additive on* $\mathrm{Reg}(\mu)$.

b) $\mathrm{Reg}(\mu)$ *is a σ-algebra if and only if μ is countably additive on the σ-algebra generated by* $\mathrm{Reg}(\mu)$.

PROOF. Only b) needs to be established. This is done in the standard fashion. Let $\{D_n\}$ be a disjoint sequence in $\mathrm{Reg}(\mu)$ with union D. Let θ_n be open with $D_n \subset \theta_n$ and $\mu(\theta_n \setminus D_n) \leqslant 2^{-n-1}\cdot\varepsilon$ for a given $\varepsilon > 0$. Let m be such that $\mu(\bigcup_{n=m+1}^\infty D_n) \leqslant \varepsilon/4$. Let $K_n \subset D_n$ for $n = 1,\ldots,m$ be compacts with $\mu(D_n \setminus K_n) < \frac{1}{4}\varepsilon m^{-1}$. We have $\mu[(\bigcup_{n=1}^\infty \theta_n) \setminus (\bigcup_{n=1}^m K_n)] \leqslant \varepsilon$ with $\bigcup_{n=1}^m K_n \subset D \subset \bigcup_{n=1}^\infty \theta_n$. Thus, $D \in \mathrm{Reg}(\mu)$. Thus, $\mathrm{Reg}(\mu)$ is a σ-algebra if μ is countably additive on the σ-algebra generated by $\mathrm{Reg}(\mu)$. The converse follows from a).

LEMMA 2. *Let* $A \in \mathrm{Reg}(\mu)$.

i) *There exists a* G_δ $A_\delta \in \mathrm{Reg}(\mu)$ *and an* F_σ $A_\sigma \in \mathrm{Reg}(\mu)$ *with* $A_\sigma \subset A \subset A_\delta$ *and* $\mu(A_\delta \setminus A_\sigma) = 0$.

ii) *There exists a* G_δ $A^\delta \in \mathfrak{B}_0 \cap \mathrm{Reg}(\mu)$ *and an* F_σ $A^\sigma \in \mathfrak{B}_0 \cap \mathrm{Reg}(\mu)$ *with* $A_\sigma \subset A^\sigma \subset A^\delta \subset A_\delta$.

iii) $\mu(A) = \mu(A_\sigma) = \mu(A_\delta) = \mu(A^\sigma) = \mu(A^\delta) = \sup\{\mu(K)$: *$K$ compact Baire* $\subset A^\sigma\} = \inf\{\mu(G)$: *$G$ open Baire* $\supset A^\delta\}$.

iv) *There is an* $A_0 \in \mathfrak{B}_0 \cap \mathrm{Reg}(\mu)$ *with* $\mu(A \Delta A_0) = 0$.

PROOF.

i) Immediate from the definition of regularity.

ii) Let $A_\sigma = \bigcup_{n=1}^\infty K_n$ and $A_\sigma = \bigcap G_n$ where K_n is compact and G_n is open for all n. By Urysohn's Theorem there is a compact G_δ, $K'_{n,m}$ satisfying $K_n \subset K'_{n,m} \subset G_m$ for all n, m. Set $K'_n = \bigcap_{m=1}^\infty K'_{n,m}$. K'_n is a compact G_δ and $K_n \subset K'_n \subset A_\delta$ for all n. Set A^σ equal to the F_σ, $\bigcup_{n=1}^\infty K'_n$. A^δ is obtained analogously as a countable intersection of open F_σ sets.

iii) From the definition of regularity the K_n in ii) may be chosen with $\mu(A) = \sup \mu(K_n) \leqslant \sup \mu(K'_n) \leqslant \sup\{\mu(K)$: K compact Baire $\subset A^\sigma\} \leqslant \mu(A^\sigma) = \mu(A)$. Thus, $\mu(A) = \sup\{\mu(K)$: K compact Baire $\subset A^\sigma\}$. Similarly, $\mu(A) = \inf\{\mu(G)$: G open Baire $\supset A^\sigma\}$.

iv) Set $A_0 = A^\delta$ or A^σ.

Plachky, [20], shows that if ν is a finitely additive probability on a Boolean algebra $\mathcal{Q}_1$ and $BA_1^+(\mathcal{Q}_1, \nu, \mathcal{Q}_2)$ denotes the convex compact set of extensions of ν to a probability measure on a larger algebra $\mathcal{Q}_2$ then $\mu \in BA_1^+(\mathcal{Q}_1, \nu, \mathcal{Q}_2)$ is extreme if and only if for all $A_2 \in \mathcal{Q}_2$ and $\varepsilon > 0$ there is an $A_1 \in \mathcal{Q}_1$ with $\mu(A_1 \Delta A_2) < \varepsilon$. Thus, in Lemma 2, μ, on $\mathrm{Reg}(\mu)$, is an extreme extension of its restriction to $\mathfrak{B}_0 \cap \mathrm{Reg}(\mu)$.

PROPOSITION 3. *If* $\mu \in BA^+(\mathfrak{B})$ *is such that* $\mathrm{Reg}(\mu)$ *is a* σ*-algebra then* $\mathfrak{B}_0 \subset \mathrm{Reg}(\mu)$.

To establish this we first consider the case $X = [0, 1]$. Let Y denote those $x \in (0, 1)$ so that $\inf\{\mu(\theta)$: $x \in \theta$ open$\} = 0$. The complement of Y is at most countably hence Y is dense. Each $\{x\}$ with $x \in Y$ is in $\mathrm{Reg}(\mu)$ with $\mu(\{x\}) = 0$. For $\varepsilon > 0$ let θ be an open set containing $x \in Y$ with $\mu(\theta_\varepsilon) < \varepsilon$, $K_\varepsilon^- = [0, x) \setminus \theta_\varepsilon$ and $K_\varepsilon^+ = (x, 1] \setminus \theta_\varepsilon$. Both K_ε^- and K_ε^+ are compact. It is easily verified that $\lim_{\varepsilon \to 0} \mu(K_\varepsilon^-) = \mu([0, x))$ and $\lim_{\varepsilon \to 0} \mu(K_\varepsilon^+) = \mu((x, 1])$. Thus, $\{[0, x), (x, 1]\} \subset \mathrm{Reg}(\mu)$. It follows that all intervals, open, closed, or half open, whose endpoints are chosen from Y belong to $\mathrm{Reg}(\mu)$. The σ-algebra generated by these intervals is $\mathfrak{B}_0 = \mathfrak{B}$. Since $\mathrm{Reg}(\mu)$ is a σ-algebra $\mathfrak{B}_0 = \mathrm{Reg}(\mu)$. This establishes this case.

Let X be arbitrary and let f: $X \to [0, 1]$ be continuous. Let ν be the finitely additive Borel measure on $[0, 1]$ which is the image of μ under f. Thus, for Borel $A \subset [0, 1]$, $\nu(A) = \mu(f^{-1}(A))$. Just as in the countably additive case $A \in \mathrm{Reg}(\nu)$ if and only if $f^{-1}(A) \in \mathrm{Reg}(\mu)$. Consequently, $\mathrm{Reg}(\nu)$ is a σ-algebra hence is equal to the Borel algebra of $[0, 1]$ by the special case just established. Thus, f is measurable for the σ-algebra $\mathrm{Reg}(\mu)$. Since f is arbitrary it follows that all $f \in \mathcal{C}(X)$ are $\mathrm{Reg}(\mu)$-measurable. Thus, since $\mathfrak{B}_0$ is the smallest σ-algebra so that all $f \in \mathcal{C}(X)$ are $\mathfrak{B}_0$-measurable, $\mathfrak{B}_0 \subset \mathrm{Reg}(\mu)$. This establishes the proposition.

In [4], Babiker and Knowles define a space X to be *completion regular* if and only if every $\mu \in CA^+(\mathfrak{B}_0)$ is completion regular in the sense of Berberian [5]. That is, each $\mu \in CA^+(\mathfrak{B}_0)$ has a unique extension in $BA^+(\mathfrak{B})$. Alternatively X is completion regular if and only if $\mathfrak{B}$ is the μ-completion of $\mathfrak{B}_0$ for all $\mu \in CA^+(\mathfrak{B}_0)$. Examples of completion regular spaces include all perfectly normal compact Hausdorff spaces X. In [5] Berberian notes that if X is completion regular all points must be G_δ's. Under the assumption that the continuum is real valued measurable an example may be constructed of a non-completion regular X each of whose points is a G_δ. In order that X be completion regular it is necessary and sufficient that every Borel set be regular with respect to the paving of compact G_δ's for all countably additive Borel measures. This corollary is easily deduced from the definition of completion regularity.

COROLLARY 3.1. *Let X be completion regular. The following are equivalent for* $\mu \in BA^+(\mathfrak{B})$

a) $\mathrm{Reg}(\mu)$ *is a σ-algebra*

b) $\mathrm{Reg}(\mu) = \mathfrak{B}$

c) $\mu \in CA^+(\mathfrak{B}) = CA_t^+(\mathfrak{B})$.

COROLLARY 3.2. *If X is an infinite compact Hausdorff space there is a* $\mu \in BA^+(\mathfrak{B})$ *so that* $\mathrm{Reg}(\mu)$ *is not a σ-algebra.*

PROOF. Any extension μ to $\mathfrak{B}$ of a member of $BA^+(\mathfrak{B}_0) \setminus CA^+(\mathfrak{B}_0)$ will do. The non-emptiness of $BA^+(\mathfrak{B}_0) \setminus CA^+(\mathfrak{B}_0)$ is a special case of Proposition 4.

We are interested in determining for which infinite Boolean algebras $\mathcal{C}$ every element of $BA^+(\mathcal{C})$ is countably additive. If no infinite strictly decreasing sequence in $\mathcal{C}$ has a lower bound then, automatically, $BA^+(\mathcal{C}) = CA^+(\mathcal{C})$. Such Boolean algebras are termed *Cantor separable* in [28]. Cantor separable Boolean algebras $\mathcal{C}$ are characterized in terms of their Stone space $X_{\mathcal{C}}$ by the fact that each non-empty zero set has a non-empty interior. Completely regular spaces X with the aforementioned property are called *almost P-spaces* in [17] and have been studied in [7], [10] and [27]. Thus, $\mathcal{Q}$ is Cantor separable if $X_{\mathcal{Q}}$ is an almost P-space. Notice that if $\mathcal{Q}$ is σ-complete it is not Cantor separable if it is infinite. $\beta N \setminus N$ is the most familiar example of an almost P-space [9, 65.8]. Graves and Wheeler in [10] give a method for producing a large class of almost P-spaces. The following proposition was pointed out by R. F. Wheeler.

PROPOSITION 4. *The following are equivalent for an infinite Boolean Algebra* $\mathcal{Q}$

a) $\mathcal{Q}$ *is Cantor separable*

b) $X_{\mathcal{Q}}$ *is an almost P-space*

c) $BA^+(\mathcal{Q}) = CA^+(\mathcal{Q})$.

PROOF. We already have a) $\Leftrightarrow$ b) $\Rightarrow$ c). Let us assume c) and see that this implies b). Notice that all $\{0, 1\}$-valued elements of $BA^+(\mathcal{Q})$ are countably additive. Phrased in terms of the corresponding ultrafilters on $\mathcal{Q}$ this says that if $\{A_n: n \in \mathbf{N}\}$ is a decreasing sequence in an ultrafilter then $\varnothing \neq \inf_n A_n$. That is, there is an $A_\infty \in \mathcal{Q}$ with $\varnothing \neq A_\infty \subset A_n$ for all n. Since every decreasing sequence of non-empty elements of $\mathcal{Q}$ lies in an ultrafilter this says that no decreasing sequence of non-empty elements of $\mathcal{Q}$ has $\varnothing$ as infimum. In particular, regarding $\mathcal{Q}$ as the clopen algebra of $X_{\mathcal{Q}}$, the intersection of a decreasing sequence of non-empty clopen sets (that is, a zero set) has non-empty interior. Thus, c) implies both a) and b).

REMARK. We use the term *δ-ultrafilter* for an arbitrary Boolean algebra to denote any ultrafilter whose corresponding $\{0, 1\}$-valued measure is countably additive.

A compact Hausdorff space X is called *Borel regular* [19], or *Radon*, [21], if and only if $CA^+(\mathfrak{B}) = CA_t^+(\mathfrak{B})$ if and only if every $\mu \in CA^+(\mathfrak{B}_0)$ has a unique extension, the regular extension, to $\mathfrak{B}$ belonging to $CA^+(\mathfrak{B})$. If $\mu \in CA_1^+(\mathfrak{B}) \setminus CA_t^+(\mathfrak{B})$ then $\mathrm{Reg}(\mu)$ is a super-σ-algebra of $\mathfrak{B}_0$ properly contained in $\mathfrak{B}$. The canonical example of a non-Borel regular space is the compact ordinal space $[0, \omega_1]$ where ω_1 is the first uncountable ordinal. There are countably additive $\{0, 1\}$-valued extensions of the Dirac measure δ_{ω_1} from $\mathfrak{B}_0$ to $\mathfrak{B}$ other than the regular extension [9, ex. 53.10a]. An example of a Borel regular space X which is not completion regular is the one point compactification $D \cup \{\infty\}$ of a discrete space D with uncountable non-real-valued measurable cardinal, [8, ex. 6.2]. The Dirac measure δ_∞ has extensions from $\mathfrak{B}_0$ to $\mathfrak{B}$ other than the regular one but all must be purely finitely additive [2], [13], since they induce on D finitely additive, diffuse [2], probability measures. We shall be primarily concerned with $\mathrm{Reg}(\mu)$ for μ non-countably additive yet with μ countably additive on $\mathfrak{B}_0$ but occasionally with μ countably additive and non-regular on $\mathfrak{B}$. In any case, μ_{reg} will denote the unique element of $CA_t^+(\mathfrak{B})$ agreeing with μ on $\mathfrak{B}_0$.

PROPOSITION 5. *Let $\mu \in BA^+(\mathfrak{B})$ be countably additive on $\mathfrak{B}_0$. On* $\mathrm{Reg}(\mu)$, *μ and μ_{reg} coincide.*

PROOF. Let $A \in \mathrm{Reg}(\mu)$. One can, in the proof of Lemma 2, find A_σ an F_σ in $\mathrm{Reg}(\mu)$ and A_δ a G_δ in $\mathrm{Reg}(\mu)$, so that $A_\sigma \subset A \subset A_\delta$ and so that $\mu(A_\delta \setminus A_\sigma) = \mu_{\mathrm{reg}}(A_\delta \setminus A_\sigma) = 0$. Let $\{A^\sigma, A^\delta\} \subset \mathfrak{B}_0 \cap \mathrm{Reg}(\mu)$ with $A_\sigma \subset A^\sigma \subset A^\delta \subset A_\delta$. Then, $\mu(A) = \mu(A^\sigma) = \mu_{\mathrm{reg}}(A^\sigma) = \mu_{\mathrm{reg}}(A)$.

In the remainder of the paper we will be dealing fairly exclusively with extensions μ of Dirac measures δ_x for $x \in X$ from $\mathfrak{B}_0$ to $\mathfrak{B}$. All such extensions must be $\{0, 1\}$-valued on $\mathrm{Reg}(\mu)$. If $A \in \mathrm{Reg}(\mu)$ then $\mu(A) = 0$ if and only if $x \notin A$.

PROPOSITION 6. *Let $\Sigma_1 \subset \Sigma_2$ be σ-algebras of subsets of a set Ω. Let $\delta \in CA_1^+(\Sigma_1)$ be $\{0, 1\}$-valued. Let η be the σ-ideal in Σ_2 of sets of outer measure* 0 *under δ.*

i) *If the quotient algebra Σ_2/η is finite then $BA_1^+(\Sigma_1, \delta, \Sigma_2)$ is a finite dimensional subset of $CA_1^+(\Sigma_2)$.*

ii) *If Σ_2/η is infinite there is a family $\{\mu_t^1\} \subset BA_1^+(\Sigma_1, \delta, \Sigma_2)$ of mutually singular, non-atomic, purely finitely additive measures whose cardinality is 2^c where c is the continuum.*

PROOF. There is an affine bijection from $BA_1^+(\Sigma_1, \delta, \Sigma_2)$ to $BA_1^+(\Sigma_2/\eta)$. If $\mu \in BA_1^+(\Sigma_1, \delta, \Sigma_2)$ then $\mu(A) = 0$ for all $A \in \eta$ hence μ induces on Σ_2/η an element, also denoted by μ, in the usual fashion. This gives the affine bijection.

ii) If Σ_2/η is infinite it is an infinite F-algebra as in [3]. By Corollary 3.2.3 of [3] there is a family $\{\mu_t\}$, of cardinality 2^c, of mutually singular non-atomic probability measures on Σ_2/η all with the same negligible sets. Pulling back under the affine bijection from $BA_1^+(\Sigma_1, \delta, \Sigma_2)$ to $BA_1^+(\Sigma_2/\eta)$ one obtains the same sort of family in $BA^+(\Sigma_1, \delta, \Sigma_2)$. If $\mu_s \in \{\mu_t\}$ is countably additive there can be no other countably additive $\mu_r \in \{\mu_t\}$ for $\mu_r \perp \mu_s$ and both have the same nullsets. Delete μ_s if necessary so that no element of $\{\mu_t\}$ is countably additive. Each μ_t has a non-trivial purely finitely additive part which is a multiple of a purely finitely additive μ_t' which is easily verified to belong to $BA_1^+(\Sigma_1, \delta, \Sigma_2)$. Furthermore, μ_t' must be non-atomic for each t. This establishes ii).

i) Suppose that Σ_2/η is finite and has n atoms $\{a_1,\ldots,a_n\}$. Corresponding to each a_i is an $A_i \in \Sigma_2$ which is such that if $A \in \Sigma_2$ then $A_i \setminus A \in \eta$ or $A \cap A_i \in \eta$. The $\{0,1\}$-valued measure δ_i on Σ_2/η or in $BA_1^+(\Sigma_1, \delta, \Sigma_2)$ corresponding to a_i is an extreme point of $BA_1^+(\Sigma_2/\eta)$ and $BA_1^+(\Sigma_2/\eta) = \mathrm{conv}(\delta_1,\ldots,\delta_n)$. To show that $BA_1^+(\Sigma_1, \delta, \Sigma_2) \subset CA^+(\Sigma_2)$ it suffices to show that each δ_i, considered as an element of $BA_1^+(\Sigma_1, \delta, \Sigma_2)$, is in $CA^+(\Sigma_2)$. To this end let $\{E_n\}$ be an increasing sequence in Σ_2 with $\delta_i(E_n) = 0$ for all n. We have $E_n \cap A_i \in \eta$ for all n hence, by the σ-completeness of η, we have $(\bigcup_n E_n) \cap A_i \in \eta$. Thus, $\delta_i(\bigcup_n E_n) = 0$. This establishes countable additivity of δ_i hence establishes i).

Remarks. Recall from [2] that a measure μ is *strongly finitely additive* if and only if there is a partition $\{A_n\colon n \in N\}$ with $\mu(A_n) = 0$ for all n. Any purely finitely additive probability measure is the sum of countably many strongly finitely additive measures, [2]. In ii) purely finitely additive measures may be replaced by strongly finitely additive measures.

Actually ii) asserts only that such a family of probabilities exists in $BA(\Sigma_2/\eta)$. This is true if η is replaced by the ideal generated by the null sets of a non $\{0,1\}$-valued measure or Σ_2/η by an arbitrary F-algebra.

Corollary 6.1. *If Σ_2/η is infinite there exist c purely finitely additive $\{0,1\}$-valued elements of $BA_1^+(\Sigma_1, \delta, \Sigma_2)$.*

Proof. There is a strongly finitely additive non-atomic $\mu \in BA_1^+(\Sigma_1, \delta, \Sigma_2)$. Let $\{A_n\} \subset \Sigma_2$ be an increasing sequence with $\mu(A_n) = 0$ for all n and with $\bigcup_n A_n = \Omega$. Let $\mathcal{Q}$ denote the algebra Σ_2/η and let $X_{\mathcal{Q}}$ be its Stone space. $BA_1^+(\Sigma_1, \delta, \Sigma_2)$ is affinely homeomorphic to the Bauer simplex of Radon probability measures on $X_{\mathcal{Q}}$. Let $\tilde{\mu}$ be the Radon measure on $X_{\mathcal{Q}}$ corresponding to μ so that if $A \in \Sigma_2/\eta$ or if $A \in \Sigma_2$ then $\mu(A) = \tilde{\mu}([A])$ where $[A]$ is the clopen set in $X_{\mathcal{Q}}$ corresponding to A. We have $\mu(A) = \int \chi_{[A]}(x)\tilde{\mu}(dx) = \int \chi_x(A)\tilde{\mu}(dx)$ (where $x \in X_{\mathcal{Q}}$ are considered as ultrafiters on $\mathcal{Q}$). If there were a set Z with outer measure $\tilde{\mu}^*(Z) > 0$ of δ-ultrafilters $x \in X_{\mathcal{Q}}$ (so that each χ_x is countably additive on $\mathcal{Q}$), it would follow that $0 = \lim_{n\to\infty}\mu(A_n) = \lim_{n\to\infty}\int \chi_{[A_n]}(x)\tilde{\mu}(dx) \geqslant \tilde{\mu}^*(Z) > 0$. Since this is impossible $\tilde{\mu}$-almost all $x \in X_{\mathcal{Q}}$ have χ_x purely finitely additive. Since $\tilde{\mu}$ is non-atomic there is a compact perfect set $Y \subset \mathrm{supp}(\tilde{\mu}) \subset X_{\mathcal{Q}}$ so that if $x \in Y$ then χ_x is purely finitely additive. Y contains at least c elements.

Corollary 6.2. *If $\mathcal{Q}$ is a Boolean algebra then $\mu \in BA_1^+(\mathcal{Q})$ is purely finitely additive with corresponding measure $\tilde{\mu}$ on the Stone space $X_{\mathcal{Q}}$ only if μ-almost all $x \in X_{\mathcal{Q}}$ are not δ-ultrafilters.*

We may apply the preceding results to the case where $\mathfrak{B}_0 = \Sigma_1$ and $\mathfrak{B} = \Sigma_2$. A $\{0,1\}$-valued measure δ on $\mathfrak{B}_0$ is a Dirac measure δ_x. η will be denoted by η_x. η_x consists of those Borel sets in X contained in a σ-compact subset of $X' = X \setminus \{x\}$. We are only interested in the case where $\mathfrak{B}/\eta_x = \mathfrak{B}_x$ has cardinality larger than 2 so that $\{x\}$ is not a G_δ.

PROPOSITION 7. *Let x be a non-G_δ-point in X.*

i) *If $\mathfrak{B}_x$ is finite the elements of $BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ form a finite dimensional simplex in $CA_1^+(\mathfrak{B})$. In this case there is a $\mu \in BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ with* $\mathrm{Reg}(\mu) = \hat{\mathfrak{B}}_0^{\delta_x} = \hat{\mathfrak{B}}_0^{\mu}$.

ii) *If $\mathfrak{B}_x$ is infinite there is a family of cardinality 2^c of singular non-atomic purely finitely additive elements of $BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ and a family of cardinality c of $\{0,1\}$-valued purely finitely additive elements.*

PROOF. We need only find in case i) a $\mu \in BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ with $\mathrm{Reg}(\mu) = \hat{\mathfrak{B}}_0^{\mu}$. Let $\{\delta_x, \delta_1, \ldots, \delta_n\}$ denote the extreme points of $BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ where δ_x is the usual Dirac measure on $\mathfrak{B}$. We assert that $\mu = \frac{1}{n}(\delta_1 + \cdots + \delta_n)$ has $\mathrm{Reg}(\mu) = \hat{\mathfrak{B}}_0^{\mu}$. Suppose not. Note that $\hat{\mathfrak{B}}_0^{\mu} = \hat{\mathfrak{B}}_0^{\delta_x}$ is the largest subalgebra of $\mathfrak{B}$ to which δ_x has a unique extension. Note also that δ_x agrees with μ on $\mathrm{Reg}(\mu)$ by Proposition 5. There is an extreme extension δ of δ_x from $\hat{\mathfrak{B}}_0^{\mu}$ to $\mathrm{Reg}(\mu)$ other than δ_x hence other than μ. This extreme extension δ is the restriction of one of $\{\delta_1, \ldots, \delta_n\}$ to $\mathrm{Reg}(\mu)$, say δ_1. Since all extreme extensions of δ_x to $\mathrm{Reg}(\mu)$ are $\{0,1\}$-valued there is an $A \in \mathrm{Reg}(\mu)$ with $0 = \delta_x(A) = \mu(A)$ and $\delta_1(A) = 1$. But $\mu(A) = \frac{1}{n}(\delta_1(A) + \cdots + \delta_n(A)) \geq \frac{1}{n}$ which is impossible. Thus, $\mathrm{Reg}(\mu) = \hat{\mathfrak{B}}_0^{\mu}$.

COROLLARY 7.1. *If $\mathfrak{B}_x$ is infinite and $\mu \in BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ has* $\mathrm{Reg}(\mu) \neq \hat{\mathfrak{B}}_0^{\mu}$ *there is a $\nu \in BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ with* $\mathrm{Reg}(\nu)$ *a proper subset of* $\mathrm{Reg}(\mu)$.

REMARK. We know of no case in which x is a non-G_δ-point for which i) holds in Proposition 7. For the case $X = [0, \omega_1]$ and $x = \omega_1$ one may set A_0 equal to the relatively closed set in $[0, \omega_1)$ consisting of limit ordinals, and set $A_n = \{\alpha + 1: \alpha \in A_{n-1}\}$ for $n \in \omega$. Then $[0, \omega_1) = \bigcup_n A_n$. Each A_n is in $\mathfrak{B} \setminus \eta_x$ hence $\mathfrak{B}_x$ is infinite. A similar argument shows that if D is an infinite discrete set with uncountable cardinality then $X = D \cup \{\infty\}$ has $\mathfrak{B}_x$ infinite then $x = \infty$.

COROLLARY 7.2. *If $\mathfrak{B}_x$ is finite there is a closed set $E \subset X'$ whose complement is σ-compact and is such that E has a partition $\{E_1, \ldots, E_n\}$ with each E_i closed. Within each E_i the set $\mathfrak{F}_i$ of non-σ-compact closed sets forms a δ-ultrafilter of closed sets. If $E_i \cup \{x\} = X_i$ is considered as the one point compactification of E_i then δ_x*

has a one dimensional simplex of extensions to the Borel sets of X_i. The extreme extension δ_i is defined by $\delta_i(A) = 1$ if and only if A contains an element of $\mathfrak{F}_i$ for $i = 1,\dots,n$.

PROOF. Let $\{\delta_0, \delta_1,\dots,\delta_n\}$ be the extreme elements of $BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ with δ_0 the regular extension. For each $i = 1,\dots,n$ there is a δ-ultrafilter $\mathfrak{F}_i$ of closed subsets of X' so that $\delta_i(A) = 1$ if and only if A meets each element of $\mathfrak{F}_i$. One may find $\{F_1,\dots,F_n\}$ so that $F_i \in \mathfrak{F}_i$ for $i = 1,\dots,n$ and so that $F_i \cap F_j \in \eta_x$ for all $i \neq j$. One may find an open σ-compact $\theta \subset X'$ with $F_i \cap F_j \subset \theta$ for all i, j. Let $E_i = F_i \setminus \theta$ for all i and let $E = \bigcup_{i=1}^n E_i = X' \setminus \theta$. Any extension δ of δ_x to the Borel sets of X_i with $\delta(x) = 0$ may be extended to an element of $BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ with $\delta(E_i) = 1$. We must have $\delta = \delta_i$ which establishes the corollary.

COROLLARY 7.3. *If $\mathfrak{B}_x$ is finite every closed set in X' contains a dense σ-compact subset.*

PROOF. We may, by Corollary 7.2, assume that $BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B}) = \{\delta_x, \delta\}$ so that $\mathfrak{F} = \{F \text{ closed in } X'\colon \delta(F) = 1\}$ is the set of non-σ-compact closed sets in X'.

Assume that $X' \neq \bar{E}$ for any $E \in \eta_x$. If this is the case then $E \in \eta_x$ implies that $\bar{E} \in \eta_x$. To see this note that if $\bar{E} \notin \eta_x$ then $\bar{E} \in \mathfrak{F}$ and $\bar{E}^c \in \eta_x$. Since X is the closure of $E \cup \bar{E}^c \in \eta_x$ one has a contradiction.

Let $\{\theta_\alpha\} \subset \eta_x$ be a sequence indexed by ordinals α defined by transfinite induction so that $\bar{\theta}_\alpha$ is a proper subset of $\theta_{\alpha+1}$ and so that $\theta_\alpha = \bigcup_{\beta<\alpha}\theta_\beta$ if α is a limit ordinal. The last element θ_λ of this sequence occurs for a limit ordinal λ so that $\bar{\theta}_\lambda \in \mathfrak{F}$ hence so that $\theta_\lambda \notin \eta_x$. Since η_x is σ-complete λ is of uncountable cofinality. Let $\psi_\alpha = \theta_{\alpha+1} \setminus \bar{\theta}_\alpha$ for $\alpha < \lambda$ and let $\psi_\lambda = X' \setminus \bar{\theta}_\lambda$. We have $X' = [\bigcup\{\psi_\alpha\colon \alpha \leqslant \lambda\}] \cup [\bigcup\{\partial\theta_\alpha\colon \alpha < \lambda\}]$. The open set $\bigcup\{\psi_\alpha\colon \alpha \leqslant \lambda\}$ is dense in X' hence is not in η_x. The closed set $\bigcup\{\partial\theta_\alpha\colon \alpha < \lambda\}$ is σ-compact hence is in an open $\theta_\infty \in \eta_x$. Let $D = \{\alpha \leqslant \lambda\colon \psi_\alpha \setminus \theta_\infty \neq \varnothing\}$. The open sets $\{\psi_\alpha\colon \alpha \in D\}$ together with θ_∞ cover X'. Thus, $\operatorname{card}(D) \geqslant \aleph_1$. If K is a compact set in X' it is covered by θ_∞ together with finitely many ψ_α with $\alpha \in D$ hence a σ-compact set is covered by θ_∞ together with countably many ψ_α with $\alpha \in D$. Let $\{D_n\colon n \in N\}$ be a countable partition of D into uncountable sets. For each n let $U_n = \bigcup\{\psi_\alpha\colon \alpha \in D_n\}$. The family $\{U_n\colon n \in N\}$ is a disjoint family of open sets with $\bigcup\{U_n\colon n \in N\} = \bigcup\{\psi_\alpha\colon \alpha \in D\}$. Since a σ-compact F meets only countably many ψ_α, no U_n is in η_x. Thus, $\mathfrak{B}_x$ is infinite which is impossible. Thus, $X' = \bar{E}$ for some $E \in \eta_x$. This demonstration also establishes, if $F \in \mathfrak{F}$ replaces X', that $F = \bar{E}$ for some $E \in \eta_x$, which establishes the corollary.

In the unlikely event that $\mathfrak{B}_x$ be finite for some non-G_δ-point x, Proposition 7 gives a countably additive $\mu \in BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ with $\operatorname{Reg}(\mu) = \hat{\mathfrak{B}}_0^\mu$. We conclude by giving an example where $\operatorname{Reg}(\mu)$ is always larger than $\hat{\mathfrak{B}}_0^\mu$.

EXAMPLE 8. Let X be the one point compactification $D \cup \{x\}$ of an uncountable discrete space. $\mathfrak{B}_0$ consists of countable sets in D and their complements in X, $\mathfrak{B} = 2^X$ and η_x consists of countable sets in D hence is a maximal ideal in $\mathfrak{B}_0$ and $\mathfrak{B}_0$ is μ-complete for any $\mu \in BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$. The $\mu \in BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ with $\mu(\{x\}) = 0$ are identified with elements of $BA_1^+(2^D/\eta_x)$ or with elements of $BA_1^+(2^D)$ which annihilate η_x hence are those $\mu \in BA_1^+(2^X)$ with $\mu(A) = 0$ if A is countable in X. If $\mu \in BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ then μ agrees with δ_x on $\operatorname{Reg}(\mu)$. If $A \subset D$ has $\mu(A) = 0$ then $A \in \operatorname{Reg}(\mu)$ since A is open whereas $A \cup \{x\} \notin \operatorname{Reg}(\mu)$. Thus, $\operatorname{Reg}(\mu)$ consists of $A \subset D$ with $\mu(A) = 0$ and the complements in X of these A. Let η_μ denote the ideal in 2^D of μ-negligible sets. η_μ is a maximal ideal in $\operatorname{Reg}(\mu)$ and $2^D/\eta_\mu$ satisfies the countable chain condition. On the other hand $2^D/\eta_x$ does not satisfy the countable chain condition since D has an uncountable partition into uncountable sets. Thus, $\eta_x \neq \eta_\mu$ and $\hat{\mathfrak{B}}_0^\mu \neq \operatorname{Reg}(\mu)$.

Note that if the cardinality of D is not real-valued measurable, [1], [2], then all elements μ of $BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ with $\mu(\{x\}) = 0$ must be purely finitely additive. If the cardinality of D is real-valued measurable any countably additive diffuse measure m on 2^D gives an element of $CA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ singular to δ_x and $\operatorname{Reg}(\mu)$ is guaranteed to be strictly between $\mathfrak{B}_0$ and $\mathfrak{B}$. If $\mu \in BA^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$ is purely finitely additive it is a countable convex combination $\Sigma\{\lambda_n\mu_n\colon n \in \mathbf{N}\}$ of strongly finitely additive $\{\mu_n\} \subset BA_1^+(\mathfrak{B})$. Each μ_n must be in $BA_1^+(\mathfrak{B}_0, \delta_x, \mathfrak{B})$. From the definition of strong finite additivity there exist $\{A_m^n\colon m \in \mathbf{N}\} \subset \eta_{\mu_n}$ which partition D. We have $\{A_m^n\colon m \in \mathbf{N}\} \subset \operatorname{Reg}(\mu_n)$. Since $D \notin \operatorname{Reg}(\mu_n)$ it is impossible for $\operatorname{Reg}(\mu_n)$ to be σ-algebra even though $\mathfrak{B}_0 \subset \operatorname{Reg}(\mu_n)$.

REMARK. Karel Prikry and Richard Gardner pointed out Example 8.

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Department of Mathematical Sciences
Northern Illinois University
De Kalb, Illinois 60115
U.S.A.