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THE DEGREE OF APPROXIMATION BY POSITIVE LINEAR OPERATORS ON COMPACT CONNECTED ABELIAN GROUPS

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Abstract

In 1953 P. P. Korovkin proved that if (T_n) is a sequence of positive linear operators defined on the space C of continuous real 2π-periodic functions and $\lim T_n f = f$ uniformly for $f = 1$, cos and sin, then $\lim T_n f = f$ uniformly for all $f \in C$. Quantitative versions of this result have been given, where the rate of convergence is given in terms of that of the test functions 1, cos and sin, and the modulus of continuity of f. We extend this result by giving a quantitative version of Korovkin's theorem for compact connected abelian groups.

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Throughout G will denote a compact Hausdorff abelian group, Γ its character group, and $C(G)$ the space of continuous functions on G with the uniform norm $\|\cdot\|$. A linear operator T on $C(G)$ will be called positive if $Tf \geqslant 0$ whenever $f \geqslant 0$. It is well known that such an operator takes real functions into real functions, and $|Tf| \leqslant T|f|$ for all $f \in C(G)$. In particular T is continuous with $\|T\| = \|T1\|$, where 1 denotes the constant function with value 1.

The so-called Korovkin theory is concerned with deducing convergence properties of a sequence (T_n) of positive linear operators from those of $(T_n f)$ for f belonging to a (small) subset $S(G)$ of $C(G)$. We refer to $S(G)$ as a test set (for (T_n)). Korovkin [6] proved that when G is taken to be the circle group $\mathbf{T}$ then $\{1, e_1\}$ serves as a test set, where e_1: $e^{ix} \to e^{ix}$. Subsequently this result was given in a quantitative form by Shisha and Mond (see [10], Theorem 3 and [11]) in which the rate of convergence of $(T_n f)$ is estimated in terms of that of $(T_n 1)$ and $(T_n e_1)$ and the modulus of continuity of f. Censor (see [3], Theorem 2 and the remarks immediately preceding it) gave a version of this result for the multi-dimensional torus $\mathbf{T}^n$. For other results along these lines, including the case for algebraic polynomials on the unit interval, see [4], [6], [7], [8], [10] and [11]. In a new direction Nishishiraho [9] has given a quantitative version of Korovkin's theorem for compact subsets of a locally convex Hausdorff space, which includes the case where the underlying space is real Euclidean space.

In [1] we considered the Korovkin theory on a locally compact abelian group G, with test set $S(G)$ given by a set of continuous characters generating Γ. Here we shall make use of the ideas of Nishishiraho to recast these results in a quantitative form. Our results will include those of Shisha and Mond, and Censor for the periodic case. We shall also derive a corresponding result for the infinite dimensional torus.

For a nonempty subset Λ of Γ denote by $\langle \Lambda \rangle$ the subgroup of Γ generated by Λ, and by $A(G, \Lambda)$ the annihilator of Λ in G (see [5], (23.23)). We define the modulus of continuity of $f \in C(G)$ with respect to Λ by

$$\omega(f, \Lambda, \delta) = \sup\{|f(x) - f(y)| : |\gamma(x) - \gamma(y)| \leqslant \delta \text{ for all } \gamma \in \Lambda\},$$

where $\delta \geqslant 0$. It is clear that $\omega(f, \Lambda, \delta)$ is a nondecreasing function of δ.

LEMMA 1. *For each subset Λ of Γ and each $f \in C(G)$ the function $\delta \to \omega(f, \Lambda, \delta)$ is continuous at* 0.

PROOF. Write $K = A(G, \Lambda) = \bigcap\{C_\delta : \delta > 0\}$, where

$$C_\delta = \{x \in G : |\gamma(x) - 1| \leqslant \delta \text{ for all } \gamma \in \Lambda\}.$$

We first show that for any open neighbourhood V of 0 there exists $\delta > 0$ such that $C_\delta \subset K + V$. Indeed if not then $\{C_\delta \setminus (K + V) : \delta > 0\}$ is a family of closed sets with the finite intersection property and, since G is compact,

$$K \setminus (K + V) = \bigcap \{C_\delta \setminus (K + V) : \delta > 0\} \neq \varnothing,$$

a contradiction.

Now choose $\varepsilon > 0$ and an open neighbourhood V of 0 such that $|f(x) - f(y)| < \varepsilon$ whenever $x - y \in V$, and $\delta > 0$ satisfying $C_\delta \subset K + V$. Then

$$\begin{aligned}\omega(f, \Lambda, \delta) &\leqslant \sup\{|f(x)-f(y)|: x-y \in K+V\}\\ &= \sup\{|f(x)-f(x+y)+f(x+y)-f(x+y+z)|:\\ &\qquad\qquad\qquad x \in G,\ -y \in K,\ -z \in V\}\\ &\leqslant \sup\{|f(x)-f(x+y)|: x \in G,\ -y \in K\}\\ &\quad + \sup\{|f(x)-f(x+z)|: x \in G,\ -z \in V\}\\ &\leqslant \omega(f, \Lambda, 0)+\varepsilon\end{aligned}$$

and, since $\omega(f, \Lambda, \delta)$ is nondecreasing as a function of δ, this establishes the result.

In order to show that $\omega(f, \Lambda, \delta)$ is subhomogeneous in δ we require a preliminary result concerning characters of compact connected abelian groups.

LEMMA 2. *Let G be connected and choose $n \in \mathbf{N}$ (the set of positive integers). Then $\bigcap\{\gamma_i^{-1}(\xi_i): i = 1,2,\ldots,n\}$ is nonempty for every independent subset $\Lambda = \{\gamma_1, \gamma_2,\ldots,\gamma_n\}$ of Γ and for every $\{\xi_1, \xi_2,\ldots,\xi_n\} \subset \mathbf{T}$.*

PROOF. Since Λ is independent we have that each γ_i is nonconstant and, using the connectedness of G, that $\gamma_i(G) = \mathbf{T}$. Choose x_1 such that $\gamma_1(x_1) = \xi_1$ and suppose that x_k has been chosen satisfying $\gamma_i(x_k) = \xi_i$ for $i = 1,2,\ldots,k$. We show how to choose x_{k+1} such that $\gamma_i(x_{k+1}) = \xi_i$ for $i = 1,2,\ldots,k+1$.

First note that by [5], (24.10),

$$\langle\gamma_1, \gamma_2,\ldots,\gamma_k\rangle = A\left\{\Gamma, \bigcap_{i=1}^{k} \ker(\gamma_i)\right\}$$

and, since Λ is independent and its elements have infinite order ([5], (24.25), we must have $\gamma_{k+1}^m \notin \langle\gamma_1, \gamma_2,\ldots,\gamma_k\rangle$ for all $m \in \mathbf{N}$. It follows that for each $n \in \mathbf{N}$ there exists $z_n \in \bigcap_{i=1}^k \ker(\gamma_i) \setminus \ker(\gamma_{k+1}^{n!})$. In particular, since $z_n \notin \ker(\gamma_{k+1}^{n!})$, it follows that the $\gamma_{k+1}(jz_n)$ are pairwise distinct for $j = 1,2,\ldots,n$. Hence the subgroup of $\mathbf{T}$ generated by $\{\gamma_{k+1}(jz_n): j = 1,2,\ldots,n,\ n \in \mathbf{N}\}$ is infinite and thus dense in $\mathbf{T}$. Consequently we can choose a sequence $(y_n) \subset \bigcap_{i=1}^k \ker(\gamma_i)$ such that $\lim_n \gamma_{k+1}(y_n) = \xi_{k+1}\bar{\gamma}_{k+1}(x_k)$. Using the compactness of $\bigcap_{i=1}^k \ker(\gamma_i)$ we have the existence of $y \in \bigcap_{i=1}^k \ker(\gamma_i)$ and a subnet (y_{n_α}) of (y_n) such that $\lim_\alpha y_{n_\alpha} = y$. From the continuity of γ_{k+1} it follows that $\gamma_{k+1}(y) = \xi_{k+1}\bar{\gamma}_{k+1}(x_k)$, and $\gamma_{k+1}(y + x_k) = \xi_{k+1}$. We also have $\gamma_i(y + x_k) = \gamma_i(y)\gamma_i(x_k) = \xi_i$ for $i = 1,2,\ldots,k$, and thus $x_{k+1} = y + x_k$ satisfies the required condition.

It should be noted that if for some compact abelian group G, $\bigcap\{\gamma_i^{-1}(\xi_i): i = 1,2,\ldots,n\}$ is nonempty for some subset $\Lambda = \{\gamma_1, \gamma_2,\ldots,\gamma_n\}$ of Γ and every

$\{\xi_1, \xi_2, \ldots, \xi_n\} \subset \mathbf{T}$ then Λ must be independent with all its elements having infinite order. Indeed suppose there exist integers $m_1, m_2, \ldots, m_n$ not all zero such that $\gamma_1^{m_1}\gamma_2^{m_2}\cdots\gamma_n^{m_n} = 1$, and choose $\xi_1, \xi_2, \ldots, \xi_n \in \mathbf{T}$ satisfying $\xi_1^{m_1}\xi_2^{m_2}\cdots\xi_n^{m_n} \neq 1$. By the assumption on Λ we have the existence of $x \in G$ such that $\gamma_i(x) = \xi_i$ for $i = 1, 2, \ldots, n$. Then

$$1 = \gamma_1^{m_1}\gamma_2^{m_2}\cdots\gamma_n^{m_n}(x) = \xi_1^{m_1}\xi_2^{m_2}\cdots\xi_n^{m_n},$$

contradicting our choice of the ξ_i. Thus the γ_i do not satisfy any nontrivial relation, so that in particular each γ_i has infinite order and Λ is independent. It follows that if every finite subset of a generating set of Γ satisfies the above condition then Γ is torsion free which, by [5], (24.25), implies that G is connected.

We can now prove:

LEMMA 3. *Let G be connected and let Λ be an independent subset of Γ. Then, for any $f \in C(G)$,*

$$\omega(f, \Lambda_0, \lambda\delta) \leqslant \pi(1 + \lambda)\omega(f, \Lambda_0, \delta)$$

for all $\lambda, \delta \geqslant 0$ and every finite nonempty subset Λ_0 of Λ.

PROOF. Firstly we show that for any $n \in \mathbf{N}$,

$$\omega(f, \Lambda_0, n\delta) \leqslant \pi n\omega(f, \Lambda_0, \delta).$$

For $n = 1$ the inequality is evident, so take $n \geqslant 2$ and suppose that $x, y \in G$ satisfy $|\gamma(x) - \gamma(y)| \leqslant n\delta$ for all $\gamma \in \Lambda_0$. Writing $m = [\frac{1}{2}\pi n] + 2$, where $[\lambda]$ denotes the greatest integer not exceeding λ, we can choose $\xi_1(\gamma), \xi_2(\gamma), \ldots, \xi_m(\gamma) \in \mathbf{T}$ such that $\xi_1(\gamma) = \gamma(x)$, $\xi_m(\gamma) = \gamma(y)$ and

$$|\xi_j(\gamma) - \xi_{j+1}(\gamma)| = n^{-1}|\gamma(x) - \gamma(y)| \leqslant \delta$$

for $j = 1, 2, \ldots, m - 1$. Using Lemma 2 we have the existence of $x_1, x_2, \ldots, x_m \in G$ with $x_1 = x$, $x_m = y$ and $\gamma(x_j) = \xi_j(\gamma)$ for all $\gamma \in \Lambda_0$ and each $j = 1, 2, \ldots, m$. Then

$$|f(x) - f(y)| = \left|\sum_{j=1}^{m-1} \left(f(x_{j+1}) - f(x_j)\right)\right|$$
$$\leqslant (m - 1)\omega(f, \Lambda_0, \delta) \leqslant \pi n\omega(f, \Lambda_0, \delta).$$

Now take any $\lambda \geqslant 0$ and put $n = [\lambda] + 1$. Since $\omega(f, \Lambda_0, \delta)$ is nondecreasing as a function of δ we have

$$\omega(f, \Lambda_0, \lambda\delta) \leqslant \omega(f, \Lambda_0, n\delta) \leqslant \pi n\omega(f, \Lambda_0, \delta) \leqslant \pi(1 + \lambda)\omega(f, \Lambda_0, \delta).$$

For any positive linear operator T on $C(G)$ and nonempty $\Lambda \subset \Gamma$, write

$$\tau(\Lambda) = \sup\left\{ \sum \left[T|\gamma - \gamma(x)|^2(x) : \gamma \in \Lambda \right] : x \in G \right\}^{1/2}.$$

When T is a convolution operator, given by $Tf = \mu * f$ for some nonnegative bounded Radon measure μ, then using

$$|\gamma - \gamma(x)|^2 = 2 - \bar{\gamma}(x)\gamma - \gamma(x)\bar{\gamma}$$

and applying T, we obtain

$$\tau(\Lambda) = \left(\sum \{2\hat{\mu}(1) - \hat{\mu}(\gamma) - \hat{\mu}(\bar{\gamma}) : \gamma \in \Lambda\} \right)^{1/2}.$$

We define the modulus of continuity of $f \in C(G)$ with respect to T and Λ by

$$\Omega(f) = \inf\{\omega(f, \Lambda_0, \tau(\Lambda_0)) : \Lambda_0 \subset \Lambda \text{ is finite and nonempty}\}.$$

For a net (T_ρ) of positive linear operators on $C(G)$, τ_ρ and Ω_ρ will be defined as above with respect to each T_ρ.

LEMMA 4. *Let $\Lambda \subset \Gamma$ with $\langle \Lambda \rangle = \Gamma$ and let (T_ρ) be a net of positive linear operators on $C(G)$ such that* $\lim \tau_\rho(\gamma) = 0$ *for each $\gamma \in \Lambda$. Then* $\lim \Omega_\rho(f) = 0$ *for all $f \in C(G)$.*

PROOF. Let V be any open neighbourhood of 0 in G. We show that there exists a finite nonempty subset Λ_0 of Λ and $\delta > 0$ such that

$$\{x \in G : |\gamma(x) - 1| < \delta \text{ for all } \gamma \in \Lambda_0\} \subset V. \tag{1}$$

First note that by the duality of G and Γ, (1) holds for some $\delta' > 0$ and finite subset Λ' of Γ (replacing δ, Λ_0 respectively). Now take Λ_0 to be a finite subset of Λ such that $\Lambda' \subset \langle \Lambda_0 \rangle$ and $\delta = n^{-1}\delta'$, where

$$n = \max\{i : \gamma \in \Lambda', \gamma = \gamma_1\gamma_2 \cdots \gamma_i, \gamma_j^{\varepsilon_j} \in \Lambda_0 \text{ for } j = 1, 2, \ldots, i \text{ and some choice of } \varepsilon_j = \pm 1\};$$

in the above representation for γ a word of minimum length appears. Then (1) holds for this choice of Λ_0 and δ, using the inequality

$$|\gamma_1\gamma_2 \cdots \gamma_i - 1| \leqslant \sum_{j=1}^{i} |\gamma_i - 1|.$$

Next consider $f \in C(G)$, $\varepsilon > 0$ and choose an open neighbourhood V of 0 in G such that $x - y \in V$ implies $|f(x) - f(y)| < \varepsilon$, and then $\Lambda_0 \subset \Lambda$ finite nonempty and $\delta > 0$ satisfying (1). Since $\lim \tau_\rho(\gamma) = 0$ for each $\gamma \in \Lambda$ we have the existence of ρ_0 such that

$$\sum \{\tau_\rho(\gamma) : \gamma \in \Lambda_0\} < \delta$$

for all $\rho \geqslant \rho_0$, and it follows that for this range of ρ,

$$\Omega_\rho(f) \leqslant \omega\left(f, \Lambda_0, \tau_\rho(\Lambda_0)\right) \leqslant \omega\left(f, \Lambda_0, \sum \left\{\tau_\rho(\gamma) : \gamma \in \Lambda_0\right\}\right) < \varepsilon.$$

Our main result is an estimate of the rate of convergence of $(T_\rho(f))$ in terms of that of $(\Omega_\rho(f))$.

THEOREM. *Let G be connected, let T be a positive linear operator on $C(G)$, and take Λ to be any independent set generating Γ. Then, for $f, g \in C(G)$,*

$$\|Tf - fg\| \leqslant \|f\|\,\|T1 - g\| + \pi\left\|T1 + (T1)^{1/2}\right\|\Omega(f). \tag{2}$$

If (T_ρ) is a net of positive linear operators on $C(G)$ such that $\lim \tau_\rho(\gamma) = 0$ *for all* $\gamma \in \Lambda$ *and* $\lim T_\rho 1 = g$ *then* $\lim T_\rho f = fg$ *for all* $f \in C(G)$.

PROOF. Choose Λ_0 to be any finite nonempty subset of Λ.

If $\tau(\Lambda_0) > 0$ then, using Lemma 3, we have for any $x, y \in G$,

$$\begin{aligned}
|f(x) - f(y)| &\leqslant \omega\left(f, \Lambda_0, \left(\sum \left\{|\gamma(x) - \gamma(y)|^2 : \gamma \in \Lambda_0\right\}\right)^{1/2}\right) \\
&\leqslant \pi\left[1 + \tau(\Lambda_0)^{-1}\left(\sum \left\{|\gamma(x) - \gamma(y)|^2 : \gamma \in \Lambda_0\right\}\right)^{1/2}\right]\omega(f, \Lambda_0, \tau(\Lambda_0)).
\end{aligned}$$

Now apply T and evaluate at y to obtain

$$\begin{aligned}
&|Tf(y) - f(y)T1(y)| \\
&\leqslant \pi\left[T1(y) + \tau(\Lambda_0)^{-1}T\left(\sum \left\{|\gamma - \gamma(y)|^2 : \gamma \in \Lambda_0\right\}\right)^{1/2}(y)\right]\omega(f, \Lambda_0, \tau(\Lambda_0)) \\
&\leqslant \pi\left[T1(y) + (T1(y))^{1/2}\right]\omega(f, \Lambda_0, \tau(\Lambda_0)),
\end{aligned}$$

the second step following from the Cauchy-Schwarz inequality for positive linear functionals. Since these inequalities hold for all $y \in G$,

$$\|Tf - fT1\| \leqslant \pi\left\|T1 + (T1)^{1/2}\right\|\omega(f, \Lambda_0, \tau(\Lambda_0)). \tag{3}$$

If $\tau(\Lambda_0) = 0$ then, for any $\varepsilon > 0$ and $x, y \in G$,

$$|f(x) - f(y)| \leqslant \pi\left[1 + \varepsilon^{-1}\left(\sum \left\{|\gamma(x) - \gamma(y)|^2 : \gamma \in \Lambda_0\right\}\right)^{1/2}\right]\omega(f, \Lambda_0, \varepsilon).$$

Applying T as above we have

$$|Tf(y) - f(y)T1(y)| \leqslant \pi T1(y)\omega(f, \Lambda_0, \varepsilon)$$

and, by Lemma 1, the same inequality holds with 0 replacing ε.

Thus (3) holds for all finite nonempty $\Lambda_0 \subset \Lambda$, from which (2) follows. The last statement of the theorem is proved by appealing to Lemma 4.

Of particular interest is the following special case of the theorem.

COROLLARY. *Let G be connected and take Λ to be any independent set generating Γ. For any net (T_ρ) of positive linear operators on $C(G)$ satisfying $T_\rho 1 = 1$ for all ρ we have*

$$\|T_\rho f - f\| \leqslant 2\pi\Omega_\rho(f) \tag{4}$$

for all $f \in C(G)$. In particular if $\lim \tau_\rho(\gamma) = 0$ *for all* $\gamma \in \Lambda$ *then* $\lim T_\rho f = f$ *for all* $f \in C(G)$.

There is no possibility of extending the result to groups that are not connected, as the following example shows.

EXAMPLE. Consider the Cantor group $\mathbf{D} = \prod_{i=1}^{\infty} \mathbf{Z}(2)$, where $\mathbf{Z}(2)$ denotes the cyclic group of order two, and for each $n \in \mathbf{N}$ write G_n for the open subgroup given by

$$G_n = \{(x_i) \in \mathbf{D}\colon x_i = 0 \text{ for } i \leqslant n\}.$$

Denoting the characteristic function of G_n by 1_n we see that the

$$k_n = 2^n 1_n = \sum \{\gamma\colon \gamma \in A(\Gamma_{\mathbf{D}}, G_n)\}$$

are nonnegative trigonometric polynomials, and $T_n f = k_n * f$ defines a sequence (T_n) of positive convolution operators on $C(\mathbf{D})$. We observe that $T_n 1 = 1$ for all $n \in \mathbf{N}$ and, since (k_n) is a bounded approximate unit for $L^1(\mathbf{D})$, $\lim T_n f = f$ for each $f \in C(\mathbf{D})$. We show that there exists $f \in C(G)$ and $n_0 \in \mathbf{N}$ such that (T_n) cannot satisfy (4) for all $n \geqslant n_0$.

It is well known that $\Gamma_{\mathbf{D}} \cong \prod_{i=1}^{\infty}{}^{*} \mathbf{Z}(2)$ (the countable weak direct product of the groups $\mathbf{Z}(2)$). Writing γ_i for the continuous character of $\mathbf{D}$ that (under the above isomorphism) has 1 in the ith entry and zero elsewhere, we see that $\Lambda = \{\gamma_1, \gamma_2, \ldots\}$ is an independent set generating $\Gamma_{\mathbf{D}}$. Put $f = \sum_{i=1}^{\infty} 2^{-i^2}\gamma_i$. Then $f \in C(\mathbf{D})$ and

$$\|T_n f - f\| = \left\| \sum_{i=n+1}^{\infty} 2^{-i^2}\gamma_i \right\| = \sum_{i=n+1}^{\infty} 2^{-i^2},$$

using the property

$$T_n\gamma = \begin{cases} \gamma, & \gamma \in A(\Gamma_{\mathbf{D}}, G_n), \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore

$$\begin{aligned} \Omega_n(f) &\leqslant \omega(f, \{\gamma_1, \gamma_2, \ldots, \gamma_{n+1}\}, \tau(\{\gamma_1, \gamma_2, \ldots, \gamma_{n+1}\})) \\ &= \omega(f, \{\gamma_1, \gamma_2, \ldots, \gamma_{n+1}\}, 2^{1/2}). \end{aligned}$$

But the continuous characters of $\mathbf{D}$ take the values ± 1 only, so that $|\gamma(x) - \gamma(y)| < 2$ implies that $x - y \in \ker(\gamma)$. It follows that

$$\Omega_n(f) \leqslant \sup\left\{\left|\sum_{i=n+2}^{\infty} 2^{-i^2}(\gamma_i(x) - \gamma_i(y))\right| : x, y \in \mathbf{D}\right\}$$
$$= 2 \sum_{i=n+2}^{\infty} 2^{-i^2}.$$

Since $\lim_n \sum_{i=n+1}^{\infty} 2^{-i^2}(\sum_{i=n+2}^{\infty} 2^{-i^2})^{-1} = \infty$ it is impossible for (4) to hold.

Our theorem includes the known quantitative estimates for convergence of a sequence of positive linear operators on $C(\mathbf{T})$. Indeed $\Lambda = \{e_1\}$ is an independent set generating $\mathbf{Z}$, the character group of $\mathbf{T}$. The modulus of continuity of $f \in C(\mathbf{T})$ with respect to Λ is

$$\omega(f, \Lambda, \delta) = \sup\{|f(x) - f(y)| : |x - y| \leqslant \delta\},$$

which is just the usual modulus of continuity. Now an easy calculation gives

$$\tau(e_1) = 2\sup\{T(\sin^2 \tfrac{1}{2}(x - t))(x) : x \in \mathbf{T}\}^{1/2},$$

and substituting into (2) gives Theorem 3 in [10].

We now consider the convergence of positive linear operators on $C(\mathbf{T}^n)$. Take $\Lambda = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, where γ_i: $\mathbf{T}^n \to \mathbf{T}$ is the projection onto the ith coordinate, $i = 1, 2, \ldots, n$. Then Λ is an independent set generating the character group of $\mathbf{T}^n$. The modulus of continuity of $f \in C(\mathbf{T}^n)$ with respect to Λ is

$$\omega(f, \Lambda, \delta) = \sup\{|f(x) - f(y)| : |x_i - y_i| \leqslant \delta \text{ for } i = 1, 2, \ldots, n\},$$

where $x = (x_1, x_2, \ldots, x_n)$. We have

$$\tau(\Lambda) = 2\sup\left\{\sum_{i=1}^{n} T(\sin^2 \tfrac{1}{2}(x_i - t_i))(x) : x \in \mathbf{T}^n\right\}^{1/2},$$

and substituting into (2) gives [3], Theorem 2.

Finally we present an application of our results to the infinite dimensional torus $\mathbf{T}^\infty$, the direct product of countably many copies of $\mathbf{T}$. Take $\Lambda = \{\gamma_1, \gamma_2, \ldots\}$, where γ_i is the ith coordinate projection of $\mathbf{T}^\infty$ onto $\mathbf{T}$. Let (k_n) be a sequence of nonnegative continuous functions on $\mathbf{T}$ with $\|k_n\|_1 = 1$ for all $n \in \mathbf{N}$, and define $(K_n) \subset C(\mathbf{T}^\infty)$ by

$$K_n(x) = \prod_{i=1}^{n} k_n(x_i), \qquad x = (x_i) \in \mathbf{T}^\infty;$$

for a similar construction see [5], (44.53). We shall examine the convergence of the positive convolution operators T_n given by $T_n f = K_n * f$, $f \in C(\mathbf{T}^\infty)$. Suppose that $\lim_n \hat{k}_n(e_1) = 1$, so that by Korovkin's theorem, $\lim_n \hat{k}_n(e_m) = 1$ for all $m \in \mathbf{Z}$, where $e_m = e_1^m$. We have

$$\tau_n(\{\gamma_1, \gamma_2, \ldots, \gamma_n\}) = n^{1/2}\big(2 - \hat{k}_n(e_1) - \hat{k}_n(e_{-1})\big)^{1/2},$$

using the property that $\hat{K}_n(\gamma_i) = \hat{k}_n(e_1)$ for $i \leqslant n$. Thus

$$\begin{aligned}\|K_n * f - f\| &\leqslant 2\pi\Omega_n(f)\\ &\leqslant 2\pi\omega\Big(f, \{\gamma_1, \gamma_2, \ldots, \gamma_n\}, n^{1/2}\big(2 - \hat{k}_n(e_1) - \hat{k}_n(e_{-1})\big)^{1/2}\Big)\\ &= 2\pi \sup\Big\{|f(x) - f(y)| : |x_i - y_i| \leqslant n^{1/2}\big(2 - \hat{k}_n(e_1) - \hat{k}_n(e_{-1})\big)^{1/2}\\ &\qquad\qquad \text{for } i = 1, 2, \ldots, n\Big\}.\end{aligned}$$

In particular if (k_n) is the Fejér-Korovkin kernel (see [2], 1.6.1) then $\hat{k}_n(e_1) = \hat{k}_n(e_{-1}) = \cos \pi/(n+2)$ and

$$\|K_n * f - f\| \leqslant 2\pi \sup\big\{|f(x) - f(y)| : |x_i - y_i| \leqslant \pi n^{-1/2} \text{ for } i = 1, 2, \ldots, n\big\}.$$

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