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GENERALIZED PRODUCTS OF WEAKLY $m - n$ COMPACT SPACES, I

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Abstract

A topological space X is said to be weakly-Lindelöf if and only if every open cover of X has a countable sub-family with dense union. We know that products of two Lindelöf spaces need not be weakly-Lindelöf. In this paper we obtain non-trivial sufficient conditions on small sub-products to ensure the producitivity of the property weakly-Lindelöf with respect to arbitrary products.

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Introduction

Let n be an infinite cardinal. A topological space X is said to be *weakly $\infty - n$ compact* if and only if every open cover of X has a sub-family of cardinality strictly less than n with dense union. Weakly $\infty - \aleph_1$ compact spaces are called weakly-Lindelöf spaces.

Let $X = \prod\{X_i: i \in I\}$ and let $X_{I'} = \prod\{X_i: i \in I'\}$ where each X_i is a topological space and $I' \subset I$. The topology generated by the sets of the form $W = \prod\{W_i: i \in I\}$ where each W_i is open in X_i and $|R(W)| < k$ where $R(W) = \{i \in I: W_i \neq X_i\}$ is called the *k-box topology* on the product X and we denote it by $(\prod X_i)_k$ where $\aleph_0 \leqslant k \leqslant |I|^+$ (= cardinal successor of $|I|$). If $k = \aleph_0$, we get the usual product-topology and if $k = |I|^+$, we get the box topology (see [2]).

If $(\Pi X_i)_k$ is weakly-Lindelöf, then every sub-product $(X_{I'})_k$ of X is weakly-Lindelöf and the question of interest is to what extent the converse is true. In this direction we prove the following:

(i) Let $X = \Pi\{X_i: i \in I\}$ and let $n \geq \gamma \geq k \geq \aleph_0$. Suppose n is *regular* and *strongly γ-inaccessible*, then $(\Pi X_i)_k$ is weakly $\infty - n$ compact if and only if $(X_{X'})_k$ is weakly $\infty - n$ compact for all $I' \in P_\gamma(I)$ where $P_\gamma(I) = \{I' \subset I: |I|' < \gamma\}$.

(ii) Let $d(X)$ denote the *density number* of a space X (see [8]). Let $X = \Pi\{X_i: i \in I\}$ and let $n \geq \gamma \geq k \geq \aleph_0$ and let n be regular and strongly γ-inaccessible. Suppose $d(X_i) < n$ for all $i \in I$, then $(\Pi X_i)_k$ is weakly $\infty - n$ compact.

In particular if we take $\gamma = k = \aleph_0$ in (i) we obtain the following:

Let $X = \Pi\{X_i: i \in I\}$ with usual product topology and let n be a regular cardinal. Then X is weakly $\infty - n$ compact if and only if every finite sub-product of X is weakly $\infty - n$ compact (see [10]).

1. Basic terminology

In this section we shall define weakly $m - n$ compact spaces and generalized product topologies on the product $\Pi\{X_i: i \in I\}$.

A. Definition. An m-fold open cover of a topological space X is a collection of open subsets $\{U_i: i \in I\}$ of X such that $X = \cup\{U_i: i \in I\}$ and $|I| = m$.

A topological space X is said to be *weakly $m - n$ compact* if and only if every m-fold open cover of X has a sub-family of cardinality strictly less than n with dense union where m and n are infinite cardinals.

A topological space X is said to be *weakly $\infty - n$ compact* if and only if X is weakly $m - n$ compact for each $m \geq n$.

B. Special cases.

(i) Weakly $\infty - \aleph_0$ compact spaces $\equiv$ weakly-compact spaces.

(ii) Weakly $\infty - \aleph_1$ compact spaces $\equiv$ weakly-Lindelöf spaces.

(iii) Weakly $m - \aleph_0$ compact spaces $\equiv$ initially weakly m-compact spaces.

C. Definition. A subset E of X is said to be weakly $m - n$ compact relative to X if and only if every m-fold open cover $\mathfrak{U}$ of E by open subsets of X has a sub-family $\mathfrak{U}'$ of cardinality strictly less than n with $E \subseteq \overline{\cup \mathfrak{U}'}$.

D. Remark. (i) Let $E \subset X$. Suppose E is weakly $m - n$ compact with respect to its subspace topology, then E is weakly $m - n$ compact relative to X.

(ii) The converse of (i) is not necessarily true except for open subsets.

E. Example. Let E be a discrete subspace of $\beta D - D$ with D discrete $|D| = \aleph_0$ and $|E| = \aleph_1$. Let $X = D \cup E$. Then E is weakly $\aleph_1 - \aleph_1$ compact relative to X but not weakly $\aleph_1 - \aleph_1$ compact in its own right. Here E is a closed subset of X.

F. Definition. Let $X = \prod\{X_i: i \in I\}$ and let $W = \prod\{W_i: i \in I\}$ where W_i is a subset of X_i. Then $\mathfrak{R}(W) = \{i \in I: W_i \neq X_i\}$ is called the *range* of W.

Let W be a basis open set in the usual product topology on $\prod\{X_i: i \in I\}$. Then $|\mathfrak{R}(W)| < \aleph_0$. This leads to the following generalization of the product topology which we called the k-box topology on the product $\prod\{X_i: i \in I\}$.

G. Definition. The topology generated by the basic sets of the form $W = \prod\{W_i: i \in I\}$ where each W_i is open in X_i and $|\mathfrak{R}(W)| < k$ is called the k-box topology on the product $\prod\{X_i: i \in I\}$ and we denote it by $(\prod X_i)_k$ where $\aleph_0 \leqslant k \leqslant |I|^+$ (= cardinal successor of $|I|$).

If $k = \aleph_0$, we get the usual product topology on the product $\prod\{X_i: i \in I\}$ and if $k = |I|^+$, we get the box topology on the product $\prod\{X_i: i \in I\}$.

2. Weak-topological sums

In this section we shall study some properties of generalized weak-topological sums (see [3]).

A. Definition. Let $a = (a_i)$ be a fixed point in $X = \prod\{X_i: i \in I\}$. Then the γ-weak topological sum of $\{X_i: i \in I\}$ is the subspace $\{x \in X: |\{i \in I: x_i \neq a_i\}| < \gamma\}$ and we denote this by $\gamma(\prod X_i)$ where γ is an infinite cardinal.

B. Notation. For each non-empty set I' of I, we define $X_{I'} = \prod\{X_i: i \in I'\}$ and $\prod_{I'}: X \to X_{I'}$ is called the projection map onto $X_{I'}$. In particular if $I' = \{i\}$ we get the usual projection map $\prod_i: X \to X_i$ where $X = \prod\{X_i: i \in I\}$.

C. Remark. Let $W = \prod\{W_i: i \in I\}$ where each W_i is a subset of X_i. Then we have the following:

(i) $|\mathfrak{R}(\prod_{I'}(W))| \leqslant |\mathfrak{R}(W)|$.

(ii) $|\mathfrak{R}(\prod_{I'}^{-1}(W_{I'}))| = |\mathfrak{R}(W_{I'})|$.

(iii) If $V \supset W$, then $\prod_i(V) = X_i$ for all $i \in I - \mathfrak{R}(W)$.

D. PROPOSITION. *Let* $\Pi_{I'}: (X)_k \to (X_{I'})_k$. *Then* $\Pi_{I'}$ *is open, continuous and onto. Furthermore the subspace* $X(I') = \{x \in X\colon x_i = a_i \text{ for } i \in I - I'\}$ *of* $(X)_k$ *is homeomorphic to* $(X_{I'})_k$ *under the map* $\Pi_{I'}$.

PROOF. Follows from Remark C.

E. PROPOSITION. *The subspace* $\gamma(\Pi X_i)$ *is dense in* $(\Pi X_i)_k$ *provided* $k \leqslant \gamma$.

PROOF. Let $W = \Pi\{W_i\colon i \in I\}$ be a basic open set in $(\Pi X_i)_k$ consider the point $p = (p_i)$ where

$$\begin{aligned} p_i &= a_i \text{ for } i \in I - \mathfrak{R}(W), \\ &= w_i \in W_i \text{ for } i \in \mathfrak{R}(W). \end{aligned}$$

Then $|\{i \in I\colon p_i \neq a_i\}| \leqslant |\mathfrak{R}(W)| < k \leqslant \gamma$. Hence $p \in W \cap \gamma(\Pi X_i)$ and we are done.

F. EXAMPLE. If X_i is a discrete space for $i = 1, 2, \ldots$ and $d_i \neq a_i$ for $i = 1, 2, \ldots$, then $d = (d_i) \notin \aleph_0(\Pi X_i)$ and hence $\aleph_0(\Pi X_i)$ is not dense in $(\Pi X_i)_{\aleph_1}$.

G. DEFINITION. A property 'p' is said to be densely defined in a topological space X if whenever one of its dense subsets has the property 'P', then X has the property 'P'.

H. EXAMPLE. (i) Weak $m - n$ compactness is a densely defined property.
(ii) Let E be the Sorgenfrey line. Then $E \times E$ is weakly-Lindelöf but not Lindelöf.

3. Machinery

In this section we shall establish two special cases of the main theorem.

A. PROPOSITION. *Let* $U = \Pi\{U_i\colon i \in I\}$ *and* $V = \Pi\{V_i\colon i \in I\}$ *where* U_i, V_i *are subsets of* X_i *for* $i \in I$. *Then the following are equivalent*:
(i) $U \cap V = \varnothing$.
(ii) $U_i \cap V_i = \varnothing$ *for some* $i \in \mathfrak{R}(U) \cap \mathfrak{R}(V)$.
(iii) $\mathfrak{R}(U) \cap \mathfrak{R}(V) \neq \varnothing$ *and* $\Pi_{I'}(U) \cap \Pi_{I'}(V) = \varnothing$ *where* $I \supseteq I' \supseteq \mathfrak{R}(U) \cap \mathfrak{R}(V)$.

PROOF. (i) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). Let $I' = \mathfrak{R}(U) \cap \mathfrak{R}(V)$, then $U_i \cap V_i = \varnothing$ for some $i \in \mathfrak{R}(U) \cap \mathfrak{R}(V)$ and hence $U \cap V = \varnothing$.

B. DEFINITION. Let β be a base for a topological space X. Then X is said to be *β-weakly $m - n$ compact* if and only if for every cover $\mathcal{U} \in \mathcal{P}(\beta)$ of X with $|\mathcal{U}| = m$ there exists a $\mathcal{V} \in \mathcal{P}(\mathcal{U})$ such that $|\mathcal{V}| < n$ and $X = \overline{\bigcup \mathcal{V}}$.

We note the following facts about β-weakly $m - n$ spaces:

(i) If X is weakly $m - n$ compact then X is β-weakly $m - n$ compact.

(ii). X is β-weakly $\infty - n$ compact if and only if X is weakly $\infty - n$ compact.

C. PROPOSITION. *Let n and γ be infinite cardinals such that $n \geqslant \gamma$. Let $\bar{\gamma} = \gamma$ if γ is regular and let $\bar{\gamma} = \gamma^+$ if γ is singular. Suppose n is regular and strongly γ-inaccessible, then $\bar{\gamma} \leqslant n$.*

PROOF. (i) If γ is regular, then $\bar{\gamma} = \gamma$ and hence $\bar{\gamma} \leqslant n$.

(ii) If γ is singular, then since n is regular and strongly γ-inaccessible, we have $\gamma < n$, $\mathrm{cf}(\gamma) < \gamma$ and $\gamma^{\mathrm{cf}(\gamma)} < n$. Hence $\bar{\gamma} = \gamma^+ \leqslant \gamma^{\mathrm{cf}(\gamma)} < n$.

D. REMARK. $\bar{\gamma}$ in Proposition C is regular cardinal.

E. PROPOSITION. (i) Let $k < \mathrm{cf}(n)$. Then the k-fold union of weakly $m - n$ compact subsets of a given topological space X is weakly $m - n$ compact relative to X.

(ii) Let f: $X \to Y$ be a continuous onto map. If X is weakly $m - n$ compact, then Y is weakly $m - n$ compact.

PROOF. Straightforward.

F. NOTATIONS. (i) $|I|^{\gamma}_{-} = \Sigma\{|I|^k : k < \gamma\}$.

(ii) $P_\gamma(I) = \{I' \subset I : |I'| < \gamma\}$.

G. LEMMA. *Let $X = \prod\{X_i : i \in I\}$ and let $m \geqslant n \geqslant \mathrm{cf}(n) > |I|^{\gamma}_{-} \geqslant \gamma \geqslant k \geqslant \aleph_0$. If $(X_{I'})_k$ is weakly $m - n$ compact for all $I' \in P_\gamma(I)$, then $\gamma(\prod X_i)$ is weakly $m - n$ compact relative to $(\prod X_i)_k$.*

PROOF. We note that $\gamma(\prod X_i) = \bigcup\{X(I') : I' \in P_\gamma(I)\}$ and since $X(I')$ is homeomorphic to $(X_{I'})_k$, $\gamma(\prod X_i)$ is the $|I|^{\gamma}_{-}$-fold union of weakly $m - n$ compact subspaces of $(\prod X_i)_k$. Hence we have the lemma by $E - (i)$.

The sets $W = \Pi\{W_i: i \in I\}$ where each W_i is open in X_i and $|\mathfrak{R}(W)| < k$ form the canonical basis for $(\Pi X_i)_k$. Let $A \subset \Pi\{X_i: i \in I\}$, then the canonical basis for 'A' consists of all sets of the form $A \cap W$ with W as above. In this terminology we rephrase Lemma G as follows:

H. LEMMA. *Let $X = \Pi\{X_i: i \in I\}$ and let $m \geqslant n \geqslant \mathrm{cf}(n) > |I|^{\gamma}_{-} \geqslant \gamma \geqslant k \geqslant \aleph_0$. If $(X_{I'})_k$ is weakly $m - n$ compact with respect to its canonical basis for all $I' \in P_\gamma(I)$, then $\gamma(\Pi X_i)$ is weakly $m - n$ compact with respect to its canonical basis.*

I. THEOREM. *Let $X = \Pi\{X_i: i \in I\}$ and let $m \geqslant n > |I| \geqslant \gamma \geqslant k \geqslant \aleph_0$. If n is regular and strongly γ-inaccessible and if $(X_{I'})_k$ is weakly $m - n$ compact for all $I' \in P_\gamma(I)$, then $\gamma(\Pi X_i)$ is weakly $m - n$ compact relative to $(\Pi X_i)_k$.*

PROOF. Consider $|I|^{\gamma}_{-} = \Sigma\{|I|^k: k < \gamma\} \geqslant |I| \geqslant \gamma$. Since n is regular and strongly γ-inaccessible we have $n = \mathrm{cf}(n)$ and $|I|^{\gamma}_{-} < n$. Hence $m \geqslant n = \mathrm{cf}(n) > |I|^{\gamma}_{-} \geqslant \gamma \geqslant k \geqslant \aleph_0$ and therefore we can apply Lemma G to obtain the theorem.

In the above theorem there is a restriction on the cardinality of the index set I and we wish to relax this condition in the next section.

4. Weakly $m - n$ compact spaces

In this section we shall study the productivity of weak $m - n$ compactness in a general setting. Our argument here closely parallels one from [3].

A. THEOREM. *Let $m \geqslant n \geqslant \gamma \geqslant k \geqslant \aleph_0$ and let n be regular and strongly γ-inaccessible. Let $X = \Pi\{X_i: i \in I\}$ and let $\mathfrak{U} = \{U = \Pi\{U_i: i \in I\}$: each U_i is open in X_i and $|\mathfrak{R}(U)| < k\}$ where $|\mathfrak{U}| = m$. If $(X_{I'})_k$ is weakly $m - n$ compact for all $I' \in P_\gamma(I)$ and $\underline{\gamma(\Pi X_i)} \subset \overline{\cup \mathfrak{U}}$, then there exists a $\mathfrak{U}' \subset \mathfrak{U}$ such that $|\mathfrak{U}'| < n$ and $\gamma(\Pi X_i) \subseteq \overline{\cup \mathfrak{U}'}$.*

PROOF. Let $\bar{\gamma} = \gamma$, γ = regular and $\bar{\gamma} = \gamma^+$, γ = singular. Then $\bar{\gamma}$ is regular and $\bar{\gamma} \leqslant n$. By Theorem 3-I, $\gamma(X_{I'})$ is weakly $m - n$ compact relative to $(X_{I'})_k$ for all $I' \in P_\gamma(I)$. We note that $\Pi_{I'}(\gamma(\Pi X_i)) = \gamma(X_{I'})$ and hence $\{\Pi_{I'}(U): U \in \hat{\mathfrak{U}}\}$ is an m-fold open cover of $\gamma(\dot{X}_{I'})$ where $I' \subset I$. Let $I' \in P_{\bar{\gamma}}(I)$. Then there exists a $\mathfrak{U}_{I'} \subset \mathfrak{U}$ such that $|\mathfrak{U}_{I'}| < n$ and

$$\gamma(X_{I'}) \subseteq \mathrm{cl}_{(X_{I'})_k}\left(\cup \{\Pi_{I'}(U): U \in \mathfrak{U}_{I'}\}\right). \tag{4.1}$$

Let $I_1 \subset I$ with $|I_1| < n$ and let $\mathfrak{F}_1 = \{\mathfrak{U}_{I'}: I' \in P_{\bar{\gamma}}(I_1)$ where $\mathfrak{U}_{I'}$ has the property (4.1)$\}$. Let $I_2 = I_1 \cup \mathfrak{R}(\mathfrak{F}_1)$ where $\mathfrak{R}(\mathfrak{F}_1) = \bigcup\{\mathfrak{R}(U): U \in \mathfrak{U}_{I'}$ and $\mathfrak{U}_{I'} \in \mathfrak{F}_1\}$. Trivially $\mathfrak{R}(\mathfrak{F}_1) \subset I$ and therefore $I_2 \subset I$.

We note the following:

(i) $|\mathfrak{R}(U)| < k \leqslant n$.

(ii) $\mathfrak{U}_{I'} \subset \mathfrak{U}$, $|\mathfrak{U}_{I'}| < n$ for all $I' \in P_{\bar{\gamma}}(I_1)$.

(iii) $|P_{\bar{\gamma}}(I_1)| \leqslant |I_1|^{\gamma}_{-}$, $\gamma =$ regular and $|P_{\bar{\gamma}}(I_1)| \leqslant |I_1|^{\gamma^+}_{-}$, $\gamma =$ singular.

Since n is strongly γ-inaccessible $|P_{\bar{\gamma}}(I_1)| < n$ for all $\gamma \leqslant n$. Hence we have the following:

$$\text{(4.2)} \qquad \begin{aligned} &\text{(i) } |\mathfrak{R}(\mathfrak{F}_1)| < n, \\ &\text{(ii) } |I_2| < n. \end{aligned}$$

Inductively we define $\mathfrak{F}_\alpha = \bigcup\{\mathfrak{U}_{I'}: I' \in P_{\bar{\gamma}}(I_\alpha)\}$ and $I_{\alpha+1} = I_\alpha \cup \mathfrak{R}(\mathfrak{F}_\alpha)$ for $\alpha < \bar{\gamma}$. Let $I^* = \bigcup\{I_\alpha: \alpha < \bar{\gamma}\}$ and $\mathfrak{U}' = \bigcup\{\mathfrak{F}_\alpha: \alpha < \bar{\gamma}\}$. Since n is regular and $|I_\alpha| < n$ for all $\alpha < \bar{\gamma}$ we have

$$\text{(4.3)} \qquad |I^*| < n \quad \text{and} \quad |\mathfrak{U}'| < n.$$

Each $\mathfrak{U}_{I'} \subset \mathfrak{U}$ and therefore each $\mathfrak{F}_\alpha \subset \mathfrak{U}$ and hence $\mathfrak{U}' \subset \mathfrak{U}$. We shall prove that

$$\gamma(\Pi X_i) \subseteq \overline{\bigcup \mathfrak{U}'}.$$

Let $x \in \gamma(\Pi X_i)$ and let $V = \Pi\{V_i: i \in I\}$ be a basic open neighborhood of x in $(\Pi X_i)_k$. Then we have $|\mathfrak{R}(V)| < k \leqslant \gamma \leqslant \bar{\gamma}$ and hence there exists a $\alpha < \bar{\gamma}$ such that

$$\text{(4.4)} \qquad \mathfrak{R}(V) \cap I^* = R(V) \cap I_\alpha.$$

Let $H = \mathfrak{R}(V) \cap I_\alpha$, then $H \subset I_\alpha \subset I$ and $|H| < k < \bar{\gamma}$. By (4.1), there exists a $U \in \mathfrak{U}_H \subset \mathfrak{F}_\alpha$ such that

$$\text{(4.5)} \qquad \Pi_H(U) \cap \Pi_H(V) \neq \varnothing.$$

Since $U \in \mathfrak{F}_\alpha$, $\mathfrak{R}(U) \subset I_{\alpha+1}$ and by (4),

$$\text{(4.6)} \qquad \begin{aligned} \mathfrak{R}(U) \cap \mathfrak{R}(V) &= (\mathfrak{R}(U) \cap I_{\alpha+1}) \cap \mathfrak{R}(V) \\ &= \mathfrak{R}(U) \cap (I_{\alpha+1} \cap I^*) \cap \mathfrak{R}(V) \\ &= \mathfrak{R}(U) \cap (\mathfrak{R}(V) \cap I^*) \cap I_{\alpha+1} \\ &= \mathfrak{R}(U) \cap \mathfrak{R}(V) \cap I_\alpha \\ &\subseteq H. \end{aligned}$$

Hence by 3-A, $U \cap V \neq \varnothing$ and therefore $V \cap (\bigcup \mathfrak{U}') \neq \varnothing$. This is true for every neighborhood V of x and therefore we have $\gamma(\Pi X_i) \subseteq \overline{\bigcup \mathfrak{U}'}$.

For $k \leqslant \gamma$, $\gamma(\Pi X_i)$ is a dense subspace of $(\Pi X_i)_k$ and we are ready to give the main theorem.

B. THEOREM. *Let* $m \geqslant n \geqslant \gamma \geqslant k \geqslant \aleph_0$ *and let* n *be regular and strongly* γ-*inaccessible. Let* $X = \prod\{X_i: i \in I\}$ *and suppose* $(X_{I'})_k$ *is weakly* $m - n$ *compact for all* $I' \in P_\gamma(I)$. *If* β *is the canonical base for the product space* $(\prod X_i)_k$, *then* $(\prod X_i)_k$ *is* β-*weakly* $m - n$ *compact.*

5. Weakly $\infty - n$ compact spaces

We recall that the concept of β-weak $\infty - n$ compactness is equivalent to the weak $\infty - n$ compactness and hence we obtain product theorems for weakly $\infty - n$ compact spaces as special cases of Theorem 4B.

A. THEOREM. *Let* $X = \prod\{X_i: i \in I\}$ *and let* $n \geqslant \gamma \geqslant k \geqslant \aleph_0$. *Suppose* n *is regular and strongly* γ-*inaccessible, then* $(\prod X_i)_k$ *is weakly* $\infty - n$ *compact if and only if* $(X_{I'})_k$ *is weakly* $\infty - n$ *compact for all* $I' \in P_\gamma(I)$.

B. COROLLARY. *Let* $X = \prod\{X_i: i \in I\}$ *with the usual product topology. Let* n *be a regular cardinal then* X *is weakly* $\infty - n$ *compact if and only if every finite sub-product of* X *is weakly compact* (*see* [10]).

PROOF. '$\Rightarrow$'. This follows from the fact that $\prod_{I'}: X \to X_{I'}$ is continuous for every $I' \subset I$.

'$\Leftarrow$'. We note that regular cardinals are infinite and every infinite cardinal is strongly $\aleph_0$-inaccessible. Hence taking $\gamma = k = \aleph_0$ in Theorem A, we obtain the corollary.

C. DEFINITION. Let k be any cardinal. A space X is said to be k-separable if and only if X contains a dense subset of cardinality k.

The density of a space X, which we denoted by $d(X) = \min\{|D|: D \subset X, \bar{D} = X\} = \min\{k: X \text{ is } k\text{-separable}\}$.

D. REMARKS. (i) Let $\mathbf{R}$ = reals, X = discrete space, Y = indiscrete space. Then $d(\mathbf{R}) = \aleph_0$, $d(X) = |X|$, $d(Y) = 1$.

(ii) If $d(X) < n$, then X is weakly $\infty - n$ compact.

(iii) Let $X = \prod\{X_i: i \in I\}$ and let $A = \prod\{A_i: i \in I\}$. Then

$$\mathrm{cl}_{(\prod X_i)_k} A = \prod\{\mathrm{cl}_{X_i} A_i: i \in I\}$$

where $|I|^+ \geqslant k \geqslant \aleph_0$.

If n is strongly γ-inaccessible and if $d(X_i) < n$ for each $i \in I$, then each X_i has a dense subset A_i with $|A_i| < n$ and hence $A_{I'}$ is a dense subset of $(X_{I'})_k$ and

$|A_{I'}| < n$ for all $I' \in P_\gamma(I)$ where $\gamma \leqslant n$ and $A = \prod\{A_i \colon i \in I\}$. Therefore $d(X_{I'}) < n$ and hence $(X_{I'})_k$ is weakly $\infty - n$ compact for all $I' \in P_\gamma(I)$. Thus we have the following theorem:

E. THEOREM. *Let* $X = \prod\{X_i \colon i \in I\}$ *and let* $n \geqslant \gamma \geqslant k \geqslant \aleph_0$. *Suppose* n *is regular and strongly* γ*-inaccessible. If* $d(X_i) < n$ *for all* $i \in I$, *then* $(\prod X_i)_k$ *is weakly* $\infty - n$ *compact.*

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