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WEYL FUNCTIONS AND THE A_p CONDITION ON COMPACT LIE GROUPS

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Abstract

Let G be a compact, simple, simply connected Lie group. The L^p-norm of a central trigonometric polynomial reduces naturally to a weighted L^p-norm of a trigonometric polynomial on a maximal torus T. The weight is $|\Delta|^{2-p}$, where Δ is the usual Weyl function. If $p \geqslant 2$, we prove that $|\Delta|^{2-p}$ satisfies Muckenhoupt's A_p condition if and only if the L^p-norms of the irreducible characters of G are uniformly bounded.

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1

Suppose G is a compact, simple, simply connected, n-dimensional Lie group; T a (l-dimensional) maximal torus of G and R a central trigonometric polynomial over G. Via Weyl's integration and character formulas, the $L^p(G)$-norm of R reduces to a weighted L^p-norm over T,

$$\|R\|_{L^p(G)} = \|\tilde{R}\|_{L^p_{|\Delta|^{2-p}}(T)}$$

(that is $\int_G |R|^p = \int_T |\tilde{R}|^p \cdot |\Delta|^{2-p}$). Here Δ is the usual Weyl function and $\tilde{R}$ is a trigonometric polynomial over T which is odd under the action of the Weyl group and has the following property: the coefficients of $\tilde{R}$ contained in the fundamental Weyl chamber are exactly (up to a translation) the coefficients of the corresponding irreducible characters of the polynomial R.

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Hence a central multiplier of G is $L^p(G) - L^p(G)$ bounded provided an analogous multiplier operator on T is $L^p_{|\Delta|^{2-p}}(T) - L^p_{|\Delta|^{2-p}}(T)$ bounded. This remark leads to the theory of weighted norm inequalities and mainly to the A_p condition.

DEFINITION. *A nonnegative locally integrable function ω on $\mathbf{R}^d$ satisfies the A_p condition if*

$$\operatorname{Sup}\left(\frac{1}{m(B)}\int_B \omega\right)\cdot\left(\frac{1}{m(B)}\int_B \omega^{-1/(p-1)}\right)^{p-1} < \infty$$

where the supremum is taken over all the balls B (here $m(B)$ denotes the measure of the ball).

We recall that the A_p condition is a necessary and sufficient condition for certain weighted norm inequalities: for the maximal function, for the classical singular integrals, etcetera (see Muckenhoupt (1972) and Coifman and Fefferman (1974)).

Let $p \geqslant 2$, in this paper we prove that the weight $|\Delta|^{2-p}$ satisfies the A_p condition if and only if $p < 2n/(n-l)$

2

We point out that a related, but fairly different, problem was considered in Stanton (1976), where the author studied the $L^p(G)$-convergence of polyhedral partial sums of central $L^p(G)$-integrable functions. As we did before, Stanton reduced the problem to a weighted inequality over T and had to consider the weight $|\Delta|^{2-p}$. But (as in the abelian case) the polyhedral partial sum operator is essentially a composition with a finite number of Hilbert transforms in directions perpendicular to the faces of the polyhedron chosen. Then Stanton established the values of p for which $|\Delta|^{2-p}$ satisfies the A_p condition in one variable, uniformly in the other variables, and did not consider the question: *when does $|\Delta|^{2-p}$ satisfy A_p?* In other words, his problem was essentially 1-dimensional and his results are different from ours: Stanton proved that, for $p \geqslant 2$, $|\Delta|^{2-p}$ satisfies the A_p condition in one variable, uniformly in the other variables, provided $p < p_0$, where, in general, p_0 is strictly less than $2n/(n-l)$. This result cannot be improved: indeed, as pointed out in Stanton and Tomas (1978, page 489), in general $|\Delta|^{2-p}$ does not satisfy A_p in one variable for $p > p_0$.

It is worth noting that the condition $p < 2n/(n-l)$ is also necessary and sufficient to ensure the $L^p(G)$-norm uniform boundedness of the irreducible characters of G: see Clerc (1976), Stanton and Tomas (1978), Giulini, Soardi and Travaglini (1982). See also Stanton (1976), Dooley (1979), Giulini, Soardi and Travaglini (1979), Dooley (1980), Fournier and Ross (to appear) for related problems about central L^p-integrable functions on compact Lie groups. Throughout all these papers the index $2n/(n-l)$ seems to have a basic role: for $2 \leqslant p < 2n/(n-l)$ one obtains some theorems which resemble the classical abelian ones, while for $p \geqslant 2n/(n-l)$ one finds results of an entirely different type. The purpose of this paper is to prove a result which may help to clarify this difference:

THEOREM. *Let* $p \geqslant 2$, *then* $|\Delta|^{2-p}$ *satisfies the* A_p *condition if and only if* $p < 2n/(n-l)$.

The "if" part of the theorem is new, while the "only if" part is a consequence of a theorem proved in Giulini, Soardi and Travaglini (1982), which we shall prove again in the sequel, for completeness.

The idea of the proof is to show the equivalence of the following conditions: (i) $|\Delta|^{2-p}$ satisfies A_p, (ii) $|\Delta|^{2-p}$ belongs to $L^1(T)$. This leads to a natural question in the classical A_p condition theory, which will be stated at the end of the paper.

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In the next section we shall fix the notation.

3

Throughout the paper G will denote a compact, simple, simply connected Lie group with Lie algebra $\mathfrak{g}$ and T a maximal torus with Lie algebra $\mathfrak{t}$. The complexification $\mathfrak{t}_{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$ and we denote by Φ the set of roots of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$. We choose in Φ a system Φ^+ of positive roots. If $\mathfrak{t}^*_{\mathbf{C}}$ denotes the dual space of $\mathfrak{t}_{\mathbf{C}}$, we transfer the Killing form to a nondegenerate bilinear form $(\cdot, \cdot)$ on $\mathfrak{t}^*_{\mathbf{C}} \times \mathfrak{t}^*_{\mathbf{C}}$ via the natural isomorphism of $\mathfrak{t}_{\mathbf{C}}$ with $\mathfrak{t}^*_{\mathbf{C}}$. The norm induced by $(\cdot, \cdot)$ on $\mathfrak{t}^*_{\mathbf{C}}$ is denoted by $|\cdot|$. Let Δ denote the usual Weyl function, that is $\Delta(H) = \prod_{\alpha \in \Phi^+} (-2i) \sin \frac{1}{2}\alpha(iH)$ where $H \in \mathfrak{t}$. We recall that the semilattice Σ of the dominant weights is in one-to-one correspondence with the dual object $\hat{G}$ (a maximal set of pairwise inequivalent unitary irreducible representations of G).

We refer to Varadarajan (1974) for the above facts.

4

Let $\mathfrak{Q} \subset \mathfrak{t}$ be a fundamental domain. We need the following lemma.

LEMMA. *The following conditions are equivalent*:

(a) $$p < \frac{2n}{n-l},$$

(b) $$\int_{\mathfrak{Q}} |\Delta|^{2-p} < \infty.$$

The implication (a) → (b) was proved in Clerc (1976) (see also Stanton and Tomas (1978)). The converse is an easy consequence of Theorem 3 in Giulini, Soardi and Travaglini (1982), where it is proved that, if $\eta = 2n/(n-l)$, then $\|\chi_{N\beta}\|_\eta^\eta > \text{const} \cdot \log N$, where $\beta = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$, and $\chi_{N\beta}$ is the character of the representation associated to the dominant weight $N\beta$. For completeness we give an outline of this part of the proof of the lemma.

Let $2 + s = 2n/(n-l)$. We apply Weyl's formulas and choose as a basis in $\mathfrak{t}_{\mathbf{C}}$ the fundamental weights, then

$$\|\chi_{(N-1)\beta}\|_{2+s}^{2+s} = I \geqslant \text{const} \cdot \int_0^\pi dt_1 \cdots \int_0^\pi dt_l \prod_{i=1}^{|\Phi^+|} \frac{|\sin \gamma_i(Nt)|^{2+s}}{|\sin \gamma_i(t)|^s},$$

where $|\Phi^+| = \text{card}\,\Phi^+$, $t = (t_1, \ldots, t_l)$ belongs to T and the γ_i are linear functions with nonnegative integral coefficients. Changing variables the above integral becomes

$$\int_0^{N\pi} dt_1 \cdots \int_0^{N\pi} dt_l \prod_{i=1}^{|\Phi^+|} \frac{|\sin \gamma_i(t)|^{2+s}}{\gamma_i(t)^s}.$$

Set $\bar{n} = (n_1, \ldots, n_l)$ where the n_k are positive integers $(k = 1, \ldots, l)$ and denote by J_N the set of multi-indices $\bar{n}$ such that $1 \leqslant n_k \leqslant N$ for all k. Also set

$$R_{\bar{n}} = \{t\colon (n_k - 1)\pi \leqslant t_k \leqslant n_k\pi,\ k = 1, \ldots, l\}.$$

We observe that the integral

$$\int_{R_{\bar{n}}} \prod_{i=1}^{|\Phi^+|} |\sin \gamma_i(t)|^{2+s}\, dt$$

does not depend upon $\bar{n}$. Therefore

$$I \geqslant \text{const} \cdot \sum_{\bar{n} \in J_N} \int_{R_{\bar{n}}} \prod_{i=1}^{|\Phi^+|} \frac{|\sin \gamma_i(t)|^{2+s}}{\gamma_i(t)^s} \, dt$$
$$\geqslant \text{const} \cdot \sum_{\bar{n} \in J_N} \prod_{i=1}^{|\Phi^+|} \gamma_i(\bar{n})^{-s}$$

and some computations give $I \geqslant \text{const} \cdot \log N$.

PROOF OF THE THEOREM. Clearly A_p implies (b), we have to prove the converse. Suppose A_p fails. Hence there exists a sequence of balls B_j such that

$$(*) \qquad \lim_{j \to \infty} \left(\frac{1}{m(B_j)} \int_{B_j} |\Delta|^{2-p} \right) \cdot \left(\frac{1}{m(B_j)} \int_{B_j} |\Delta|^{(p-2)/(p-1)} \right)^{p-1} = \infty.$$

From now on, $B(Y, s)$ will denote the ball centered at Y of radius s. Let $B_j = B(X_j, \varepsilon_j)$. Now Δ is periodic, so we may suppose $\operatorname{Sup} \varepsilon_j < \infty$. Then $\varepsilon_j \to 0$, otherwise (because of (b)) no term of the product in $(*)$ can diverge. Moreover, since Δ is periodic, we may suppose that each ball B_j is contained in $\mathfrak{Q}$ and $X_j \to X$ in $\mathfrak{Q}$. Again, we may suppose that the balls B_j are contained in a ball E centered at X with the following property: if $\bar{\alpha}(\bar{H}) \in 2\pi i\mathbf{Z}$ for some $\bar{\alpha} \in \Phi^+$ and $\bar{H} \in E$, then $\bar{\alpha}(\bar{H}) = \bar{\alpha}(X)$.

Let $\Phi_X^+ = \{\alpha \in \Phi^+ : \alpha(X) \in 2\pi i\mathbf{Z}\}$. Passing, if necessary, to a subsequence, we may suppose that for each $\alpha \in \Phi_X^+$ one and only one of the following three conditions is satisfied:

(i) for all j, $\alpha(X_j) \in 2\pi i\mathbf{Z}$,

(ii) for all j, $\alpha(X_j) \notin 2\pi i\mathbf{Z}$ and $\dfrac{\varepsilon_j}{|\alpha(iX_j) - \alpha(iX)|} \to 0$,

(iii) for all j, $\alpha(X_j) \notin 2\pi i\mathbf{Z}$ and $\dfrac{\varepsilon_j}{|\alpha(iX_j) - \alpha(iX)|} \geqslant \text{const} > 0$.

The above conditions lead to a natural partition of the set Φ_X^+,

$$\Phi_X^+ = \Phi_i \cup \Phi_{ii} \cup \Phi_{iii}.$$

For each $\alpha \in \Phi_X^+$, let $P_\alpha = \{Y \in \mathfrak{t} \mid \alpha(Y) = \alpha(X)\}$. We observe that if $\alpha \in \Phi_i$, then $X_j \in P_\alpha$, while if $\alpha \in \Phi_{iii}$, then the distance between X_j and P_α is less than ε_j (up to a constant). Then, for large j, there exists a ball $\bar{B}_j = B(\bar{X}_j, \bar{\varepsilon}_j)$ contained in E such that for all $\alpha \in \Phi_i \cup \Phi_{iii}$ we have $\alpha(\bar{X}_j) = \alpha(X)$, $B_j \subseteq \bar{B}_j$ and $\bar{\varepsilon}_j \leqslant \text{const} \cdot \varepsilon_j$ (one can choose $\bar{X}_j$ to be the point in P_α closest to X_j). Hence, if we change any B_j

into the corresponding $\bar{\bar{B}}_j$, $(*)$ is again true, while (iii) never occurs. Now we denote by $B_j = B(X_j, \varepsilon_j)$ the new balls $\bar{B}(\bar{X}_j, \bar{\varepsilon}_j)$ and we may suppose $\Phi_X^+ = \Phi_i \cup \Phi_{ii}$.

We have

$$\left(\frac{1}{m(B_j)}\int_{B_j}|\Delta|^{2-p}\right)\cdot\left(\frac{1}{m(B_j)}\int_{B_j}|\Delta|^{(p-2)/(p-1)}\right)^{p-1}$$

$$\leqslant \text{const}\cdot\left(\frac{1}{m(B_j)}\int_{B_j}\prod_{\alpha\in\Phi^+}\left|\sin\frac{\alpha(iH)}{2}\right|^{2-p}dH\right)$$

$$\cdot\left(\frac{1}{m(B_j)}\int_{B_j}\prod_{\alpha\in\Phi^+}\left|\sin\frac{\alpha(iH)}{2}\right|^{(p-2)/(p-1)}dH\right)^{p-1}$$

$$\leqslant \text{const}\cdot\left(\frac{1}{m(B_j)}\int_{B_j}\prod_{\alpha\in\Phi_i}|\alpha(iH)-\alpha(iX)|^{2-p}\prod_{\alpha\in\Phi_{ii}}|\alpha(iH)-\alpha(iX)|^{2-p}\,dH\right)$$

$$\cdot\left(\frac{1}{m(B_j)}\int_{B_j}\prod_{\alpha\in\Phi_i}|\alpha(iH)-\alpha(iX)|^{(p-2)/(p-1)}\right.$$

$$\left.\cdot\prod_{\alpha\in\Phi_{ii}}|\alpha(iH)-\alpha(iX)|^{(p-2)/(p-1)}\,dH\right)=K.$$

We observe that if $\alpha\in\Phi_{ii}$ and $H\in B_j$ we have (for large j)

$$\tfrac{1}{2}|\alpha(iX_j)-\alpha(iX)|\leqslant|\alpha(iH)-\alpha(iX)|\leqslant 2\,|\alpha(iX_j)-\alpha(iX)|\,.$$

Then, by the above inequalities, K is less than

$$\text{const}\cdot\left(\prod_{\alpha\in\Phi_{ii}}|\alpha(iX_j)-\alpha(iX)|^{2-p}\cdot\frac{1}{m(B_j)}\int_{B_j}\prod_{\alpha\in\Phi_i}|\alpha(iH)-\alpha(iX)|^{2-p}\,dH\right)$$

$$\cdot\left(\prod_{\alpha\in\Phi_{ii}}|\alpha(iX_j)-\alpha(iX)|^{\frac{p-2}{p-1}}\cdot\frac{1}{m(B_j)}\int_{B_j}\prod_{\alpha\in\Phi_i}|\alpha(iH)-\alpha(iX)|^{\frac{p-2}{p-1}}dH\right)^{p-1}.$$

Now we observe that inside the integrals we have only roots α such that, for every j, $\alpha(iX_j)=\alpha(iX)$. Then we translate X_j to zero and change any B_j into

$B_j^0 = B(0, \varepsilon_j)$. Hence we have to evaluate

$$\left(\frac{1}{m\left(B_j^0\right)}\int_{B_j^0}\prod_{\alpha\in\Phi_i}|\alpha(iH)|^{2-p}\,dH\right)\cdot\left(\frac{1}{m\left(B_j^0\right)}\int_{B_j^0}\prod_{\alpha\in\Phi_i}|\alpha(iH)|^{\frac{p-2}{p-1}}dH\right)^{p-1}$$

$$\leqslant\left(\frac{1}{m\left(B_j^0\right)}\int_{B_j^0}\prod_{\alpha\in\Phi^+}|\alpha(iH)|^{2-p}\prod_{\alpha\in\Phi^+\setminus\Phi_i}|\alpha(iH)|^{p-2}\,dH\right)$$

$$\cdot\operatorname*{Sup}_{H\in B_j^0}\prod_{\alpha\in\Phi_i}|\alpha(iH)|^{p-2}$$

$$\leqslant \text{const}\cdot\varepsilon_j^{-l}\cdot\left(\int_{B_j^0}\prod_{\alpha\in\Phi^+}|\alpha(iH)|^{2-p}\,dH\right)$$

$$\cdot\operatorname*{Sup}_{H\in B_j^0}\prod_{\alpha\in\Phi^+\setminus\Phi_i}|\alpha(iH)|^{p-2}\cdot\varepsilon_j^{|\Phi_i|(p-2)}$$

$$\leqslant \text{const}\cdot\varepsilon_j^{|\Phi^+|(p-2)-l}\cdot\int_{B_j^0}\prod_{\alpha\in\Phi^+}|\alpha(iH)|^{2-p}\,dH = M.$$

Now we recall that $p < 2n/(n-l)$ and we argue exactly as in Proposition 1 in Stanton and Tomas (1978). We obtain

$$M \leqslant \text{const}\cdot\varepsilon_j^{|\Phi^+|(p-2)-l}\cdot\varepsilon^{|\Phi^+|(2-p)+l} = \text{const}$$

which contradicts (*).

The proof is now complete.

The above theorem gives a necessary and sufficient condition, but in our case one cannot obtain divergence results. Indeed (see the introduction) we are not working with arbitrary trigonometric polynomials over T, but with Weyl group invariant polynomials. See Stanton and Tomas (1978), page 489 for a related discussion.

5

The technique of the above proof suggests the following question: *suppose $P_1,\ldots,P_r$ are polynomials defined on the unit cube in $\mathbf{R}^d$ and $\alpha_1,\ldots,\alpha_r$ are real numbers. Let ω be the periodic extension of $|P_1^{\alpha_1}\cdot\cdots\cdot P_r^{\alpha_r}|$ to the whole of $\mathbf{R}^d$. Are the following equivalent?*

(i) *ω satisfies A_p,*

(ii) *ω and $\omega^{-1/(p-1)}$ belong to $L_{\text{loc}(\mathbf{R}^d)}$.*

References

J. L. Clerc (1976), "Localisation des sommes de Riesz sur un groupe de Lie compact', *Studia Math.* **55**, 21–26.

R. R. Coifman and C. Fefferman (1974), 'Weighted norm inequalities for maximal functions and singular integrals', *Studia Math.* **51**, 241–250.

A. H. Dooley (1979), 'Norm of characters and lacunarity for compact Lie groups', *J. Functional Analysis* **32**, 254–267.

A. H. Dooley (1980), 'Random Fourier series for central functions on compact Lie groups', *Illinois J. Math.* **24**, 545–553.

J. J. F. Fournier and K. A. Ross, 'Random Fourier series on compact abelian hypergroups', to appear.

S. Giulini, P. M. Soardi and G. Travaglini (1979), 'A Cohen type inequality for compact Lie groups', *Proc. Amer. Math. Soc.* **77**, 359–364.

S. Giulini, P. M. Soardi and G. Travaglini (1982), 'Norms of characters and Fourier series on compact Lie groups', *J. Functional Analysis*, to appear.

B. Muckenhoupt (1972), 'Weighted norm inequalities for the Hardy maximal function', *Trans. Amer. Math. Soc.* **165**, 207–226.

R. J. Stanton (1976), 'On mean convergence of Fourier series on compact Lie groups', *Trans. Amer. Math. Soc.* **218**, 61–87.

R. J. Stanton and P. A. Tomas (1978), 'Polyhedral summability of Fourier series on compact Lie groups', *Amer. J. Math.* **100**, 477–493.

V. S. Varadarajan (1974), *Lie groups, Lie algebras and their representations* (Prentice-Hall, Englewood Cliffs, N. J.).

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