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INVARIANT SUBSPACE LATTICES OF LAMBERT'S WEIGHTED SHIFTS

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Abstract

Let $B(H)$ be the Banach algebra of all (bounded linear) operators on an infinite-dimensional separable complex Hilbert space H and let $\{a_m\}_{m=0}^{\infty}$ be a bounded sequence of positive real numbers. For a given injective operator A in $B(H)$ and a non-zero vector f in H, we put $w_m = a_m \| A^{m+1}f \| / \| A^m f \|$, $m = 0, 1, 2, \ldots$. We define a weighted shift T_w with the weight sequence $w = \{w_m\}_{m=0}^{\infty}$ on the Hilbert space 1^2 of all square-summable complex sequences $x = \{x_0, x_1, x_2, \ldots\}$ by $T_w(x) = \{0, w_0 x_0, w_1 x_1, w_2 x_2, \ldots\}$. The main object of this paper is to characterize the invariant subspace lattice of T_w under various nice conditions on the operator A and the sequence $\{a_m\}_{m=0}^{\infty}$.

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1. Introduction

Let H be an infinite-dimensional separable complex Hilbert space and $B(H)$ the algebra of all (bounded linear) operators from H into H. If H is 1^2, that is, the Hilbert space of all square-summable complex sequences $x = \{x_0, x_1, x_2, \ldots\}$ with the norm

$$\| x \| = \left(\sum_{m=0}^{\infty} | x_m |^2 \right)^{1/2},$$

and if $\alpha = \{\alpha_m\}_{m=0}^{\infty}$ is a bounded sequence of non-zero complex numbers, then the operator T_α on 1^2 defined by

$$T_\alpha\{x_0, x_1, x_2, \ldots\} = \{0, \alpha_0 x_0, \alpha_1 x_1, \alpha_2 x_2, \ldots\}$$

is called a (unilateral forward) weighted shift on l^2 with the weight sequence $\alpha = \{\alpha_m\}_{m=0}^{\infty}$. We may, and shall assume, without any loss of generality that the weights α_m are positive real numbers [3, Problem 2]. The invariant subspaces of this class of operators have been extensively studied by many authors; see, for example, Donoghue [1], Nikolskiĭ [7], [8], [9], Kelley [5], Nordgren [10], Harrison [4] and Shields [13].

By an invariant subspace M of T we shall mean a closed linear manifold of l^2 such that $TM \subset M$. By Lat T we shall denote the lattice of invariant subspaces of T. The object of this paper is to characterize the lattice of a rather specialised class of weighted shifts. Such weighted shifts have recently been studied for their subnormality by Lambert [6].

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Let A be an injective operator on H and suppose that $\{a_m\}_{m=0}^{\infty}$ is a bounded sequence of positive real numbers. For each non-zero vector f in H, let T_w be the weighted shift on l^2 with the weight sequence $w = \{w_m\}_{m=0}^{\infty}$, where

$$w_m = a_m \frac{\|A^{m+1}f\|}{\|A^m f\|}. \tag{1}$$

A vector x in l^2 is called a cyclic vector of T_w if

$$l^2 = \bigvee_{n=0}^{\infty} \{T_w^n x\},$$

the subspace spanned by $x, T_w x, T_w^2 x, \ldots$.

A sequence $\{a_m\}_{m=0}^{\infty}$ is said to be of bounded variation if

$$\sum_{m=0}^{\infty} |a_m - a_{m+1}| < \infty.$$

It is easy to see that if $\{a_m\}_{m=0}^{\infty}$ is monotonically decreasing, then it is of bounded variation, but the converse is not true. We shall say that $\{a_m\}_{m=0}^{\infty}$ is in the class $BV(*)$ if it is of bounded variation and satisfies the condition:

$$\Delta = \sup_{m \geq 2,\, n} \sum_{k=0}^{\infty} \left(\frac{a_{k+m} \cdots a_{k+n}}{a_m a_{m+1} \cdots a_n} \right)^2 < \infty. \tag{$*$}$$

An operator A in $B(H)$ is power-bounded if

$$\|A^n\| \leq \delta \tag{2}$$

for all $n = 1, 2, 3, \ldots$, where δ is a constant. We first prove

Lemma 2.1. *Let A be power-bounded and such that for every non-zero vector f in H, $A^n f \nrightarrow 0$ as $n \to \infty$. If the sequence $\{a_m\}_{m=0}^{\infty}$ is in $BV(*)$, then any vector $x = \{x_m\}_{m=0}^{\infty}$ in 1^2 with $x_0 \neq 0$ is a cyclic vector of T_w.*

Proof. We first observe that

$$\inf_{n \geqslant 0} \|A^n f\| = \mu(f) > 0 \quad \text{for all } f \neq 0. \tag{3}$$

In fact $\mu(f) = 0$ implies that there exists, for every $\varepsilon > 0$, an $n_0 = n_0(f, \varepsilon)$ such that $\|A^{n_0} f\| < \varepsilon/\delta$; and hence

$$\|A^n f\| = \|A^{n-n_0} A^{n_0} f\| \leqslant \delta \|A^{n_0} f\| < \varepsilon$$

for $n \geqslant n_0$. This contradicts our hypothesis that $A^n f \nrightarrow 0$.

Let $\{e_m\}_{m=0}^{\infty}$ be the standard orthonormal basis of 1^2. As

$$T_w^n x = \{\underbrace{0, 0, \ldots, 0}_{n},\ x_0 w_0 w_1 \cdots w_{n-1}, x_1 w_1 w_2 \cdots w_n, \ldots\},$$

we have

$$\begin{aligned}
\left\| \frac{T_w^n x}{x_0 w_0 w_1 \cdots w_{n-1}} - e_n \right\|^2 &= \sum_{m=0}^{\infty} \left(\frac{w_{m+1} \cdots w_{m+n}}{w_0 w_1 \cdots w_{n-1}} \right)^2 \left| \frac{x_{m+1}}{x_0} \right|^2 \\
&= \sum_{m=0}^{\infty} \left(\frac{a_{m+1} \cdots a_{m+n}}{a_0 a_1 \cdots a_{n-1}} \right)^2 \frac{\|A^{m+n+1} f\|^2 \|f\|^2}{\|A^{m+1} f\|^2 \|A^n f\|^2} \left| \frac{x_{m+1}}{x_0} \right|^2 \qquad \text{(by (1))} \\
&\leqslant \frac{\|A^n\|^2 \|f\|^2}{\|A^n f\|^2} \sum_{m=0}^{\infty} \left(\frac{a_{m+1} \cdots a_{m+n}}{a_0 a_1 \cdots a_{n-1}} \right)^2 \left| \frac{x_{m+1}}{x_0} \right|^2 \\
&\leqslant \frac{\delta^2 \|f\|^2 \|x\|^2}{(\mu(f))^2 |x_0|^2} \sum_{m=0}^{\infty} \left(\frac{a_{m+1} \cdots a_{m+n}}{a_0 a_1 \cdots a_{n-1}} \right)^2 \qquad \text{(by (2) and (3))} \\
&= \frac{\delta^2 \|f\|^2 \|x\|^2 a_n^2}{(\mu(f))^2 |x_0|^2 (a_0 a_1)^2} \sum_{m=0}^{\infty} \left(\frac{a_{m+2} \cdots a_{m+n}}{a_2 \cdots a_n} \right)^2 a_{m+1}^2 \\
&= C a_n^2 \sum_{m=0}^{\infty} \sum_{k=0}^{m} \left(\frac{a_{k+2} \cdots a_{k+n}}{a_2 \cdots a_n} \right)^2 \left(a_{m+1}^2 - a_{m+2}^2 \right) \\
&\qquad\qquad \text{(by Abel's transformation [15])} \\
&\leqslant C \Delta a_n^2 \sum_{m=0}^{\infty} \left(a_{m+1}^2 - a_{m+2}^2 \right) \qquad \text{(by } (*)) \\
&\leqslant 2aC\Delta a_n^2 \sum_{m=0}^{\infty} |a_{m+1} - a_{m+2}| \leqslant C a_n^2,
\end{aligned}$$

where $a = \sup_m\{a_m\}$ and C denotes a constant not necessarily the same everywhere.

Since $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis in l^2, and by $(*)$ $\sum_{n=0}^{\infty} a_n^2 < \infty$, it follows by the Paley-Wiener theorem [12, page 208] that the system

$$\left\{ \frac{T_w^n x}{x_0 w_0 w_1 \cdots w_{n-1}} \right\}_{n=0}^{\infty}$$

is a Riesz basis in l^2, whence we conclude that

$$\bigvee_{n=0}^{\infty} \{T_w^n x\} = l^2.$$

An operator A in $B(H)$ is said to belong to the class $C_{1 \cdot}$ if it is a contraction (that is $\|A\| \leqslant 1$) and $A^n f \nrightarrow 0$ for all $f \neq 0$. The class $C_{1 \cdot}$ plays an important role in the study of general contractions [14, page 72]. The following special case of Lemma 2.1 is worth mention:

Corollary 2.2. *Lemma* 2.1 *holds if* $\{a_m\}_{m=0}^{\infty}$ *is a monotonically decreasing square-summable sequence and* $A \in C_{1 \cdot}$.

Define

$$M_k = \left\{ x = \{x_m\}_{m=0}^{\infty} \in l^2;\, x_m = 0,\, m < k \right\}, \qquad k = 1, 2, \ldots.$$

It is obvious that $M_k \supset M_{k+1}$ and $M_k \in \operatorname{Lat} T_w$ for all $k = 1, 2, \ldots$. We show that the M_k are the only non-trivial invariant subspaces of T_w when A and $\{a_m\}_{m=0}^{\infty}$ satisfy the hypothesis of Lemma 2.1.

Theorem 2.3. *Let A be power-bounded and such that for every non-zero vector f in H, $A^n f \nrightarrow 0$ as $n \to \infty$. If the sequence $\{a_m\}_{m=0}^{\infty}$ is in $BV(*)$, then* Lat T_w *is order-isomorphic to* $1 + {}^*\omega$, *where* ${}^*\omega$ *denotes the order-type of the negative integers* [11, *page* 26].

Proof. Let M be a non-trivial invariant subspace of T_w. If $x = \{x_m\}_{m=0}^{\infty}$ is any vector in M, then, in view of Lemma 2.1, $x_0 = 0$. Suppose that k is the least positive integer such that $x_k \neq 0$. We first show that

$$\bigvee_{n=0}^{\infty} \{T_w^n x\} = M_k.$$

Recalling that $\{e_m\}_{m=k}^{\infty}$ is an orthonormal basis of M_k and following the proof of Lemma 2.1, we have

$$\begin{aligned}
&\left\| \frac{T_w^n x}{x_k w_k w_{k+1} \cdots w_{k+n-1}} - e_{n+k} \right\|^2 \\
&\quad = \sum_{m=0}^{\infty} \left(\frac{a_{m+k+1} \cdots a_{m+k+n}}{a_k a_{k+1} \cdots a_{k+n-1}} \right)^2 \frac{\| A^{k+m+n+1} f \|^2 \, \| A^k f \|^2}{\| A^{k+m+1} f \|^2 \, \| A^{k+n} f \|^2} \left| \frac{x_{m+k+1}}{x_k} \right|^2 \\
&\quad \leqslant \frac{\| A^n \|^2 \, \| A^k f \|^2}{\| A^{k+n} f \|^2} \sum_{m=0}^{\infty} \left(\frac{a_{m+k+1} \cdots a_{m+k+n}}{a_k a_{k+1} \cdots a_{k+n-1}} \right)^2 \left| \frac{x_{m+k+1}}{x_k} \right|^2 \\
&\quad \leqslant \frac{\delta^4 \| f \|^2 \, \| x \|^2 \, a_{k+n}^2}{(\mu(f))^2 \, | x_k |^2 \, (a_k a_{k+1})^2} \sum_{m=0}^{\infty} \left(\frac{a_{m+k+2} \cdots a_{m+k+n}}{a_{k+2} \cdots a_{k+n}} \right)^2 a_{m+k+1}^2 \\
&\quad \leqslant C a_{k+n}^2 .
\end{aligned}$$

It is now immediate by the Paley-Wiener theorem that $\bigvee_{n=0}^{\infty} \{T_w^n x\} = M_k$.

Since the span of any number of M_k is again an M_k, we conclude that $M = M_k$. Consequently, we have

$$\operatorname{Lat} T_w = \left\{ \{0\}, \ldots, M_3, M_2, M_1, 1^2 \right\}$$

and thus Lat T_w is order-isomorphic to $1 + {}^*\omega$.

Corollary 2.4. *Let A be invertible with both A and A^{-1} power-bounded. If the sequence $\{a_m\}_{m=0}^{\infty}$ is in $BV(*)$, then* Lat T_w *is order-isomorphic to* $1 + {}^*\omega$.

Proof. If $\| A^n \| \leqslant \delta$, $n = 0, \pm 1, \pm 2, \ldots$, then $\| A^n f \| \geqslant (1/\delta) \| f \|$, so $f \neq 0 \Rightarrow A^n f \nrightarrow 0$.

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We now consider the Hilbert space $1^2(\mathbf{C}^k)$, $k \geqslant 1$ of norm-square-summable sequences of vectors of the k-dimensional unitary spaces $\mathbf{C}^k$. Thus $1^2(\mathbf{C}^k)$ consists of sequences

$$x = \{x_m\}_{m=0}^{\infty}, \qquad x_m \in \mathbf{C}^k$$

such that $\sum_{m=0}^{\infty} \| x_m \|_*^2 < \infty$, where $\| x_m \|_*$ is the norm of x_m in $\mathbf{C}^k$ and

$$\| x \| = \left(\sum_{m=0}^{\infty} \| x_m \|_*^2 \right)^{1/2} .$$

Although we have not been able to prove the analogues of Theorem 2.3 and Corollary 2.4 for the Hilbert space $l^2(\mathbf{C}^k)$, we shall, however, show that Lemma 2.1 has an interesting extension in this case.

We shall say that a non-empty subset S of $l^2(\mathbf{C}^k)$ is a cyclic set of an operator T on $l^2(\mathbf{C}^k)$ if

$$\bigvee_{n=0}^{\infty} \{T^n x: x \in S\} = l^2(\mathbf{C}^k).$$

THEOREM 3.1. *Let A be power-bounded and such that for every non-zero vector f in H, $A^n f \nrightarrow 0$ as $n \to \infty$. If the sequence $\{a_m\}_{m=0}^{\infty}$ is in $BV(*)$, then any set of k-vectors in $l^2(\mathbf{C}^k)$ such that their first coordinates form a basis of $\mathbf{C}^k$ is a cyclic set of the weighted shift T_w on $l^2(\mathbf{C}^k)$.*

PROOF. Let $x^{(i)} = \{x_m^{(i)}\}_{m=0}^{\infty}$, $i = 1, 2, \ldots, k$, be k elements of $l^2(\mathbf{C}^k)$ such that $\{x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}\}$ is a basis in $\mathbf{C}^k$. We assume, without any loss of generality, that $\{x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(k)}\}$ is an orthonormal basis in $\mathbf{C}^k$. Then

$$T_w^n x^{(i)} = \{\underbrace{0, 0, \ldots, 0}_{n},\ w_{n-1} \cdots w_1 w_0 x_0^{(i)},\ w_n \cdots w_2 w_1 x_1^{(i)}, \ldots\}.$$

If $e_n(z)$, $z \in \mathbf{C}^k$, denotes the element of $l^2(\mathbf{C}^k)$ having z in the nth place and 0 elsewhere, we have

$$\left\| \frac{T_w^n x^{(i)}}{w_0 w_1 \cdots w_{n-1}} - e_n\left(x_0^{(i)}\right) \right\|^2 = \sum_{m=0}^{\infty} \left(\frac{w_{m+1} \cdots w_{m+n}}{w_0 w_1 \cdots w_{n-1}} \right)^2 \left\| x_{m+1}^{(i)} \right\|_*^2 \leqslant C a_n^2 \| x^{(i)} \|^2.$$

Since $\{e_n(x_0^{(i)})\}_{n \geqslant 0,\, 1 \leqslant i \leqslant k}$ is an orthonormal basis in $l^2(\mathbf{C}^k)$ and by $(*)$

$$\sum_{\substack{n \geqslant 0, \\ 1 \leqslant i \leqslant k}} a_n^2 \| x^{(i)} \|^2 < \infty,$$

it follows that the system

$$\left\{ \frac{T_w^n x^{(i)}}{w_0 w_1 \cdots w_{n-1}} \right\}_{n \geqslant 0,\, 1 \leqslant i \leqslant k}$$

is a Riesz basis in $l^2(\mathbf{C}^k)$ and consequently $\{x^{(i)}\}_{i=1}^{k}$ is a cyclic set of T_w.

COROLLARY 3.2. *Let A be invertible with both A and A^{-1} power-bounded and suppose that $\{a_m\}_{m=0}^{\infty}$ is in $BV(*)$. Then any set of k-vectors in $l^2(\mathbf{C}^k)$ such that their first coordinates form a basis of $\mathbf{C}^k$ is a cyclic set of T_w.*

A strictly cyclic operator algebra $\mathcal{A}$ on H is a uniformly closed subalgebra of $B(H)$ such that $\mathcal{A} f_0 = H$ for some vector f_0 in H. In this case f_0 is called a strictly cyclic vector for $\mathfrak{A}$. Moreover, if $Af_0 = 0$, $A \in \mathfrak{A}$ implies that $A = 0$, we say that f_0 is a separating vector for $\mathfrak{A}$. The following lemma is due to Embry [2]:

LEMMA 3.3. *Let f_0 be a strictly cyclic separating vector for $\mathfrak{A}$. Then there exists a constant C such that*

$$\|A\| \leqslant C\|Af_0\|$$

for every A in $\mathfrak{A}$.

THEOREM 3.4. *Let $\mathfrak{A}$ be a strictly cyclic operator algebra with a strictly cyclic separting vector f_0, and let $A \in \mathfrak{A}$. If the sequence $\{a_m\}_{m=0}^{\infty}$ is in $BV(*)$, then any set of k-vectors in $1^2(\mathbf{C}^k)$ such that their first coordinates form a basis of $\mathbf{C}^k$ is a cyclic set of T_w, where the weight sequence $w = \{w_m\}_{m=0}^{\infty}$ is defined by $w_m = a_m\|A^{m+1}f_0\|/\|A^m f_0\|$.*

PROOF. Following the proof of Theorem 3.1, it suffices to observe that

$$\begin{aligned}\left\|\frac{T_w^n x^{(i)}}{w_0 w_1 \cdots w_{n-1}} - e_n\left(x_0^{(i)}\right)\right\|^2 &\leqslant \frac{\|A^n\|^2\|f_0\|^2}{\|A^n f_0\|^2}\sum_{m=0}^{\infty}\left(\frac{a_{m+1}\cdots a_{m+n}}{a_0 a_1 \cdots a_{n-1}}\right)^2\|x_{m+1}^{(i)}\|_*^2 \\ &\leqslant C\|f_0\|^2\sum_{m=0}^{\infty}\left(\frac{a_{m+1}\cdots a_{m+n}}{a_0 a_1\cdots a_{n-1}}\right)^2\|x_{m+1}^{(i)}\|_*^2 \quad \text{(by Lemma 3.3)} \\ &\leqslant Ca_n^2\|x^{(i)}\|^2.\end{aligned}$$

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