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ZERO-SET ULTRAFILTERS AND G_δ-CLOSURES IN UNIFORM SPACES

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Abstract

The paper examines the classes $\mathscr{K}_1$ and Γ_1 of Hausdorff uniform spaces which are G_δ-closed in their Samuel compactifications, or completions. It is shown that the classes are epi-reflective, the reflections k_1 and γ_1 are described, $\mathscr{K}_1$ and Γ_1 are represented as epi-reflective hulls, membership in the classes is described by fixation of certain zero-set ultrafilters, and it is shown that $k_1 = \gamma_1$ exactly on spaces without discrete sets of measurable power. The results include familiar facts about realcompact and topologically complete topological spaces and are closely connected with the theory of metric-fine uniform spaces.

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This paper examines the classes $\mathscr{K}_1$ and Γ_1 of Hausdorff uniform spaces which are G_δ-closed in their Samuel compactifications, or completions. It is shown that the classes are epi-reflective, the reflections k_1 and γ_1 are described, $\mathscr{K}_1$ and Γ_1 are represented as epi-reflective hulls, membership in the classes is characterized by fixation of certain uniform z-ultrafilters, and it is shown that $k_1 = \gamma_1$ exactly on spaces without discrete sets of measurable power.

For a Tychonoff space equipped with its finest compatible uniformity, membership in $\mathscr{K}_1$ is equivalent to realcompactness, and in Γ_1 to topological completeness (but uniform spaces not in $\mathscr{K}_1$ (or Γ_1) frequently have realcompact (or topologically complete) topology), so the results here include results from topology (for the most part, familiar ones). The analogy is quite faithful, as the results show.

Rather, though, than from this analogy, the classes $\mathscr{K}_1$ and Γ_1 arose originally and naturally in the studies of metric-fine uniform spaces of Hager (1974) and M. D. Rice (1975a, b). The connection is explained in Section 5, and we shall reprove here, more directly, some of the results from these papers.

1. Epi-reflections and G_δ-closures

'Space' means Hausdorff uniform space, and spaces will be denoted X, $Y, \ldots$. Unif stands for the category with these objects, and maps are uniformly continuous functions. Essentially, we shall assume that the notion of an epi-reflective subcategory is a familiar one. For our purposes, Section 1 of Hager (1975) is a good introduction.

$\mathscr{K}$ is the subcategory of compact spaces. This is epi-reflective, and the reflection k_x: $X \to kX$ is the Samuel compactification. Each reflection morphism k_x is a uniformly continuous homeomorphism with dense image (which is an embedding if and only if X is precompact).

Γ is the subcategory of complete spaces. This is epi-reflective, and the reflection γ_x: $X \to \gamma X$ is the completion. Each γ_x is a dense embedding.

Now let $\mathscr{R}$ be an epi-reflective subcategory of Unif for which each reflection morphism r_x is a homeomorphism (equivalently, $\mathscr{R} \supset \mathscr{K}$)—such as $\mathscr{K}$ or Γ. Given such $\mathscr{R}$, let $\mathscr{R}_1$ be the class of spaces X for which $r_x(X)$ (or less precisely, X) is G_δ-closed in rX (that is, if $p \in rX - r_x(X)$, then there is a G_δ-set G with $p \in G$ and $G \cap r_x(X) = \varnothing$).

1.1. Proposition. (a) *$\mathscr{R}_1$ is productive and closed-hereditary, hence epi-reflective.*

(b) *The reflection $r_1 X$ is, as a set, the G_δ-closure of X in rX, and the uniformity is the finest (that is, largest) one for which the relativization to X is coarser than the uniformity of X.*

Regarding the proof of 1.1: the verification that $\mathscr{R}_1$ is productive and closed-hereditary is routine; epi-reflectivity follows from Kennison (1965). For (b), the uniformity described is realized as the supremum of the uniformities whose relativizations to X are coarser than X's; then the required universal mapping property is verified.

1.1(b) does not convey much feeling for what the uniformity of $r_1 X$ really is, but we have not a more incisive description. The difficulties appear even for $k_1 X$, as the following illustrates. (Ultimately, a good description of $k_1 X$ will be obtained, but this is involved.)

1.2. COROLLARY. (a) *If* $r_X\colon X\to rX$ *is an embedding, then* $r_1 X$ *is a subspace of* rX, *and* X *is a subspace of* $r_1 X$.

(b) *For any* X, $X\subset\gamma_1 X\subset\gamma X$.

(c) *If* X *is precompact, then* $X\subset k_1 X\subset kX$.

1.3. PROPOSITION. (a) *For any* X, $k_1 X\cap\gamma X=\gamma_1 X$ *as sets.*

(b) $K_1\subset\Gamma_1$.

PROOF. It is well known that if X is dense in Y, then $kX=kY$ and $\gamma X=\gamma Y$. It follows that if X is G_δ-dense in Y (that is, each nonvoid G_δ-set in Y meets X), then $k_1 X=k_1 Y$ and $\gamma_1 X=\gamma_1 Y$. From this, (a) follows, and (b) follows from (a).

2. Closed subspaces of products

This section concerns representing $\mathscr{K}_1$ and Γ_1 as epi-reflective hulls or something similar.

The epi-reflective hull $\mathscr{R}(\mathscr{A})$ of the class $\mathscr{A}$ of spaces consists of spaces isomorphic to a closed subspace of a product of members of $\mathscr{A}$. For example: $\mathscr{K}=\mathscr{R}[0,1]$ (suppressing some parentheses); with $\mathscr{M}$ the class of metric spaces and $\bar{\mathscr{M}}$ the complete metric spaces, $\mathscr{R}(\bar{\mathscr{M}})=\Gamma$.

In what follows, $(0,1]$ carries its usual uniformity: $\gamma(0,1]=k(0,1]=[0,1]$.

2.1. PROPOSITION. (a) (Rice (1975b)) $\Gamma_1=\mathscr{R}(\mathscr{M})$.

(b) *These conditions on* X *are equivalent:*

(1) $X\in\mathscr{K}_1$.

(2) *The evaluation* $e\colon X\to\prod\{(0,1]_f : f\in U(X,(0,1])\}$ *has closed range.*

(3) *There is* $Y\in\mathscr{R}(0,1]$ *and a perfect map* $f\colon X\to Y$.

(c) $\mathscr{K}_1\cap$(*Precompact spaces*) $=\mathscr{R}(0,1]$.

We shall prove 2.1 by the technique of 'generalized perfect maps' developed in Hager (1975): For epi-reflective $\mathscr{R}\supset\mathscr{K}$, a (uniformly continuous) map $f\colon X\to Y$ is called $\mathscr{R}$-perfect if $rf(rX-r_x(X))\subset rY-r_y(Y)$. Given a class $\mathscr{A}$ of spaces, the class of all X for which there are $A\in\mathscr{A}$ and $\mathscr{R}$-perfect $f\colon X\to A$ is denoted $p_{\mathscr{R}}\mathscr{A}$ (and called the $\mathscr{R}$-perfect hull). A $\mathscr{K}$-perfect map is called just 'perfect', and $p_{\mathscr{K}}\mathscr{A}$ is familiar as the 'left-fitting hull'.

2.2. LEMMA. *Let* $\mathscr{R}$ *be epi-reflective with* $\mathscr{R}\supset\mathscr{K}$, *and suppose that* $r(0,1]=[0,1]$. *Then these conditions on* X *are equivalent:*

(a) $X\in\mathscr{R}_1$.

(b) *For each* $p\in rX-r_x(X)$, *there is* $g\in U(rX,[0,1])$ *with* $g(p)=0$ and $g(r_x(X))\subset(0,1]$.

(c) *For each* $p \in rX - r_x(X)$, *there is* $f \in U(X,(0,1])$ *with* $rf(p) = 0$.
(d) *The evaluation* $e\colon X \to \prod\{(0,1]_f \mid f \in U(X,(0,1])\}$ *is* $\mathscr{R}$*-perfect.*
(e) $X \in p_{\mathscr{R}}\mathscr{R}(0,1]$.

Proof. Sketch: (a)$\Rightarrow$(b). If G is a G_δ and $p \in G$, there is a zero-set $Z(g)$ with $p \in Z(g) \subset G$.

(b)$\Rightarrow$(c). Since $r(0,1] = [0,1]$, a g as in (b) is an rf as in (c).

(c)$\Rightarrow$(d). 2.10 of Hager (1975).

(d)$\Rightarrow$(e). Obvious.

(e)$\Rightarrow$(*a*). If $f\colon X \to Y$ is $\mathscr{R}$-perfect, then whenever G is a G_δ in rY missing $r_y(Y)$, $f^{-1}(G)$ is a G_δ in rX missing $r_x(X)$.

Proof of 2.1. (b) We take $\mathscr{R} = \mathscr{K}$ in 2.3. Then, (1)$\Rightarrow$(2) by 2.3 ((a)$\Leftrightarrow$(e)). Assuming (1), 2.3(d) holds. But a $\mathscr{K}$-perfect map has closed range (5.2 of Hager (1975)), so (2) holds. If (2) holds, then e is $\mathscr{K}$-perfect because e is a homeomorphism (2.6 of Hager (1975)).

(c) From (b) and the fact that X is precompact if and only if e is an embedding.

(d) We shall show that $\mathscr{R}(\mathscr{M}) \subset \Gamma_1 \subset p_\Gamma \mathscr{R}(\mathscr{M}) \subset \mathscr{R}(\mathscr{M})$. First, if $M \in \mathscr{M}$, then γM is metric with points G_δ-sets. Thus, $\mathscr{M} \subset \Gamma_1$; so $\mathscr{R}(\mathscr{M}) \subset \Gamma_1$ by 1.2(a). Next, we have $\Gamma_1 = p_\Gamma \mathscr{R}(0,1] \subset p_\Gamma \mathscr{R}(\mathscr{M})$ (by 2.2; then $(0,1] \in \mathscr{M}$). Finally, if $X \in p_\Gamma \mathscr{R}(\mathscr{M})$, then there are $A \in \Gamma$, $B \in \mathscr{R}(\mathscr{M})$, and a closed embedding of X in $A \times B$, by 4.4 of Hager (1975). Since $\Gamma = \mathscr{R}(\bar{\mathscr{M}}) \subset \mathscr{R}(\mathscr{M})$, we have $A \times B \in \mathscr{R}(\mathscr{M})$, hence $X \in \mathscr{R}(\mathscr{M})$.

2.3. Remark. 2.1(b), read as a statement in Tychonoff spaces, is true, but the more familiar version has the real line replacing $(0,1]$ in (2) and (3). In Unif, this replacement is not valid. Let X be the metric hedgehog with $\aleph_0$ spines. X is not precompact, but each $f \in U(X,R)$ is bounded. Then the evaluation

$$e\colon X \to \prod\{R_f \mid f \in U(X,R)\}$$

has precompact range which, if closed, would be compact; then X would be compact. Likewise, if there were perfect $f\colon X \to Y \in \mathscr{R}(R)$, again Y would be compact, and X also.

3. Zero-set ultrafilters

For f a real-valued function, $Z(f) \equiv \{x \mid f(x) = 0\}$. For $X \in \mathrm{Unif}$, a (uniform) zero-set of X is a member of $\mathscr{Z}(X) \equiv \{Z(f) \mid f \text{ bounded in } U(X,R)\}$. A z-ultrafilter on X is a maximal filter in $\mathscr{Z}(X)$.

A z-ultrafilter $\mathscr{F}$ has the countable intersection property (cIP), if ($\mathscr{F}_0 \subset \mathscr{F}$, $\mathscr{F}_0$ countable $\Rightarrow \bigcap \mathscr{F}_0 \neq \varnothing$). It is easy to see that if $\mathscr{F}$ has cIP, then $\mathscr{F}$ is *closed* under countable intersection (because $\mathscr{Z}(X)$ is).

A family $\mathscr{D}$ of subsets of X is called discrete if there is a uniformly continuous pseudometric ρ and $\varepsilon > 0$ such that $\rho(A, B) \geqslant \varepsilon$ for different $A, B \in \mathscr{D}$. If $\mathscr{D}_1, \mathscr{D}_2, \ldots$ are discrete, then $\bigcup \mathscr{D}_n$ is called σ-discrete.

If $\mathscr{B}$ is a family of subsets of X, then $\mathrm{co}\mathscr{B} \equiv \{X - B \mid B \in \mathscr{B}\}$. A z-ultrafilter $\mathscr{F}$ is said to have the co(σ-discrete) intersection property, or co(σ-disc) IP, if ($\mathscr{F}_0 \subset \mathscr{F}$, $\mathrm{co}\mathscr{F}_0$ σ-discrete $\Rightarrow \bigcap \mathscr{F}_0 \neq \varnothing$). Equivalently, if ($\mathscr{D} \subset \mathrm{co}\mathscr{F}$, $\mathscr{D}$ σ-discrete $\Rightarrow \bigcup \mathscr{D} \neq X$). Evidently, this property entails cIP. (See 3.3 regarding *closure* under such intersections.)

For $p \in kX$, let $\mathscr{F}_p = \{Z(f) \mid kf(p) = 0, f$ bounded in $U(X, R)\}$. (Note that $p \notin k_1 X$ if and only if $\varnothing \in \mathscr{F}_p$; so $\mathscr{F}_p$ is not always a filter.) A z-ultrafilter is called fixed if $\mathscr{F} = \mathscr{F}_p$ for some $p \in X$.

3.1. Proposition. *Let $\mathscr{F}$ be a z-ultrafilter on X.*

(a) *$\mathscr{F}$ has cIP if and only if there is (unique) $p \in k_1 X$ with $\mathscr{F} = \mathscr{F}_p$.*

(b) *These are equivalent:*

(1) *$\mathscr{F}$ has co(σ-disc) IP;*

(2) *$\mathscr{F}$ has cIP and is Cauchy;*

(3) *there is (unique, $p \in \gamma_1 X$ with $\mathscr{F} = \mathscr{F}_p$.*

3.2. Corollary. *$X \in \mathscr{K}_1$ (resp., Γ_1) if and only if each z-ultrafilter with cIP (resp., co(σ-disc IP), is fixed.*

Prior to the proof, a few remarks are in order. A topological zero-set (that is, of a continuous function) need not be a uniform zero-set. Let X be an uncountable set with the uniformity inherited from the one-point compactification of discrete X. Then $\mathscr{Z}(X)$ consists of finite and co-countable sets, but, topologically, every subset is zero. (Note that Chapter 15 of Gillman and Jerison (1960) speaks of topological zero sets.)

In Tychonoff spaces, the compact reflection (β) is the space of all topological z-ultrafilters, but not so in Unif. For $X \in$ Unif, the space of all z-ultrafilters is a generally much larger 'compactification' than kX, called $H(A^*(kX))$ and pretty thoroughly analysed in Hager (1969); 3.1(a) above more or less follows from 4.2 and 5.3 there. (The space is also called an 'Alexandroff compactification'; see 7.5 and Section 9 of Hager (1974).)

Proof of 3.1. (a) Let $p \in k_1 X$. We show that $\mathscr{F}_p$ is a z-ultrafilter with cIP. First, $\mathscr{F}_p$ is a z-filter: If $Z(f)$ and $Z(g) \in \mathscr{F}_p$, then $Z(f) \cap Z(g) = \ldots = Z(|kf| + |kg|) \cap X$, which is nonvoid because $p \in k_1 X$; If $Z(g) \supset Z(f) \in \mathscr{F}_p$, we can arrange it that

$0 \leqslant g \leqslant f$, hence $0 \leqslant kg \leqslant kf$, so $kg(p) = 0$ and $Z(g) \in \mathscr{F}_p$. Next, $\mathscr{F}_p$ is maximal: If $Z(f) \notin \mathscr{F}_p$, then $p \notin Z(kf)$ and there is (by complete regularity of kX) $Z(kg)$ containing p and missing $Z(kf)$; then $Z(g) \in \mathscr{F}_p$, while $Z(f) \cap Z(g) = \emptyset$. Finally, $\mathscr{F}_p$ has cIP. Let $Z(f_1), Z(f_2), \ldots \in \mathscr{F}_p$, and suppose (as we may) that $0 \leqslant f_n \leqslant 2^{-n}$; now $p \in$ each $Z(kf_n)$, hence $p \in \bigcap_n Z(kf_n) = Z(\sum kf_n) = Z(k \sum f_n)$; so

$$\bigcap Z(f_n) = X \cap Z(k \textstyle\sum f_n) = Z(\sum f_n) \in \mathscr{F}_p.$$

Conversely, suppose $\mathscr{F}$ has cIP. Then there is $p \in \bigcap \{\bar{Z} \mid Z \in \mathscr{F}\}$ (closures in kX), because the family has the finite IP and kX is compact. Let G be a G_δ containing p. We may suppose that $G = \bigcap_n Z_n$, where each Z_n is a zero-set of kX with p in its interior. Clearly then, $Z_n \cap Z \neq \emptyset$ for each $Z \in \mathscr{F}$. Thus each $Z_n \cap X \in \mathscr{F}$ (since $\mathscr{F}$ is maximal), and therefore $G \cap X = \bigcap_n (Z_n \cap X) \neq \emptyset$, by cIP. So $p \in k_1 X$. Thus, by the other half of this proof above, $\mathscr{F}_p$ is a z-ultrafilter. Since clearly, $\mathscr{F} \subset \mathscr{F}_p$, equality follows.

(b) (1)$\Rightarrow$(2). Suppose $\mathscr{F}$ has co(σ-disc) IP, and let $\mathscr{U}$ be a uniform cover of X. Choose a metric space M and uniformly continuous f: $X \to M$ such that $f^{-1}(\mathscr{S}) < \mathscr{U}$, where $\mathscr{S}$ is the collection of 1-spheres in M (see Isbell (1964)). Now choose a σ-discrete open cover $\mathscr{V}$ refining $\mathscr{S}$, by Stone (1948). It suffices to show that some member of $\mathscr{F}$ is contained in a member of $f^{-1}(\mathscr{V})$. If not, then given $V \in \mathscr{V}$, $f^{-1}(V) \not\supset F$ for each $F \in \mathscr{F}$; that is, $(X - f^{-1}(V)) \bigcap F \neq \emptyset$. In M, each closed set is a zero-set, so $X - f^{-1}(V) \in \mathscr{F}$. But $\bigcap_V (X - f^{-1}(V)) = \emptyset$, a contradiction.

(2)$\Rightarrow$(3). Let $\mathscr{F}$ be Cauchy, with CIP. Then $\mathscr{F} = \mathscr{F}_p$ for $p \in k_1 X$, by (a). If $\mathscr{F}^*$ is the filter in the power set of γX generated by $\mathscr{F}$, then $\mathscr{F}^*$ is Cauchy on γX, and converges to some $q \in \gamma X$. If $q \neq p$, then choose a zero-set neighbourhood Z_q of q in kX not containing p, then a zero-set Z_p of kX containing p and missing Z_q. Then $Z_p \cap X$ and $Z_q \cap X$ are disjoint members of $\mathscr{F}_p$, a contradiction. Thus $p = q$, and $p \in \gamma X \cap k_1 X = \gamma_1 X$ (using 1.3).

(3)$\Rightarrow$(1). Let $p \in \gamma_1 X$ and let $\mathscr{D} \subset \mathrm{co}\mathscr{F}$ be σ-discrete. So $\mathscr{D} = \bigcup \mathscr{D}_n$, where for each n there is a uniformly continuous pseudometric ρ_n and $\varepsilon_n > 0$ such that $\mathscr{D}_n$ is ε_n-discrete for ρ_n. Each ρ_n has an extension to a uniformly continuous pseudometric $\gamma\rho_n$ on γX. Let $Z_n = \{x \in X \mid \gamma\rho_n(p, x) = 0\}$. Clearly, $Z_n \in \mathscr{F}_p$. Then $Z \equiv \bigcap_n Z_n \in \mathscr{F}_p$ as well.

Now suppose $X = \bigcup \{D \mid D \in \bigcup \mathscr{D}_n\}$. Set $X_n = \bigcup \{D \mid D \in \mathscr{D}_n\}$, so $X = \bigcup X_n$. Fixing n, if $Z \cap X_n \neq \emptyset$ then there is a single $D_0^n \in \mathscr{D}_n$ with $Z \cap X_n \subset D_0^n$. (For, choose $p_0 \in Z \cap X_n$, then unique $D_0^n \in \mathscr{D}_n$ with $p_0 \in D_0^n$. Then $\rho_n(p_0, D) \geqslant \varepsilon_n > 0$ for any $D \in \mathscr{D}_n$ with $D \neq D_0^n$. Thus $\emptyset = D \cap Z_n \supset D \cap Z$.) Thus, $A = \bigcup_n (Z \cap X_n) \subset \bigcup_n D_0^n$. But $\bigcup_n D_0^n \in \mathrm{co}\mathscr{F}_p$ so $X - \bigcup_n D_0^n$ and Z are disjoint members of $\mathscr{F}_p$, a contradiction.

It is frequently convenient to know that a z-ultrafilter with cIP is *closed* under countable intersection (as in the proof of 3.1), and so one naturally wonders about closure conditions for z-ultrafilters with co(σ-disc) IP. The difficulty is that a union of a discrete family of uniform cozero sets need not be cozero. Indeed, if X has such a discrete family $\mathscr{D}$, then for any $p \in X - \bigcup \mathscr{D}$, $\mathscr{F}_p$ has co(σ-disc) IP of course, $\mathscr{D} \subset \mathrm{co}\mathscr{F}_p$ but $\bigcup \mathscr{D} \notin \mathrm{co}\mathscr{F}_p$. However:

3.3. Proposition. *In a uniform space X*

(a) *These properties of a discrete family $\mathscr{D}$ are equivalent.*

(1) $\bigcup \mathscr{D}$ *is cozero.*

(2) *For each $\mathscr{D}' \subset \mathscr{D}$, $\bigcup \mathscr{D}'$ is cozero.*

(3) $\mathscr{D} = \{\mathrm{coz} g \mid g \in \mathscr{S}\}$ *for some equiuniformly continuous family $\mathscr{S} \subset U(X, R)$.*

(b) *A z-ultrafilter $\mathscr{F}$ has co(σ-disc) IP if and only if $\mathscr{F}$ has cIP, and $\bigcup \mathscr{D} \in \mathrm{co}\mathscr{F}$ whenever $\mathscr{D}$ is as in* (a).

Proof. (a) (2)$\Rightarrow$(1) is obvious. (1)$\Rightarrow$(3). Let $\mathscr{D}$ be ε-discrete for ρ, let $\bigcup \mathscr{D} = \mathrm{coz} f$ with $0 \leqslant f \leqslant 1$, and for $D \in \mathscr{D}$, let $f_D = ((\varepsilon - \rho(\circ, D)) \vee 2\varepsilon$. Then $\mathscr{S} = \{ff_0 \mid D \varepsilon \mathscr{D}\}$ works. (3)$\Rightarrow$(2). For $\mathscr{D}' = \{\mathrm{coz} g \mid g \in \mathscr{S}'\}$, $\bigcup \mathscr{D}' = \mathrm{coz} \sum \{g \mid g \in \mathscr{S}'\}$.

(b) If $\mathscr{F}$ has co(σ-disc) IP, then $\mathscr{F}$ has cIP of course. Suppose $\mathscr{D}$ is discrete, $\mathscr{D} \subset \mathrm{co}\mathscr{F}$, $\bigcup \mathscr{D}$ is cozero, but $\bigcup \mathscr{D} \notin \mathrm{co}\mathscr{F}$. Then $Z = X - \bigcup \mathscr{D} \notin \mathscr{F}$, so there is $Z_0 \in \mathscr{F}$ with $Z \cap Z_0 = \varnothing$. Then $\{\{X - Z_0\}\} \cup \mathscr{D} \subset \mathrm{co}\mathscr{F}$ is σ-discrete, but

$$(X - Z_0) \cup \bigcup \mathscr{D} = X.$$

Conversely, if $\mathscr{F}$ has the stated properties, then condition (2) in 3.1 can be derived exactly as (1) $\Rightarrow$ (2) there.

4. $\mathscr{K}_1$ versus Γ_1

The main result here is a uniform version of the Katětov–Shirota Theorem. It is almost exactly analogous to the (topological) version presented in Curzer and Hager (1976); so we can be brief.

Recall that a set S has measurable power if there is a countably additive $\{0, 1\}$-valued measure μ on the power set of S, with $\mu(\{p\}) = 0$ for each $p \in S$ and $\mu(S) = 1$; equivalently, if each ultrafilter in the power set with cIP is fixed (12.2 of Gillman and Jerison (1960)).

Let $\mathscr{N}$ be the class of spaces which have no discrete set of measurable power; equivalently, each uniform cover has a uniform refinement of nonmeasurable power. It is not hard to see that $\mathscr{N}$ is productive and hereditary, hence epi-reflective in Unif; let n: Unif$\to \mathscr{N}$ denote the reflecting functor. It can be shown that a base for nX is the set of all uniform coverings of X of nonmeasurable power. (See Isbell (1964), p. 52.)

4.1. PROPOSITION. *$\mathscr{K}_1 = \Gamma_1 \cap \mathscr{N}$ and $k_1 = \gamma_1 \circ n$.*

4.2. LEMMA. *For any X, if $\mathscr{F}$ is a z-ultrafilter with cIP, then whenever $\mathscr{D} \subset \mathrm{co}\mathscr{F}$, $\mathscr{D}$ is discrete of nonmeasurable power, and $\bigcup \mathscr{D}'$ is cozero for each $\mathscr{D}' \subset \mathscr{D}$, then $\bigcup \mathscr{D} \in \mathrm{co}\mathscr{F}$.*

Hence, if $X \in \mathscr{N}$, then each z-ultrafilter with cIP has co(σ-disc) IP.

PROOF. For the first part, the proof of 7 of Curzer and Hager (1976) can be copied. (The reference is to the corresponding statement for Tychonoff spaces. There, the hypothesis that the sets $\bigcup \mathscr{S}'$ be cozero is unnecessary.)

The second part follows using 3.3.

PROOF OF 4.1. Let $X \in \Gamma_1 \cap \mathscr{N}$, and let $\mathscr{F}$ be a z-ultrafilter with cIP. By 4.2, $\mathscr{F}$ has co(σ-disc) IP. Since $X \in \Gamma_1$, $\mathscr{F}$ is fixed, and thus $X \in \mathscr{K}_1$ (using 3.2 twice).

Now, $\mathscr{K}_1 \subset \Gamma_1$, by 1.3. Let $X \in \mathscr{K}_1$ and let D be discrete in X. Then D is closed, so by 1.1 $D \in \mathscr{K}_1$. The relative uniformity on D is discrete, so $\mathscr{Z}(D)$ is the power set $\mathscr{P}$(D). So saying that $D \in \mathscr{K}_1$ is saying that D has nonmeasurable power. Thus $\mathscr{K}_1 \subset \mathscr{N}$.

We show that $k_1 = \gamma_1 \circ n$. Given X, $\gamma_1 nX \in \Gamma_1$ since γ_1 operates last. It is trivial to check that $\gamma\mathscr{N} \subset \mathscr{N}$; thus $\gamma nX \in \mathscr{N}$. Since $\mathscr{N}$ is hereditary and $\gamma_1 nX \subset \gamma nX$ (1.2), $\gamma_1 nX \in \mathscr{N}$ as well. Thus $\gamma_1 nX \in \mathscr{K}_1$. Now let $Y \in \mathscr{K}_1$, and let $f \in U(X, Y)$. Then $nf \in U(nX, Y)$ (since $Y \in \mathscr{N}$), and $\gamma_1 nf \in U(\gamma_1 nX, Y)$ (since $Y \in \Gamma_1$). Uniqueness of $\gamma_1 nf$ (as a 'lift' of f over $X \to nX \to \gamma_1 nX$) is easy. Thus, $k_1 X = \gamma_1 nX$ for every X.

4.3. COROLLARY. $\mathscr{K}_1 = \mathscr{R}(\mathscr{M} \cap \mathscr{N})$.

PROOF. First note that $\mathscr{R}(\mathscr{M}) \cap \mathscr{R}(\mathscr{N}) = \Gamma_1 \cap \mathscr{N} = \mathscr{K}_1$ by 2.1 and 4.1. Now $\mathscr{R}(\mathscr{M} \cap \mathscr{N}) \subset \mathscr{R}(\mathscr{M}) \cap \mathscr{R}(\mathscr{N})$ is obvious. If $X \in \mathscr{R}(\mathscr{M})$, there is a closed embedding $e\colon X \to \prod_\alpha M_\alpha$ ($M_\alpha \in \mathscr{M}$). Then $e\colon X \to \prod_\alpha \pi_\alpha(e(X))$ is also a closed embedding, and each $\pi_\alpha(e(X)) \in \mathscr{M}$. But for $X \in \mathscr{N}$, each $\pi_\alpha(e(X)) \in \mathscr{N}$. ($\mathscr{N}$ is closed under uniformly continuous images: a discrete set in the image induces one in the domain.) Thus, $\mathscr{R}(\mathscr{M}) \cap \mathscr{N} \subset \mathscr{R}(\mathscr{M} \cap \mathscr{N})$.

4.4. REMARK. For more general $\mathscr{R} \supset \mathscr{K}$, we have this: If $\mathscr{R} \supset \Gamma$, then each r_x is an embedding, so $X \subset r_1 X \subset rX$, by 1.2. If $\mathscr{R} \subset \Gamma \cap \mathscr{N}$, then $\mathscr{K}_1 \subset \mathscr{R}_1 \subset \Gamma_1 \cap \mathscr{N} = \mathscr{K}_1$; that is, $\mathscr{R}_1 = \mathscr{K}_1$. Thus, knowledge of $\mathscr{R}_1$ (and r_1) is complete except when $\Gamma \not\subset \mathscr{R} \not\subset \Gamma \cap \mathscr{N}$.

5. Metric-fine spaces

We shall discuss briefly the theory of these spaces, and what it has to do with $\mathscr{K}_1$ and Γ_1.

A space X is called metric-fine if each uniformly continuous map of X to any metric space M remains uniformly continuous when M is re-equipped with its finest compatible uniformity. These spaces were introduced in Hager (1971), and systematically examined in Hager (1974), Frolík (1974), Rice (1975a), which see.

The property is coreflective in Unif; the metric-fine coreflection of X is denoted mX. The space meX (eX having base of countable uniform covers of X) has base of all countable $\operatorname{coz}X$-covers, and mX has base of all $\operatorname{coz}X$-covers which are σ-discrete with respect to X. It follows that $\mathscr{Z}(mX) = \mathscr{Z}(meX) = \mathscr{Z}(eX) = \mathscr{Z}(X)$ and so the z-ultrafilters of X, eX, meX and mX are all the same. (The space of all z-ultrafilters of X is kmX, by the way (Hager (1974).)

It follows relatively easily that for a z-ultrafilter $\mathscr{F}$ on X (a) $\mathscr{F}$ has cIP if and only if $\mathscr{F}$ is Cauchy for meX, (b) $\mathscr{F}$ has co(σ-disc) IP if and only if $\mathscr{F}$ is Cauchy for mX. ((a) is implicit in 8.4 of Hager (1974), (b) is closely related to 3.5 of Rice (1975b)—using (2) of 3.1.) Thus, from 3.2 here, (a) $X \in \mathscr{K}_1$ if and only if $meX \in \Gamma$ (which is almost in 8.1 of Hager (1974)), (b) $X \in \Gamma_1$ if and only if $mX \in \Gamma$ (in Rice (1975b)).

That $\mathscr{K}_1 = \Gamma_1 \cap \mathscr{N}$ can thus be put: $meX \in \Gamma$ if and only if $mX \in \Gamma$ and $X \in \mathscr{N}$. This is almost 8.1 of Hager (1974), and is in 2.2 and 2.3 of Rice (1975b). (Each of these results uses Katětov–Shirota Theorem in its proof, of course.)

Rice (1975b) also has results related to 2.1(b) and (c).

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