

## THE ORDER-DUAL OF A TRL GROUP, I

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(Received 13 October 1976)

Communicated by E. Strzelecki

### Abstract

Conditions are found for several intrinsically defined partial orders on  $\mathfrak{Lb}$ , the vector space of order-bounded additive functionals on a commutative pogroup, to have Riesz interpolation properties, and to make  $\mathfrak{Lb}$  a TRL group.

Subject classification (Amer. Math. Soc. (MOS) 1970): primary 06 A 60, 46 A 40; secondary 54 F 05, 06 A 75.

### 1. Introduction

We begin a study of the vector lattice  $\mathfrak{Lb}$  of all order-bounded additive functionals on a commutative partially ordered group  $G$ , with particular attention to tight Riesz properties of  $\mathfrak{Lb}$ .  $G$  is assumed to be an  $l$ -group with respect to a partial order  $\leq$ , and to carry a compatible tight Riesz order and its open-interval topology. Thus besides the usual notion of positivity for a functional  $f \in \mathfrak{Lb}$  there are others, some of which (here written  $\leq$ ,  $\leq_o$ ,  $\leq_a$ ) we describe.

A fundamental theorem due to F. Riesz describes the vector-lattice structure of  $\mathfrak{Lb}$  under its principal partial order  $\leq$ . We show that two orders  $\leq$  and  $\leq_o$  are determining orders for this  $\leq$ . The main aim of the paper is to formulate conditions under which  $\leq$  on  $\mathfrak{Lb}$  is a compatible tight Riesz order for  $\leq$ . The interest in this question stems from the fact that, by Riesz's formula, the lattice operations on  $\mathfrak{Lb}$  with respect to  $\leq$  are not pointwise on  $G^+$ ; this is unlike the situation in most previously studied examples of compatible tight Riesz orders on  $l$ -groups. Two types of conditions are found; one based on compactness properties in  $G$  ( $9^\circ$  and  $5^\circ$ ), the other on properties of basic elements of  $G$  ( $10^\circ$ ). The latter are the more delicate.

We also find sufficient conditions for  $\leq$  to be non-secular ( $9^\circ$ ,  $11^\circ$ ,  $12^\circ$ ).

By examples it is shown that not all continuous additive functionals need be order-bounded ( $4^\circ$ ,  $5^\circ$ ).

Thanks are due to Robert Redfield who supplied the present form of Theorem  $10^\circ$  and the example in  $13^\circ$ , thus substantially improving an earlier version of this paper.

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Functionals  $f$  which are positive with respect to  $\leq$ , but not with respect to some compatible tight Riesz order for  $\leq$ , lie on the surface of the positive cone of  $\mathcal{Q}\mathfrak{b}$ , so a study of such orders gives information about surface structure. A subsequent paper will deal with these questions.

## 2. Preliminaries

**2.1** We summarize some definitions and results which are needed later. All order symbols  $<$ ,  $<_o$ ,  $\prec$ , ... in this paper should be read as excluding equality, with  $\leq$  meaning " $<$  or  $=$ ", and so on. The  $(m, n)$  *tight Riesz property* for a poset  $(X, \leq)$ , abbreviated  $\text{TR}(m, n)$ , asserts the following: For any set of elements  $a_i, b_j$  ( $i = 1, 2, \dots, m; j = 1, 2, \dots, n$ ) in  $X$  such that  $a_i < b_j$  for all  $i, j$ , there exists  $x \in X$  such that  $a_i < x < b_j$  for all  $i, j$ . We have

$$\text{TR}(2, 2) \Rightarrow \text{TR}(1, 2); \text{TR}(1, 2) \Rightarrow \text{TR}(1, 1); \text{TR}(1, 2) \Leftrightarrow \text{TR}(2, 1)$$

when  $X$  is a pogroup;  $\text{TR}(1, 2)$  does not imply  $\text{TR}(2, 2)$ . The loose Riesz property  $\text{LR}(m, n)$  is defined by replacing  $<$  by  $\leq$  at each occurrence. For any order  $\leq$  on  $X$ , its *associated preorder*  $\leq$  is defined thus:

$$x \leq y \text{ if and only if } (\forall u \in X) [u < x \Rightarrow u < y] \ \& \ (\forall t \in X) [t > y \Rightarrow t > x]. \quad (1)$$

When  $(X, \leq)$  is a pogroup, this is equivalent to saying:

$$z \geq 0 \text{ if and only if } a > 0 \Rightarrow a + z > 0;$$

that is, the positive wedge of  $\leq$  is got by adjoining to the positive cone of  $\leq$  all the pseudopositives of  $\leq$ . We consider only cases where  $\leq$  is a partial order. We call  $\leq$  a *determining order* for  $\leq$ . A partial order may have many determining orders.

A *tight Riesz group* (abbreviated *TR group*) is here defined to be a directed commutative† pogroup  $(G, \leq)$  with the  $\text{TR}(1, 2)$  property, and without pseudozeros, so that  $(G, \leq)$  is likewise a directed pogroup. We call  $\leq$  a compatible tight Riesz order (CTRO) for  $\leq$ . It is generally assumed that  $G \neq (0)$  and neither  $\leq$  nor  $\leq$  is trivial. The open-interval topology  $\mathcal{U}$  defined from  $\leq$  makes  $(G, \mathcal{U})$  a non-discrete Hausdorff topological group, non-compact though quite possibly locally compact. Thus a TR group has a structure  $(G, \leq, \leq, \mathcal{U})$ . By a *TR(2, 2) group* we mean a TR group for which  $\leq$  is  $\text{TR}(2, 2)$ . For elementary consequences of these various definitions see Loy and Miller (1972) or Cameron and Miller (1975). We write  $P = \{x \in G: x \geq 0\}$ ,  $P^* = P \setminus \{0\}$ ,  $G^+ = \{x: x \geq 0\}$  for the positive cones. Order-intervals are written  $(a, b) = \{x: a < x < b\}$ ,  $[a, b] = \{x: a \leq x \leq b\}$ , and similarly  $((a, b))$ ,  $[[a, b]]$  for  $\leq$ . The intervals  $(a, b)$ ,  $a < b$ , form a base for  $\mathcal{U}$ .

By a *TRL group* we mean a structure  $(G, \leq, \leq, \mathcal{U})$  in which  $(G, \leq)$  is a TR group and  $(G, \leq)$  is an  $l$ -group,  $\leq$  being of course the associated order of  $\leq$  and  $\mathcal{U}$

† We assume that all groups in this paper are commutative: "group" means "abelian group".

being the open-interval topology of  $\leq$ . (By an “ $l$ -group”, we mean a “lattice-ordered group” in the usual sense, as in Birkhoff (1967). The lattice operations of  $(G, \leq)$  are written  $\wedge, \vee$ .) For a TRL group  $G$ ,  $\leq$  is isolated and  $(G, \mathcal{U})$  has no compact subgroups other than  $(0)$ ; see Loy and Miller (1972). If  $G$  is a TR group for which  $\leq$  is LR (2, 2) (in particular, if  $G$  is a TRL group), then  $\leq$  is necessarily TR (2, 2). See Cameron and Miller (1975).

A TRL group  $G$  is called *secular* (or *androgynous*) if any of the following pairwise equivalent properties hold:

(i)  $G$  contains a pair of elements  $x, y$  satisfying

$$x > 0, \quad y > 0, \quad x \wedge y = 0.$$

(ii) The set  $\Upsilon = \{x \in G: x^+ > 0, x^- > 0\}$  is non-empty. (Here  $x^+ = x \vee 0$ ,  $x^- = -(x \wedge 0)$ .)

(iii)  $P^* \nsubseteq \mathfrak{w}$ . (Here  $\mathfrak{w} = \{w > 0: w \wedge x = 0 \Rightarrow x = 0\}$  is the set of weak units of  $(G, \leq)$ ; we may have  $\mathfrak{w} = \emptyset$ .)

There are other characterizations; see Miller (1976). The property expresses a certain relationship of  $\leq$  to its associated order, resulting in  $P^*$  occupying a greater portion of  $G^+$  than is sometimes desirable; it can lead to computational difficulties. Since  $\leq$  determines  $\leq$ , it is allowable to call  $\leq$  secular, rather than  $G$ . Secular groups are discussed in some detail in Miller (1976).

**2.2** For any pogroup  $(G, \leq)$ , its *order-dual* is the real vector space  $\mathfrak{Lb}(G)$  (briefly,  $\mathfrak{Lb}$ ) of all order-bounded additive functionals in  $G$ , that is, additive functions mapping order-intervals of  $G$  to bounded subsets of  $\mathbf{R}$ . When  $G$  is a TR group, it does not matter which of its two orders is used here: they produce the same set  $\mathfrak{Lb}$ . However, when it comes to ordering  $\mathfrak{Lb}$ , as usual by ordering functionals pointwise on the positive cone of  $G$ , several possibilities arise. For  $f \in \mathfrak{Lb}$  we shall write

$$f > 0 \quad \text{if and only if} \quad (\forall x \in G) [x > 0 \Rightarrow f(x) > 0], \quad (2)$$

$$f >_o 0 \quad \text{if and only if} \quad (\forall x \in G) [x > 0 \Rightarrow f(x) > 0], \quad (3)$$

$$f \geq 0 \quad \text{if and only if} \quad (\forall x \in G) [x > 0 \Rightarrow f(x) \geq 0], \quad (4)$$

$$f \geq_a 0 \quad \text{if and only if} \quad (\forall x \in G) [x > 0 \Rightarrow f(x) \geq 0]; \quad (5)$$

and  $f \leq g$  will mean  $g - f > 0$  or  $g = f$ , etc. These definitions make  $\mathfrak{Lb}$  a partially ordered vector space with respect to each of  $\leq, \leq_o, \leq, \leq_a$ .

When  $G$  is a TRL group it is natural to think of the  $l$ -group structure of  $(G, \leq)$  as the dominating one, since much is known about  $l$ -groups. If we accept this view then  $\leq$  in (4) is the natural order to place on  $\mathfrak{Lb}$ . Notice that the orders  $\leq$  and  $\leq$  in (2) and (4) are wholly determined by  $\leq$  on  $G$ , that is, are defined for any pogroup  $(G, \leq)$  whether or not  $\leq$  is an associated order. Our principal concern is with (2) and (4); nevertheless, the  $\leq$ -structure on  $G$  is relevant.

We write  $\mathfrak{L}^+$  for the positive cone of  $\leq$  in  $\mathfrak{Lb}$ . It is clear that

$$f > 0 \Rightarrow f >_0 0 \Rightarrow f \succ_a 0 \quad \text{and} \quad f > 0 \Rightarrow f > 0 \Rightarrow f \succ_a 0.$$

From results due to Hayes (1962) we know that  $\text{Hom}(G, \mathbf{R})$  contains non-zero elements, for any non-trivial group  $G$ . Bonsall (1954) has pointed out that if  $(G, \leq)$  is an everywhere non-archimedean pogroup, that is, if

$$G = \{x: \text{there exists } a \geq 0 \text{ such that } -a \leq nx \leq a \text{ for all } n \in \mathbf{N}\},$$

then  $\mathfrak{L}^+ = (0)$ . On the other hand, another result of Hayes (1962) shows that  $\mathfrak{L}^+$  contains non-zero elements if  $G$  has a strong unit. When  $G$  is a locally compact TR group with  $\leq$  isolated, Mackey's theorem shows that the continuous additive functionals on  $G$  are sufficiently numerous to separate points. (Compare Hewitt and Ross (1963), Theorems (24.34) and (24.35).) It is easy to construct examples in which  $\leq$  on  $\mathfrak{Lb}$  is non-trivial; see 6° below.

We note some preliminary results for the structure  $(G, \leq, \leq_a, \mathcal{U})$ .

1°. When  $G$  is a TR group, the orders  $\leq$  and  $\leq_a$  on  $\mathfrak{Lb}$  coincide, and  $f \geq 0$  implies that  $f$  is continuous. We have, for all  $f \in \mathfrak{Lb}$ ,

$$f > 0 \Rightarrow f >_0 0 \Rightarrow f > 0 \Leftrightarrow f \succ_a 0. \quad (6)$$

PROOF. Certainly  $f \geq 0$  implies  $f \geq_a 0$ . We prove that  $f \geq_a 0$  implies that  $f$  is continuous. Let  $f \geq_a 0$ . Suppose  $\ker(f)$  meets  $P^*$ , say  $f(a) = 0$  with  $a > 0$ . Then for any  $x \in G$ ,  $x + (-a, a)$  is a neighbourhood of  $x$  on which  $f$  is constant. Hence  $f$  is continuous. Suppose, on the other hand, that  $\ker(f)$  does not meet  $P^*$ , that is  $f >_0 0$ . Let  $(x_i)_{i \in I}$  be a net converging to  $x$  in  $G$ . Given any  $\varepsilon > 0$  in  $\mathbf{R}$  choose any  $a \in P^*$ , then  $n \in \mathbf{N}$  so that  $0 < f(a)/n < \varepsilon$ , then  $b \in G$  such that  $0 < b < nb < a$ , and take  $V = (x - b, x + b)$ . (The existence of such an element  $b$  is easily shown.) Eventually the net is in  $V$  and  $|f(x_i) - f(x)| < \varepsilon$ ; so again  $f$  is continuous.

Finally,  $f \geq_a 0$  implies  $f \geq 0$ . For if  $f \geq_a 0$  then when  $x \geq 0$  we can find a net  $(x_i)_{i \in I}$  in  $P^*$  converging to  $x$  and continuity of  $f$  gives  $f(x) \geq 0$ , so  $f \geq 0$ . The implication (6) is clear. //

2°. When  $(G, \leq)$  is a TR group:

- (i) If  $f > 0$  and  $f(c) \neq 0$  for some  $c > 0$ , then  $f$  does not vanish identically on  $(0, c)$ .
- (ii) If  $f > 0$  then  $\ker(f)$  is a closed convex subgroup of  $(G, \leq, \mathcal{U})$ , and  $f(b) > 0$  for some  $b > 0$ .
- (iii) For  $f \in \mathfrak{Lb}$ ,  $f >_0 0$  if and only if  $f > 0$  and  $\ker(f)$  is not open. If  $f >_0 0$  then  $\text{ran}(f)$  is dense in  $\mathbf{R}$ .

PROOF. (i) and (ii) are straightforward. (iii) Clearly  $f >_0 0$  if and only if  $f > 0$  and  $\ker(f)$  does not meet  $P^*$ . On the other hand,  $\ker(f)$  is open if and only if  $\ker(f)$  meets  $P^*$ . For if  $\ker(f)$  is open then it contains some interval  $(a, b)$ , and taking

$a < x < y < b$  we get  $y - x \in \ker(f) \cap P^*$ ; conversely, if  $\ker(f)$  meets  $P^*$ , say  $f(a) = 0$ ,  $a > 0$ , then by (ii)  $\ker(f)$  contains the open interval  $(a, 2a)$ , hence it has an interior point, hence it is open.

Suppose  $f >_0 0$ ; let  $0 < \varepsilon \in \mathbf{R}$ . As in the proof of 1° there exists  $b > 0$  with  $0 < f(b) < \varepsilon$ . (For this,  $f > 0$  does not suffice; we need  $f >_0 0$ .) So if  $0 < \alpha < \beta$  in  $\mathbf{R}$  write  $\varepsilon = \frac{1}{2}(\beta - \alpha)$  and find a corresponding  $b$ . Then  $(m-1)f(b) \leq \alpha < mf(b)$  for some  $m \in \mathbf{N}$ , and this implies  $\alpha < f(mb) < \beta$ . Thus  $f(P^*)$  is dense in  $\mathbf{R}^+$  and hence  $f(G)$  is dense in  $\mathbf{R}$ . //

3°. When  $(G, \leq)$  is a TR group, each of the orders  $\leq$  and  $\leq_0$ , if non-trivial, is a determining order on  $\mathfrak{Lb}$  for  $\leq$ , and their cones have bases. The cone  $\mathfrak{L}^+$  of  $\leq$  has a base if  $G$  has a strong unit.

The same conclusions follow for  $\leq$  and  $\leq$  on  $\mathfrak{Lb}$  if  $(G, \leq)$  is any l-group.

PROOF. Consider  $\leq_0$ . By 1°,  $f \leq_0 g$  implies  $f \leq g$ . Now if  $h > 0$  then

$$f >_0 0 \text{ implies } f + h >_0 0, \quad (7)$$

for when  $x \in P^*$  we have  $(f+h)(x) = f(x) + h(x) > 0$ ; (7) shows that  $h$  is positive in the associated order of  $\leq_0$ . Conversely, suppose  $0 \not\leq h$ , so that  $h(x) < 0$  for some  $x > 0$ . If  $f >_0 0$  then  $f(x) \geq 0$  and by multiplying  $f$  by a small positive real if necessary we can arrange that  $0 \leq f(x) < -h(x)$ , so  $0 \not\leq f+h$  and hence  $f+h \not>_0 0$ . Therefore (7) does not hold. This proves that  $\leq_0$  determines  $\leq$ . The proof for  $\leq$  is the same.

Concerning bases for the cones, by Peressini (1976), p. 26, it suffices to produce a strictly positive linear functional in each case, that is, a linear map  $\alpha: \mathfrak{Lb} \rightarrow \mathbf{R}$  such that  $f > 0 \Rightarrow \alpha(f) > 0$ , or  $f >_0 0 \Rightarrow \alpha(f) > 0$ , or  $f > 0 \Rightarrow \alpha(f) > 0$ , respectively. For the first two cases take any  $x \in P^*$  and define  $\alpha(f) = f(x)$ . In the third define  $\alpha(f) = f(s)$  where  $s$  is a strong unit of  $G$ .†

When  $(G, \leq)$  is any l-group (that is, no determining order for  $\leq$  is given) the statements about  $\leq$  and  $\leq$  on  $\mathfrak{Lb}$  still make sense and are proved in the same way. //

For any pogroup  $(G, \leq)$  (whether or not  $\leq$  is an associated order) there is F. Riesz's theorem (see, for example, Peressini (1967), §2.3):

If  $(G, \leq)$  is an LR(2, 2) directed pogroup then  $(\mathfrak{Lb}(G), \leq)$  is a complete vector lattice, the lattice operations being given (for  $a \in G^+$ ) by the formulae

$$(f \vee g)(a) = \sup \{f(x) + g(y) : x, y \in G^+, x + y = a\}, \quad (8)$$

$$(f \wedge g)(a) = \inf \{f(x) + g(y) : x, y \in G^+, x + y = a\}; \quad (9)$$

and

$$\mathfrak{Lb} = \mathfrak{L}^+ - \mathfrak{L}^+. \quad (10)$$

† The element  $s$  is a strong unit for  $(G, \leq)$  if and only if  $(\forall x \in G^+) (\exists n \in \mathbf{N}) (x \leq ns)$ . The order  $\leq$  and its associated order have the same set of strong units.

The conditions here are met when  $(G, \leq)$  is an  $l$ -group. The conclusions of the theorem are also deducible from the following modified hypothesis:  $(G, \leq)$  is a TR(2, 2) group. The proof is like that for the theorem itself, and uses also 1° above. Since  $\leq$  is TR(2, 2) it is also LR(2, 2), and the formulae (8) and (9) also hold in their modified forms

$$(f \vee g)(a) = \sup \{f(x) + g(y) : x, y \in P^*, x + y = a\} \quad (8')$$

and its dual, (9').

### 3. Examples

For a TR group  $G$  we can form the real vector space  $\mathfrak{Q} \equiv \mathfrak{Q}(G)$  of all continuous additive functionals on  $G$ . By 1° and (10) we have

$$\mathfrak{Qb} \subseteq \mathfrak{Q}.$$

The following two examples illustrate cases where  $\mathfrak{Qb} \neq \mathfrak{Q}$ . The first is due to R. H. Redfield.

4°. Let  $G = \mathbf{R} \circ \mathbf{R}$ , the lexicographic product of  $\mathbf{R}$  with itself, in which  $\langle x, y \rangle > 0$  if and only if  $x > 0$  or  $x = 0, y > 0$ . Here  $\leq$  is full and so coincides with its associated order  $\leq$ ;  $G$  is a TRL group. Let  $f$  be the map  $f\langle x, y \rangle = y$ . Then  $f \in \mathfrak{Q}$ . However, if  $A = \{\langle 0, y \rangle : y \in \mathbf{R}\}$  then  $A$  is bounded since  $\langle -1, 0 \rangle < A < \langle 1, 0 \rangle$ , but  $f(A) = \mathbf{R}$ . Thus  $f \notin \mathfrak{Qb}$ .

5°. Let  $G$  be the subgroup of  $C[0, 1]$  consisting of all continuous functions  $x$  for which the derivative  $x'(0)$  exists. Take  $\leq$  to be the weak pointwise order ( $x \geq 0$  if and only if  $x(t) \geq 0$  for all  $0 \leq t \leq 1$ ), and define  $x > 0$  to mean  $x(t) > 0$  for all  $0 < t \leq 1$ . Then  $(G, \leq)$  is an  $l$ -subgroup of  $(C[0, 1], \leq)$ , though not convex, and

$$(x \vee y)'(0) = \begin{cases} y'(0) & \text{(if } x(0) < y(0)), \\ \max \{x'(0), y'(0)\} & \text{(if } x(0) = y(0)). \end{cases}$$

Moreover,  $\leq$  is a TR(2, 2) determining order for  $\leq$ . With respect to its open-interval topology  $\mathcal{U}$ , convergence of a net  $(x_i)_{i \in I}$  to  $x$  implies that  $x_i$  converges uniformly to  $x$  on  $[0, 1]$  and  $x_i(0) = x(0)$  for  $i \geq$  some  $i_0$ .  $G$  is a nonsecular TRL group;  $\leq$  is archimedean,  $\leq$  is not eudoxian.

Let  $f$  be the map defined by  $f(x) = x'(0)$ ; we have  $f \in \mathfrak{Q} \setminus \mathfrak{Qb}$ .

Instead, let  $G$  be as above, except that  $\leq$  is now defined thus:  $x > 0$  means

$$x(t) > 0 \text{ if } 0 < t < \tfrac{1}{2}, \quad x(t) \geq 0 \text{ if } \tfrac{1}{2} \leq t \leq 1 \text{ or } t = 0.$$

We still have  $f \in \mathfrak{Q} \setminus \mathfrak{Qb}$ , but  $\leq$  is now a secular order for the TRL group  $G$ .

6°. Let  $c$  denote the sequence space of all real sequences  $\alpha = (\alpha_n)_{n \in \mathbb{N}}$  for which the limit

$$\lambda(\alpha) = \lim_{n \rightarrow \infty} \alpha_n$$

exists;  $c$  is a vector lattice, *a fortiori* a commutative  $l$ -group, with respect to the weak pointwise order  $\leq$  on sequences. For any real sequence  $\zeta = (\zeta_n)_{n \in \mathbb{N}}$  write

$$f(\alpha) = \sum_{n=1}^{\infty} \zeta_n \alpha_n. \quad (11)$$

This equation defines an element  $f \in \mathfrak{L}^+(c)$  if and only if  $\zeta \in l^1$  and  $\zeta_n \geq 0$  for all  $n$ . The general element of  $\mathfrak{L}^+$  has the form  $f + \rho\lambda$ , where  $\rho \in \mathbb{R}^+$ , so that  $\mathfrak{L}\mathfrak{b}$  can be identified with  $l^1 \oplus (\lambda)$ .

For  $f$  in (11) we have  $f > 0$  if and only if  $\zeta_n > 0$  for all  $n$ ; on the other hand,  $\lambda > 0$ ,  $\lambda \neq 0$ . The only lattice homomorphisms in  $\mathfrak{L}^+$  are  $\lambda$  and those  $f$  for which  $\text{supp}(\zeta)$  is a singleton.

Let a filter  $\mathcal{F}$  of subsets of  $\mathbb{N}$  be given; define  $\leq$  on  $c$  by

$$\alpha > 0 \quad \text{if and only if} \quad \alpha \geq 0 \text{ and } \text{supp}(\alpha) = \{n \in \mathbb{N} : \alpha_n > 0\} \in \mathcal{F}. \quad (12)$$

Then  $\leq$  is a compatible TR(2, 2) order for  $(c, \leq)$ . For  $f$  in (11) we have

$$f >_o 0 \quad \text{if and only if} \quad \text{supp}(\zeta) \text{ meets every set in } \mathcal{F};$$

we have  $\lambda \neq_o 0$ .

Every sequence  $\alpha$  in  $c^+$  for which  $\inf \alpha_n > 0$  is a strong unit of  $c$ ;  $\mathfrak{L}\mathfrak{b}$  has no strong units.

#### 4. Tight interpolation for $\leq$ and $\leq_o$

4.1 The orders  $\leq$  and  $\leq_o$  on  $\mathfrak{L}\mathfrak{b}$  are TR(1, 1), that is, order-dense, since for example if  $f < g$  then  $f < \frac{1}{2}(f+g) < g$ . If  $\leq$  is non-trivial and TR(1, 2) then since its associated order is LR(2, 2),  $\leq$  is a compatible TR(2, 2) order for  $\leq$ . The same remark applies to  $\leq_o$ . Let  $\mathcal{F}$  denote the open-interval topology of  $\leq$  on  $\mathfrak{L}\mathfrak{b}$ . We have, in view of previous remarks:

7°. If  $(G, \leq, \leq, \mathcal{U})$  is a TR(2, 2) group (or if  $(G, \leq)$  is any  $l$ -group) and if  $\leq$  on  $\mathfrak{L}\mathfrak{b}$  is non-trivial and TR(1, 2), then

$$(\mathfrak{L}\mathfrak{b}(G), \leq, \leq, \mathcal{F})$$

is a TRL group.

We ask if either order is TR(1, 2). First, we note that  $\leq_o$  need not be TR(1, 2). This failure is simply illustrated by the following example.

8°. Take  $G = \mathbb{R}^2$  with the strong and weak pointwise orders  $\leq$ ,  $\leq$ , and functionals

$$f\langle x_1, x_2 \rangle = x_1, \quad g\langle x_1, x_2 \rangle = x_2.$$

We have  $f, g \in \mathfrak{L}^+$ , in fact  $f, g >_0 0$ , and for any  $a = \langle a_1, a_2 \rangle > 0$  in  $\mathbb{R}^2$ ,

$$(f \wedge g)(a) = \inf \{f(x) + g(y) : 0 \leq x, y; x + y = a\} = 0,$$

the infimum being attained by taking  $x = \langle 0, a_2 \rangle$ ,  $y = \langle a_1, 0 \rangle$ . Thus  $f \wedge g = 0$ , and so  $0 <_0 h <_0 f, g$  is possible for no  $h \in \mathfrak{Lb}$ .

This example also shows that  $\wedge$  for  $\mathfrak{Lb}$  need not be pointwise on  $G^+$ , since  $f(a) \wedge g(a) = \min\{a_1, a_2\} > 0$ . If for any TRL group  $G$  it is the case that  $\wedge$  (and so  $\vee$ ) is pointwise on  $G^+$  then  $\leq$  and  $\leq_0$  are TR(2, 2) on  $\mathfrak{Lb}$ . For with  $0 < f, g$  in  $\mathfrak{Lb}$  and  $a > 0$  we should have  $0 < f(a) \wedge g(a) = (f \wedge g)(a)$ , so  $f \wedge g > 0$ , whence

$$0 < \frac{1}{2}(f \wedge g) < f, g$$

so  $\leq$  is TR(1, 2), hence TR(2, 2). However, the proviso is rather special, as we have seen. The counterexample shows that  $\leq_0$  is not really the appropriate order to expect to be TR(2, 2). For  $\leq$  the property is more apt, but is still a delicate matter. The remainder of this section deals with the question for  $\leq$  on  $\mathfrak{Lb}$ . We describe two cases where  $\leq$ , if non-trivial, can be shown to be TR(2, 2): when  $(G, \mathcal{U})$  is interval-compact, and when  $(G, \leq)$  has a basis.

**4.2** A TR(2, 2) group is called *interval-compact* if  $\llbracket a, b \rrbracket$  is compact for every  $a \leq b$ . We have

$$(a, b)^- = ((a, b))^- = \llbracket a, b \rrbracket \text{ whenever } a < b$$

(where  $-$  denotes closure); and equivalent formulations of the property are:  $(a, b)^-$  is compact for every  $a < b$ ;  $(0, a)^-$  is compact for every  $a > 0$ ;  $\llbracket 0, a \rrbracket$  is compact for every  $a \geq 0$ .

If  $(G, \leq, \leq, \mathcal{U})$  is an interval-compact TR(2, 2) group then  $(G, \leq)$  is a lattice-complete  $l$ -group, and  $(G, \mathcal{U})$  is locally compact. For these and related results see Loy and Miller (1972).

**9° THEOREM.** *If  $G$  is an interval-compact TR(2, 2) group then  $\leq$  on  $\mathfrak{Lb}(G)$  if non-trivial is TR(2, 2), and  $(\mathfrak{Lb}, \leq, \leq, \mathcal{T})$  is a non-secular TRL group.*

**PROOF.** Let  $f, g \in \mathfrak{L}^+$ , and  $a > 0$  in  $G$ . From (9) we have

$$(f \wedge g)(a) = g(a) - \sup \{g(x) - f(x) : 0 \leq x \leq a\} \geq 0. \quad (13)$$

There exists a net  $(x_i)_{i \in I}$  in  $\llbracket 0, a \rrbracket$  such that

$$\lim (g(x_i) - f(x_i)) = \sup \{g(x) - f(x) : 0 \leq x \leq a\}. \quad (14)$$

Suppose that  $(f \wedge g)(a) = 0$ . Then  $\lim (g(x_i) - f(x_i)) = g(a)$ . Since

$$g(x) - f(x) \leq g(x) \leq g(a)$$

for  $x \in \llbracket 0, a \rrbracket$ , it follows that

$$\lim g(x_i) = g(a), \quad \lim f(x_i) = 0. \quad (15)$$



By assumption,  $\llbracket 0, a \rrbracket$  is compact, so replacing  $(x_i)_{i \in I}$  by a subset if necessary we can assume that  $\lim x_i$  exists,  $= x_0 \in \llbracket 0, a \rrbracket$ . Since  $f$  and  $g$  are continuous by  $1^\circ$ ,  $g(a) = \lim g(x_i) = g(x_0)$  and  $0 = \lim f(x_i) = f(x_0)$ .

Now suppose that  $f > 0$  and  $g > 0$ . Since  $x_0 < a$  would imply  $g(x_0) < g(a)$  we have  $x_0 = a$  and hence  $0 < f(a) = f(x_0) = 0$ , contradiction. We have thus shown that  $f, g > 0$  implies  $(f \wedge g)(a) > 0$  for all  $a > 0$ , that is,  $f \wedge g > 0$ . Therefore  $\leq$  is TR(1, 2), and so TR(2, 2), and  $7^\circ$  shows that  $\mathfrak{Lb}$  is a TRL group.

Suppose instead that  $f > 0$  and  $g > 0$ . The above considerations in this case show that  $f(a) = 0$  whenever  $(f \wedge g)(a) = 0$  and  $a > 0$ . Thus  $f > 0$  and  $g > 0$  imply  $f \wedge g > 0$ , which means (by 2.1) that  $\mathfrak{Lb}$  is non-secular. //

REMARK: The trivial order is always TR(2, 2); on the other hand, for  $\leq$  to determine  $\leq$  on  $\mathfrak{Lb}$ ,  $\leq$  must be non-trivial.

A less direct proof of  $9^\circ$  is possible using  $10^\circ$  and  $11^\circ$  below and a result due to Wirth (1975) characterizing interval-compact tight Riesz groups.

**4.3** For any  $l$ -group  $(G, \leq)$ , a *basic element* is by definition an element  $a \in G^+ \setminus \{0\}$  such that  $\llbracket 0, a \rrbracket$  is a fully-ordered subset of  $G^+$ . Alternative characterizations are: (i) The carrier  $\tilde{a}$  determined by  $a$  is an atom of the carrier lattice  $\mathfrak{C}$  of  $G$ ; (ii) If  $0 \leq s, t \leq a$  and  $s \wedge t = 0$  then  $s = 0$  or  $t = 0$ ; (iii)  $a^{\perp\perp}$  is fully-ordered; (iv)  $a^{\perp\perp}$  is an atom of the lattice  $\text{Pol}(G)$  of all polars of  $G$ . (For any subset  $A \subseteq G$ , the *polar* of  $A$  is  $A^\perp = \{x \in G : |x| \wedge |a| = 0 \text{ for all } a \in A\}$ , and  $c^\perp = \{c\}^\perp$ . The polars form a complete Boolean algebra  $\text{Pol}(G)$  with respect to inclusion and  $^\perp$  as complementation. For  $c \in G^+$  the *carrier* determined by  $c$  is  $\tilde{c} = \{x \in G^+ : x^\perp = c^\perp\}$ . The carriers form a distributive disjunctive lattice  $(\mathfrak{C}, \leq)$  when ordered by writing  $\tilde{a} \leq \tilde{b}$  if and only if  $a^{\perp\perp} \subseteq b^{\perp\perp}$ .)

A *basis* of the  $l$ -group  $(G, \leq)$  is any subset of  $G^+ \setminus \{0\}$  which is maximal with respect to the property: each element of the subset is basic, and the elements are pairwise disjoint.  $G$  has a basis if and only if  $\mathfrak{C}$  is atomic, that is, every element  $\tilde{x} \in \mathfrak{C}$  dominates some atom (it is then the join of the atoms it dominates); equivalently, every  $x > 0$  dominates some basic element.

The following result is due to R. H. Redfield; it subsumes a number of special cases proved earlier by the author using more complicated arguments.

**$10^\circ$ . THEOREM.** *Let  $(G, \leq)$  be any  $l$ -group with a basis. Then  $\leq$  on  $\mathfrak{Lb}$ , if non-trivial, is TR(2, 2), and  $\mathfrak{Lb}$  is a TRL group.*

**PROOF.** Let  $f, g > 0$  in  $\mathfrak{Lb}$  and suppose  $u > 0$  in  $G$ . Then  $f \wedge g \geq 0$  and we wish to show that  $(f \wedge g)(u) > 0$ . By assumption there exists some basic element  $a \leq u$ . Since  $(2a)^\sim = \tilde{a}$ ,  $2a$  is also basic. Take any  $x, y \in G^+$  with  $x + y = 2a$ ; since  $\llbracket 0, 2a \rrbracket$  is fully-ordered, either  $a \leq x$  or  $a \leq y$  so either  $f(x) + g(y) \geq f(x) \geq f(a)$  or

$f(x) + g(y) \geq g(y) \geq g(a)$  and therefore by (9),

$$(f \wedge g)(2a) \geq \min\{f(a), g(a)\} > 0$$

since  $f > 0$  and  $g > 0$ . Thus in either case

$$2(f \wedge g)(u) \geq (f \wedge g)(2a) > 0.$$

This proves that  $f \wedge g > 0$  and hence that  $\leq$  is TR (2, 2). Again 7° shows that  $\mathfrak{Lb}$  is a TRL group. //

The question of whether  $\leq$  is non-secular is not so immediately settled in this case as it is in 9°. We have the following sufficient condition.

11°. *Let  $(G, \leq)$  be any  $l$ -group with a basis. Suppose that  $\leq$  on  $\mathfrak{Lb}$  is non-trivial, and that for every  $f > 0$  in  $\mathfrak{Lb}$  there exists a basic element  $a$  such that  $f(a) > 0$ . Then  $\mathfrak{Lb}$  is non-secular.*

PROOF. We have to show that in  $\mathfrak{Lb}$ ,  $f > 0$  and  $g > 0$  imply  $f \wedge g > 0$ . Suppose  $a$  is basic and  $f(a) > 0$ . Necessarily  $g(a) > 0$ , and consequently the same argument as in the proof of 10° leads to  $(f \wedge g)(a) > 0$ . This proves  $f \wedge g > 0$ . //

Call an  $l$ -group  $(G, \leq)$  *Jaffard projectable* if  $G$  has a basis, and

$$G = a^{\perp\perp} \oplus a^{\perp}$$

for every basic element. Call  $(G, \leq)$  *finitely based* if  $G$  has a basis, and for every non-zero  $x \in G^+$  there exists no infinite subset of  $\{y: 0 < y \leq x\}$  the elements of which are pairwise disjoint (equivalently: the carrier lattice  $\mathfrak{C}$  of  $G$  is atomic and each non-zero carrier  $\tilde{x}$  dominates only finitely many atoms). P. Jaffard (1953) showed that an  $l$ -group is expressible as a direct sum

$$\sum_{i \in I} \oplus H_i$$

of fully-ordered convex subgroups if and only if it is Jaffard projectable and finitely based. In this case the  $H_i$ 's are precisely the principal bipolars  $a_i^{\perp\perp}$ , where  $\tilde{a}_i$  runs through the atoms of  $\mathfrak{C}$ . Here  $I$  need not be finite. From 11° we deduce:

12°. *Let  $(G, \leq)$  be a Jaffard projectable and finitely based  $l$ -group. Then  $\leq$ , if non-trivial, makes  $\mathfrak{Lb}$  a non-secular TRL group.*

PROOF. If  $f$  vanishes on every basic element then by the representation  $G = \sum_{i \in I} \oplus a_i^{\perp\perp}$ ,  $f$  vanishes on  $G$ . Thus  $f > 0$  implies  $f(a) > 0$  for some basic element  $a$ , and 11° gives the result. //

It is reasonable to conjecture that the condition that  $\mathfrak{C}$  is finitely based can be dropped from 15°. When  $(G, \leq)$  is Jaffard projectable its basic subgroup  $B$  (the

subgroup generated by the set of all the basic elements) has  $G$  for its lattice-closure (this result is due to R. H. Redfield), so if  $g$  vanishes on  $B$  it might be supposed that it should vanish on  $G$ . However, if a continuous functional vanishes on a subset  $X$  of a TRL group it need not vanish on  $\sup(X)$ . In fact the conjecture is false, as the following counterexample (also due to Redfield) shows.

13°. EXAMPLE of a TRL group  $G$  for which  $(G, \leq)$  is order-dense, Jaffard projectable and archimedean,  $G$  is non-secular, but  $\leq$  on  $\mathfrak{Lb}$  is secular. Let  $G$  be the subset of  $\prod_{k=1}^{\infty} \mathbf{R}$  consisting of functions  $x$  of the form

$$x = s + p,$$

where  $s \in \sum_{k=1}^{\infty} \mathbf{R}$  and  $p = (p_k)_{k \in \mathbf{N}}$ ,  $p_k = 2^{-k} \alpha_x$  for some  $\alpha_x \in \mathbf{Q}$ . Clearly  $x$  determines  $s$  and  $p$  uniquely, and  $G$  is an  $l$ -subgroup of the cardinal product  $(\prod \mathbf{R}, \leq)$ . Let  $\leq$  be the strong pointwise order on  $G$ . Then  $(G, \leq, \leq)$  is a TRL group with the asserted properties. We produce two functionals  $f, g \in \mathfrak{Lb}$  such that  $f > 0$ ,  $g > 0$  and  $f \wedge g = 0$ , namely

$$f(x) = \alpha_x, \quad g(x) = \sum_{k=1}^{\infty} x_k.$$

To prove  $f \wedge g = 0$ , let  $a \in G^+$ . If  $a \in \sum \mathbf{R}$  then  $f(a) = 0$  so  $f \wedge g(a) = 0$ . Suppose  $a \notin \sum \mathbf{R}$ , and let  $\varepsilon > 0$  in  $\mathbf{R}$ ; pick  $n$  such that  $\sum_{k > n} 2^{-k} \alpha_a < \frac{1}{2} \varepsilon$ , define  $b \in \prod \mathbf{R}$  by

$$b_k = \varepsilon / 2^n \quad (1 \leq k \leq n), \quad \alpha_a / 2^k \quad (k > n),$$

and let  $c = a \vee b - b$ . Then  $0 \leq c \in \sum \mathbf{R}$ , so  $f(c) = 0$ ,  $g(b) < \varepsilon$  and therefore

$$0 \leq (f \wedge g)(a) \leq (f \wedge g)(a \vee b) \leq f(c) + g(b) < \varepsilon.$$

Thus  $f \wedge g = 0$ .

4.4 The question of whether  $\leq$  on  $\mathfrak{Lb}$  is TR(1, 2), for  $l$ -groups  $G$  not covered in 9° and 10°, can be formulated in terms of certain sets of the form

$$A_{s,t}(a) = \{x: 0 \leq x \leq a, \text{ but neither } s \leq x \text{ nor } x \leq t\}.$$

Here  $a$  is some element in  $G^+ \setminus \{0\}$ , and  $s, t \in ((0, a))$ . It is found most useful to choose  $s, t$  so that  $0 < s < t < a$ . Let  $f > 0$  and  $g > 0$  in  $\mathfrak{Lb}$ , and assume that  $(f \wedge g)(a) = 0$ , and consider the sequence  $(x_i)_{i \in I}$  in the proof of 9°, with the properties (14) and (15). If for some  $s$  we have  $0 < s \leq x_i$  for all  $i$  in some cofinal subset  $I_0$  of  $I$ , then  $0 < f(s) \leq f(x_i)$  for all  $i \in I_0$ , contradicting (15). (Note that we need  $f > 0$  here, not merely  $f > 0$ .) Similarly, if  $x_i \leq t < a$  for some  $t$  and all  $i$  in some cofinal subset we get  $g(x_i) \leq g(t) < g(a)$ , contradicting (15). Therefore, for all  $s, t \in ((0, a))$ ,  $x_i$  is eventually in  $A_{s,t}(a)$ .

If  $A_{s,t}(a) \neq \emptyset$  for all such  $s, t$ , this means roughly speaking that the net  $(x_i)_{i \in I}$  migrates towards the boundary of  $\llbracket 0, a \rrbracket$  and away from 0 and  $a$ : this can be

illustrated by considering the group  $G = \mathbb{R}^2$ , taking  $a > 0$ . If, on the contrary, it can be shown that  $A_{s,t}(a) = \emptyset$  for some pair  $s, t \in ((0, a))$  then we have a contradiction implying  $(f \wedge g)(a) > 0$ . Thus

14°. Let  $(G, \leq)$  be an  $l$ -group. For  $\leq$  on  $\mathfrak{Lb}$  to be  $TR(1, 2)$  it is sufficient that  $G$  satisfy the following condition:

[\*] For every  $a > 0$  in  $G$  there exists a pair of elements  $s, t \in ((0, a))$  such that  $A_{s,t}(a) = \emptyset$ .

The use of basic elements in 10° reduces the discussion to the case where  $[0, a]$  is fully-ordered; here [\*] is satisfied trivially by any  $s, t$  such that  $0 < s < t < a$ . Since the sets  $A_{s,t}(a)$  for fixed  $a$  do not form a filterbase, [\*] does not seem to be a necessary condition.

### 5. Another CTRO for $(\mathfrak{Lb}, \leq)$

There is another result establishing a CTRO for  $(\mathfrak{Lb}, \leq)$ , suggested by the compactness argument in 9°. It concerns not  $\leq$  but yet another partial order on  $\mathfrak{Lb}$ , which we write  $\leq_1$ . This time we assume that  $(G, \leq, \leq, \mathcal{U})$  is a locally compact TR group. In this case the set

$$D_1 = \{x \geq 0: [0, x] \text{ is compact}\}$$

is non-empty, and generates a subgroup  $G_1 = D_1 - D_1$ , for which  $G_1 \cap G^+ = D_1$ . For  $f \in \mathfrak{Lb}(G)$  write

$$f >_1 0 \text{ if and only if } f \geq 0, \text{ and } f(x) > 0 \text{ for every } x \in D_1 \setminus \{0\}. \quad (16)$$

This makes  $(\mathfrak{Lb}(G), \leq_1)$  a partially ordered vector space, and by almost the same arguments as were used in proving 3° and 9° we find that  $\leq_1$ , if non-trivial, is a  $TR(2, 2)$  determining order for  $\leq$  on  $\mathfrak{Lb}(G)$ , and  $(\mathfrak{Lb}, \leq_1, \leq)$  is a TRL group.

When  $(G, \mathcal{U})$  is locally compact, 10° is a particular case of this result.

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