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LIMITS OF HYPERCYCLIC AND SUPERCYCLIC OPERATOR MATRICES

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Abstract

An operator A on a complex, separable, infinite-dimensional Hilbert space H is hypercyclic if there is a vector $x \in H$ such that the orbit $\{x, Ax, A^2x, \ldots\}$ is dense in H. Using the character of the analytic core and quasinilpotent part of an operator A, we explore the hypercyclicity for upper triangular operator matrix

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

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1. Introduction

Throughout this paper, let H and K be infinite-dimensional separable Hilbert spaces, let $B(H, K)$ denote the set of bounded linear operators from H to K, and abbreviate $B(H, H)$ to $B(H)$. For an operator $A \in B(H)$, write A^*, $\sigma(A)$, $\rho(A)$, $\sigma_a(A)$, $\operatorname{iso}\sigma(A)$ for the adjoint, spectrum, resolvent set, approximate point spectrum, and isolated points of the spectrum $\sigma(A)$, respectively. By $n(A)$ and $d(A)$ we denote the dimension of the kernel $N(A)$ and the codimension of the range $R(A)$. If both $n(A)$ and $d(A)$ are finite, then A is called a Fredholm operator and the index of A is defined by $\operatorname{ind}(A) = n(A) - d(A)$. $A \in B(H)$ is said to be a Weyl operator if it is Fredholm of index 0. Recall that the ascent $\operatorname{asc}(A)$ of an operator A is the smallest nonnegative integer p such that $N(A^p) = N(A^{p+1})$. If such an integer does not exist we put $\operatorname{asc}(A) = \infty$. Analogously, the descent $\operatorname{des}(A)$ of A is the smallest nonnegative q such that $R(A^q) = R(A^{q+1})$ and if such an integer does not exist we put $\operatorname{des}(A) = \infty$. It is well known that if $\operatorname{asc}(A)$ and $\operatorname{des}(A)$ are finite then

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$\mathrm{asc}(A) = \mathrm{des}(A)$. If A is Fredholm with $\mathrm{asc}(A) = \mathrm{des}(A) < \infty$, we call A a Browder operator. Note that if A is Browder then A is Weyl. The Weyl spectrum $\sigma_w(A)$ and the Browder spectrum $\sigma_b(A)$ of A are defined by $\sigma_w(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Weyl}\}$ and $\sigma_b(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Browder}\}$.

For $x \in H$, the orbit of x under A is the set of images of x under successive iterates of A:

$$\mathrm{Orb}(A, x) = \{x, Ax, A^2x, \ldots\}.$$

A vector $x \in H$ is supercyclic if the set of scalar multiples of $\mathrm{Orb}(A, x)$ is dense in H, and x is hypercyclic if $\mathrm{Orb}(A, x)$ is dense. A hypercyclic operator is one that has a hypercyclic vector. We define the notion of supercyclic operator similarly. We denote by $HC(H)$ ($SC(H)$) the set of all hypercyclic (supercyclic) operators in $B(H)$ and by $\overline{HC(H)}$ ($\overline{SC(H)}$) the norm-closure of the class $HC(H)$ ($SC(H)$). Supercyclic operators were introduced by Hilden and Wallen in 1974 [13]. Many fundamental results regarding the theory of hypercyclic and supercyclic operators were established by Kitai in her thesis [14].

Hypercyclicity or supercyclicity has been studied by many authors ([2, 3, 12], and so on). In this paper, using the character of the analytic core and quasinilpotent part of an operator A, we explore the hypercyclicity or supercyclicity for operator A and for upper triangular operator matrix

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

2. Main results

For an operator $A \in B(H)$, the analytic core of A is the subspace

$$\begin{aligned} K(A) = \{x \in H : Ax_{n+1} = x_n,\ Ax_1 = x,\ \|x_n\| \\ \le c^n \|x\| (n = 1, 2, \ldots) \text{ for some } c > 0,\ x_n \in H\}, \end{aligned}$$

and the quasinilpotent part of A is the subspace

$$H_0(A) = \Big\{x \in H : \lim_{n\to\infty} \|A^n x\|^{(1/n)} = 0\Big\}.$$

The spaces $K(A)$ and $H_0(A)$ are hyperinvariant under A and satisfy $N(A^n) \subseteq H_0(A)$, $K(A) \subseteq R(A^n)$ for all $n \in \mathbb{N}$ and $AK(A) = K(A)$; see [1, 15, 16] for more information about these subspaces.

We say that A has the single-valued extension property (SVEP) at λ_0 if, for every open neighborhood U of λ_0, the only analytic function $f : U \to H$ which satisfies the equation $(A - \lambda I) f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. We say that A has the SVEP if A has the SVEP at every $\lambda \in \mathbb{C}$.

Next, we shall consider the hypercyclicity or supercyclicity for the class of operators $A \in B(H)$ and the operator matrices

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

for which the condition $\dim K(A^*) < \infty$ holds. In what follows, we suppose that A is not quasinilpotent and let $H(A)$ be the set of all complex-valued functions that are analytic in a neighborhood of the spectrum $\sigma(A)$ of A. For $f \in H(A)$, the operator $f(A)$ is defined by the well-known analytic calculus. We start with a lemma.

LEMMA 2.1. *Suppose that $K(A^*) = \{0\}$. If $f \in H(A)$ is not constant, then:*

(1) $\sigma(A) = \sigma_w(A)$ *is connected;*

(2) $\mathrm{ind}(f(A) - \lambda I) \geq 0$ *for each* $\lambda \in \rho_{SF}(f(A))$, *where* $\rho_{SF}(f(A)) = \{\lambda \in \mathbb{C}, f(A) - \lambda I$ *is semi-Fredholm*$\}$;

(3) $\sigma_w(f(A)) = f(\sigma_w(A)) = \sigma(f(A))$ *is connected.*

PROOF. (1) We only need to prove that $\sigma(A) \subseteq \sigma_w(A)$. Let $\lambda_0 \in [\sigma(A) \backslash \sigma_w(A)]$. There are two cases to consider.

Case 1. Let $\lambda_0 \neq 0$. Since $A^* - \lambda_0 I$ is Weyl and $\{0\} \neq N(A^* - \lambda_0 I) \subseteq K(A^*)$, it follows that $K(A^*) \neq \{0\}$, which is a contradiction.

Case 2. Let $\lambda_0 = 0$. Since $A - \lambda_0 I = A$ is Weyl, using the semi-Fredholm perturbation theory, $A^* - \lambda I$ is Weyl if $0 < |\lambda|$ is sufficiently small. But since

$$N(A^* - \lambda I) \subseteq K(A^*) = \{0\},$$

it follows that $A^* - \lambda I$ is invertible. Then $0 \in \mathrm{iso}\, \sigma(A^*)$. By [15, Theorem], $H = H_0(A^*) \oplus K(A^*) = H_0(A^*)$, which means that A^* is quasinilpotent. Thus A is quasinilpotent, contradicting the assumption that A is not quasinilpotent.

From the foregoing, we know that $\sigma(A) = \sigma_w(A)$. Suppose that $\sigma(A)$ is not connected. Then $\sigma(A^*)$ is not connected. Let $\sigma(A^*) = \sigma \cup \tau$, where σ, τ are closed, $\sigma, \tau \neq \emptyset$ and $\sigma \cap \tau = \emptyset$. Define $f \in H(A^*)$ such that $f \equiv 1$ on σ and $f \equiv 0$ on τ. Put $P = f(A^*)$. Then $P^2 = P$, $R(P)$ and $N(P)$ are closed, A^*-invariant subspaces and $\sigma(A^*|_{R(P)}) = \sigma$ and $\sigma(A^*|_{N(P)}) = \tau$. Since $K(A^*) = \{0\}$, it follows that $A^* F \neq F$ for each closed A^*-invariant subspace $F \neq \{0\}$ [17, Proposition 2]. Then $0 \in \sigma \cap \tau$, which is a contradiction, since $\sigma \cap \tau = \emptyset$. Thus $\sigma(A) = \sigma_w(A)$ is connected.

(2) Since $N(A^* - \lambda I) = \{0\}$ for all $\lambda \neq 0$, A has the SVEP. By [6, Theorem 1.5], $f(A^*) = f(A)^*$ has the SVEP. Therefore, $\mathrm{ind}(f(A) - \lambda I) \geq 0$ for each $\lambda \in \rho_{SF}(f(A))$ by [9, Corollary 12].

(3) Applying (2) and [18, Theorem 3.6], we know that

$$\sigma_w(f(A)) = f(\sigma_w(A)) = f(\sigma(A)) = \sigma(f(A))$$

is connected. $\square$

If $K(A^*) = \{0\}$, then for any $f \in H(A)$,

$$\sigma(f(A)) = f(\sigma(A)) = f(\sigma_w(A)) = \sigma_w(f(A)) = \sigma_b(f(A))$$

is connected. In this case, if $|f(\lambda)| = 1$ for some $\lambda \in \sigma(A)$, then $f(\lambda) \in \sigma_w(f(A)) \cap \partial\mathbb{D}$. Since $\sigma_w(f(A))$ and $\partial\mathbb{D}$ are connected, $\sigma_w(f(A)) \cup \partial\mathbb{D}$ is connected. If $\overline{H_0(A)} = H$, by $K(A^*) \subseteq H_0(A)^{\perp}$ [15], then $K(A^*) = \{0\}$. Using [12, Theorems 2.1 and 3.3], we have the following result.

THEOREM 2.2. *Suppose that* $K(A^*) = \{0\}$ *or* $\overline{H_0(A)} = H$. *Then:*

(1) $A \in \overline{HC(H)}$ *if and only if there exists* $\lambda \in \sigma(A)$ *such that* $|\lambda| = 1$;

(2) $A \in \overline{SC(H)}$;

(3) *for any* $f \in H(A)$, $f(A) \in \overline{HC(H)}$ *if and only if there exists* $\lambda \in \sigma(A)$ *such that* $|f(\lambda)| = 1$;

(4) $f(A) \in \overline{SC(H)}$ *for any* $f \in H(A)$.

COROLLARY 2.3. *Suppose that* $K(A) = \{0\}$ *and* $K(A^*) = \{0\}$. *Then:*

(1) $A \in \overline{HC(H)}$ *if and only if* $A^* \in \overline{HC(H)}$, *if and only if there exists* $\lambda \in \sigma(A)$ *such that* $|\lambda| = 1$;

(2) $A \in \overline{SC(H)}$ *and* $A^* \in \overline{SC(H)}$;

(3) *for any* $f \in H(A)$, $f(A) \in \overline{HC(H)}$ *if and only if* $f(A^*) \in \overline{HC(H)}$, *if and only if there exists* $\lambda \in \sigma(A)$ *such that* $|f(\lambda)| = 1$;

(4) $f(A) \in \overline{SC(H)}$ *and* $f(A^*) \in \overline{HC(H)}$ *for any* $f \in H(A)$.

The hypercyclicity (or supercyclicity) for operator matrices has been studied in [2]. In the following results, we continue this work.

THEOREM 2.4. *Suppose that* $\dim K(A^*) < \infty$. *Then the following statements are equivalent:*

(1) $M_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \overline{HC(H \oplus K)}$;

(2) $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \overline{HC(H \oplus K)}$ *for each* $C \in B(K, H)$;

(3) $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \overline{HC(H \oplus K)}$ *for some* $C \in B(K, H)$.

PROOF. We only prove the equivalence between (2) and (3), and so we only need to prove that (3) implies (2). Suppose that $M_{C_0} \in \overline{HC(H \oplus K)}$. Using [12, Theorem 2.1], we will prove that:

(a) $\sigma_w(M_C) \cup \partial\mathrm{D}$ is connected for each $C \in B(K, H)$.

We claim that $\sigma_w(M_C) = \sigma_w(M_{C_0})$. If fact, let $M_C - \lambda_0 I$ be Weyl. Then $A - \lambda_0 I$ is upper semi-Fredholm, $B - \lambda_0 I$ is lower semi-Fredholm and $d(A - \lambda_0 I) < \infty$ if and only if $n(B - \lambda_0 I) < \infty$. Using the perturbation theory of semi-Fredholm operators and the fact that $A^* - \lambda_0 I$ is lower semi-Fredholm, there exists $\epsilon > 0$ such that $A^* - \lambda I$ is lower semi-Fredholm, $\lambda \neq 0$ and $\mathrm{ind}(A^* - \lambda I) = \mathrm{ind}(A^* - \lambda_0 I)$ if $0 < |\lambda - \lambda_0| < \epsilon$. Since $N(A^* - \lambda I) \subseteq K(A^*)$, it follows that $n(A^* - \lambda I) < \infty$, which implies that $A^* - \lambda I$ is Fredholm. Then $A - \lambda_0 I$ is Fredholm and hence $B - \lambda_0 I$ is Fredholm. Therefore $M_{C_0} - \lambda_0 I$ is Fredholm with $\mathrm{ind}(M_{C_0} - \lambda_0 I) = \mathrm{ind}(M_C - \lambda_0 I) = 0$, that is, $M_{C_0} - \lambda_0 I$ is Weyl. Then $\sigma_w(M_{C_0}) \subseteq \sigma_w(M_C)$. The case $\sigma_w(M_C) \subseteq \sigma_w(M_{C_0})$ has the same proof. Then $\sigma_w(M_C) \cup \partial\mathrm{D} = \sigma_w(M_{C_0}) \cup \partial\mathrm{D}$ is connected for every $C \in B(K, H)$.

(b) $\sigma(M_C) = \sigma_b(M_C)$ for every $C \in B(K, H)$.

Let $M_C - \lambda_0 I$ be Browder. Then both $A - \lambda_0 I$ and $B - \lambda_0 I$ are Fredholm and $\mathrm{asc}(A - \lambda_0 I) < \infty$, $\mathrm{des}(B - \lambda_0 I) < \infty$. Using the perturbation theory of semi-Fredholm operators again, there exists $\epsilon > 0$ such that $A^* - \lambda I$ is Fredholm, $A^* - \lambda I$ is surjective, and $\operatorname{ind}(A^* - \lambda I) = \operatorname{ind}(A^* - \lambda_0 I)$ if $0 < |\lambda - \lambda_0| < \epsilon$. Since

$$N(A^* - \lambda I) \subseteq K(A^*) \quad \text{and} \quad \dim K(A^*) < \infty,$$

it follows that $A^* - \lambda I$ is bounded from below if $0 < |\lambda - \lambda_0|$ is sufficiently small (less than ϵ). Then $A^* - \lambda I$ is invertible if $0 < |\lambda - \lambda_0|$ is sufficiently small. This implies that $\lambda_0 \notin \operatorname{acc} \sigma(A)$. Then $A - \lambda_0 I$ is Browder [10, Theorem 4.7]. Therefore $B - \lambda_0 I$ is Browder and hence $M_{C_0} - \lambda_0 I$ is Browder. Since $M_{C_0} \in \overline{HC(H \oplus K)}$, $\sigma(M_{C_0}) = \sigma_b(M_{C_0})$. Then $A - \lambda_0 I$ is injective and $B - \lambda_0 I$ is surjective. But since both $A - \lambda_0 I$ and $B - \lambda_0 I$ are Browder, it follows that both $A - \lambda_0 I$ and $B - \lambda_0 I$ are invertible. Then $M_C - \lambda_0 I$ is invertible, which proves that $\sigma(M_C) = \sigma_b(M_C)$ for every $C \in B(K, H)$.

(c) For every $C \in B(K, H)$, $\operatorname{ind}(M_C - \lambda I) \geq 0$ for each $\lambda \in \rho_{SF}(A)$.

In fact, if $M_C - \lambda_0 I$ is semi-Fredholm with $\operatorname{ind}(M_C - \lambda_0 I) \leq 0$, then $A - \lambda_0 I$ is Fredholm (see the proof of (a) above). By [4, Theorem 2.1], $B - \lambda_0 I$ is upper semi-Fredholm. Thus $M_{C_0} - \lambda_0 I$ is semi-Fredholm with

$$\operatorname{ind}(M_{C_0} - \lambda_0 I) = \operatorname{ind}(M_C - \lambda_0 I) < 0.$$

It is in contradiction to the fact that $M_{C_0} \in \overline{HC(H \oplus K)}$. □

Remark 2.1.

(1) Theorem 2.4 holds for the case of supercyclicity.

(2) The condition $\dim K(A^*) < \infty$ is essential in Theorem 2.4. For example, let $H = K = \ell_2$ and $A, B, C \in B(\ell_2)$ be defined by

$$A(x_1, x_2, x_3, \ldots) = (0, x_1, 0, x_2, 0, x_3, \ldots),$$
$$B(x_1, x_2, x_3, \ldots) = (x_2, x_4, x_6, \ldots),$$
$$C(x_1, x_2, x_3, \ldots) = (0, 0, x_1, 0, x_3, 0, x_5, \ldots).$$

Then:

(i) $K(A^*) = K(B) = H$, then $\dim K(A^*) = \infty$;

(ii) $M_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \overline{HC(H \oplus K)}$;

(iii) $M_C \notin \overline{HC(H \oplus K)}$.

In fact, we can prove that M_C is bounded from below, but M_C is not invertible. This means that there exists $\lambda \in \rho_{SF}(M_C)$ such that $\operatorname{ind}(M_C - \lambda I) < 0$. Then we have $M_C \notin \overline{HC(H \oplus K)}$.

(3) Theorem 2.4 may fail if the assumption $\dim K(A) < \infty$ holds. For example, let $A \in B(H)$ be defined in (2) in this remark. We claim that $K(A) = \{0\}$. In fact, let $y = (y_1, y_2, y_3, \ldots) \in K(A)$. Using the definition of $K(A)$, there exists $\{x_n\} \subseteq H$ such that $Ax_{n+1} = x_n$ and $Ax_1 = y$. Then $A^n x_n = y$ for any $n \in \mathbb{N}$. Let $x_n = (x_{n1}, x_{n2}, x_{n3}, \ldots)$. For any $n \in \mathbb{N}$, the nth component of $A^n x_n$ is 0. This proves that for $n \in \mathbb{N}$, $y_n = 0$. Then $y = 0$. Therefore $K(A) = \{0\}$. But the result in Theorem 2.4 fails.

EXAMPLE 2.1. Let $H = K = \ell_2$ and let $A \in B(H)$ and $B \in B(K)$ be defined by

$$A(x_1, x_2, x_3, \ldots) = (x_2, x_4, x_6, \ldots),$$
$$B(x_1, x_2, x_3, \ldots) = (0, x_1, 0, x_2, 0, x_3, \ldots),$$

then $K(A^*) = \{0\}$ and $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \overline{HC(H \oplus K)}$, therefore $M_C \in \overline{HC(H \oplus K)}$ for every $C \in B(K, H)$.

The equivalent definition of $K(A)$ is:

$$K(A) = \{x \in H : \text{ there exists } (x_n)_{n=1}^{\infty} \subseteq H \text{ such that } Ax_1 = x,\ Ax_{n+1} = x_n,$$
$$\text{(for any } n \in \mathbb{N}\text{), and } \{\|x_n\|^{(1/n)}\}_{n=1}^{\infty} \text{ is bounded}\}.$$

LEMMA 2.5. *Suppose that $K(A)$ is closed. If for each eigenspace $N(A - \lambda I)$ of finite dimension, $K(A) \cap H_0(A - \lambda I)$ is closed, then* $\operatorname{asc}(A - \lambda I) < \infty$ *for any $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is upper semi-Fredholm.*

PROOF. Let $K(A) \neq \{0\}$ and suppose that $A_1 = A|_{K(A)}$. Then A_1 is surjective.

Let $\lambda_0 \in \mathbb{C}$ such that $A - \lambda_0 I$ is upper semi-Fredholm. Without loss of generality, let $\lambda_0 \notin \sigma_a(A)$. If $\lambda_0 = 0$, since $K(A) \cap H_0(A) = H_0(A_1)$ is closed, we know that A_1 has the SVEP at λ_0. Then $n(A_1) \leq d(A_1) = 0$ [9, Corollary 11], which means that A_1 is invertible. Then there exists $\epsilon > 0$ such that $N(A - \lambda I) = N(A_1 - \lambda I) = \{0\}$ if $0 < |\lambda| < \epsilon$. Since A is upper semi-Fredholm, $A - \lambda I$ is upper semi-Fredholm if $0 < |\lambda|$ is sufficiently small. Then $A - \lambda I$ is bounded from below, that is, $0 \in [\operatorname{iso} \sigma_a(A) \cup \rho_a(A)]$. Therefore $\operatorname{asc}(A - \lambda_0 I) < \infty$. In what follows, we suppose that $\lambda_0 \neq 0$.

(a) For any $m \in \mathbb{N}$, $N[(A - \lambda_0 I)^m] \subseteq K(A)$.

Let $x \in N[(A - \lambda_0 I)^m]$, that is, $(A - \lambda_0 I)^m x = 0$. Then there exists a polynomial $P(\cdot)$ such that $\lambda_0^m x = AP(A)x$, $x = A[((P(A))/(\lambda_0^m))x]$. Let

$$c = \|((P(A))/(\lambda_0^m))\| + 1, \quad x_1 = ((P(A))/(\lambda_0^m))x, \quad x_n = [(P(A))/(\lambda_0^m)]^n x,$$

for all $n \in \mathbb{N}$. Then $Ax_1 = x$, $Ax_{n+1} = x_n$, and $\|x_n\| \leq c^n \|x\|$, which implies that $x \in K(A)$. Therefore, $\alpha(A - \lambda_0 I) = \alpha(A_1 - \lambda_0 I)$.

(b) $K(A) \cap R(A - \lambda_0 I) = R(A_1 - \lambda_0 I)$.

For any $y \in K(A) \cap R(A - \lambda_0 I)$, let $y = (A - \lambda_0 I)x_0$. Since $y \in K(A) = AK(A)$, there exists $y_0 \in K(A)$ such that $(A - \lambda_0 I)x_0 = Ay_0$. Then

$$x_0 = A[(x_0 + y_0)/(\lambda_0)].$$

Using the definition of $K(A)$, there exist $c > 0$ and $\{y_n\}_{n=1}^{\infty} \subseteq X$ such that $Ay_1 = y_0$, $Ay_{n+1} = y_n$ and $\|y_n\| \leq c^n \cdot \|y_0\|$ ($\forall n \in \mathbb{N}$).

Let

$$x_1 = ((x_0 + y_0)/\lambda_0), \quad x_n = ((x_0 + y_0)/\lambda_0^n) + (y_1/\lambda_0^{n-1}) + \cdots + (y_{n-1}/\lambda_0).$$

Then $Ax_1 = x_0,\ Ax_2 = x_1, \ldots, Ax_{n+1} = x_n$ and

$$\begin{aligned}
\|x_n\| &= \left\| \frac{x_0 + y_0}{\lambda_0^n} + \frac{y_1}{\lambda_0^{n-1}} + \cdots + \frac{y_{n-1}}{\lambda_0} \right\| \\
&\le \frac{1}{|\lambda_0|^n}[\|x_0\| + \|y_0\| + |\lambda_0| \cdot \|y_1\| + \cdots + |\lambda_0|^{n-1} \cdot \|y_{n-1}\|] \\
&\le \frac{1}{|\lambda_0|^n}[\|x_0\| + \|y_0\| + |\lambda_0| \cdot c \cdot \|y_0\| + \cdots + |\lambda_0|^{n-1} \cdot c^{n-1} \cdot \|y_0\|] \\
&\le \frac{1}{|\lambda_0|^n}\|x_0\| + \frac{\|y_0\|}{|\lambda_0|^n}[1 + |\lambda_0|c + \cdots + |\lambda_0|^{n-1}c^{n-1}].
\end{aligned}$$

If $|\lambda_0| \cdot c \le 1$, then

$$\|x_n\| \le \frac{1}{|\lambda_0|^n} \cdot \|x_0\| + \frac{\|y_0\|}{|\lambda_0|^n} \cdot n,$$

$$\|x_n\|^{(1/n)} \le \frac{1}{|\lambda_0|} \cdot \|x_0\|^{(1/n)} + \frac{1}{|\lambda_0|} \cdot (n \cdot \|y_0\|)^{(1/n)}.$$

Since

$$\lim_{n\to\infty}[(1/(|\lambda_0|)) \cdot \|x_0\|^{(1/n)} + (1/(|\lambda_0|)) \cdot (n \cdot \|y_0\|)^{(1/n)}] = 2/|\lambda_0|,$$

it follows that $\{\|x_n\|^{(1/n)}\}_{n=1}^{\infty}$ is bounded.

If $|\lambda_0| \cdot c > 1$,

$$\begin{aligned}
\|x_n\| &\le \frac{1}{|\lambda_0|^n} \cdot \|x_0\| + \frac{\|y_0\|}{|\lambda_0|^n} \cdot \frac{1 - |\lambda_0|^n \cdot c^n}{1 - |\lambda_0| \cdot c} \\
&\le \frac{1}{|\lambda_0|^n} \cdot \|x_0\| + \frac{\|y_0\|}{|\lambda_0|^n} \cdot \frac{|\lambda_0|^n \cdot c^n}{|\lambda_0| \cdot c - 1} \\
&= \frac{1}{|\lambda_0|^n} \cdot \|x_0\| + \frac{\|y_0\|}{|\lambda_0| \cdot c - 1} \cdot c^n,
\end{aligned}$$

then

$$\|x_n\|^{(1/n)} \le \frac{1}{|\lambda_0|} \cdot \|x_0\|^{(1/n)} + \left(\frac{\|y_0\|}{|\lambda_0| \cdot c - 1}\right)^{(1/n)} \cdot c.$$

Also $\{\|x_n\|^{(1/n)}\}_{n=1}^{\infty}$ is bounded. Using the equivalent definition of $K(A)$, we know $x_0 \in K(A)$. Then $K(A) \cap R(A - \lambda_0 I) = R(A_1 - \lambda_0 I)$. Hence $A_1 - \lambda_0 I$ is upper semi-Fredholm. Since $H_0(A_1 - \lambda_0 I) = K(A) \cap H_0(A - \lambda_0 I)$ is closed, it follows that A_1 has the SVEP at λ_0. Then $\alpha(A - \lambda_0 I) = \alpha(A - \lambda_0 I) < \infty$.

Suppose that $K(A) = \{0\}$. Let $A - \lambda_0 I$ be upper semi-Frehdolm. Then there exists $\epsilon > 0$ such that $A - \lambda I$ is upper semi-Fredholm, $\lambda \ne 0$, if $0 < |\lambda - \lambda_0|$ is sufficiently small. Since $N(A - \lambda I) \subseteq K(A)$, $N(A - \lambda I) = \{0\}$. Then $A - \lambda I$ is bounded from below, and therefore $\lambda_0 \in \operatorname{iso} \sigma_a(A)$. This also implies that A has the SVEP at λ_0. Then $\operatorname{asc}(A - \lambda_0 I) < \infty$. □

Let $\sigma_d(A)$ denote the surjective spectrum of A. From the statements in Remark 2.1, we know the result in Theorem 2.4 is not true if we suppose that $K(A)$ is closed. However, the following theorem holds.

THEOREM 2.6. *Let $K(A)$ be closed. Suppose that for each eigenspace $N(A-\lambda I)$ of finite dimension, $K(A) \cap H_0(A-\lambda I)$ is closed.*

(1) *If $\sigma_{ab}(M_C) = \sigma_{ab}(A) \cup \sigma_{ab}(B)$ for any $C \in B(K, H)$ and $M_{C_0} \in \overline{HC(H \oplus K)}$ for some $C_0 \in B(K, H)$, then $M_C \in \overline{HC(H \oplus K)}$ for any $C \in B(K, H)$.*

(2) *If $\sigma(A) = \sigma_a(A)$ or $\sigma(B) = \sigma_d(B)$, then the converse of (1) is true.*

PROOF. (1) (i) $\sigma_w(M_C) \cup \partial\mathrm{D}$ is connected for each $C \in B(K, H)$.

We claim that $\sigma_w(M_C) = \sigma_w(M_{C_0})$. If fact, let $M_C - \lambda_0 I$ be Weyl. Then $A - \lambda_0 I$ is upper semi-Fredholm, $B - \lambda_0 I$ is lower semi-Fredholm and $d(A - \lambda_0 I) < \infty$ if and only if $n(B - \lambda_0 I) < \infty$. Then $\mathrm{asc}(A - \lambda_0 I) < \infty$. If $d(A - \lambda_0 I) = \infty$, then by [5, Theorem 2.1] there exists $C_1 \in B(K, H)$ such that $\lambda_0 \notin \sigma_{ab}(M_{C_1})$. Therefore $\lambda_0 \notin \sigma_{ab}(A) \cup \sigma_{ab}(B)$, which implies that $n(B - \lambda_0 I) < \infty$, which is a contradiction. Then both $A - \lambda_0 I$ and $B - \lambda_0 I$ are Fredholm. Therefore $M_{C_0} - \lambda_0 I$ is Fredholm with $\mathrm{ind}(M_{C_0} - \lambda_0 I) = \mathrm{ind}(M_C - \lambda_0 I) = 0$, that is, $M_{C_0} - \lambda_0 I$ is Weyl. Then $\sigma_w(M_{C_0}) \subseteq \sigma_w(M_C)$. The case $\sigma_w(M_C) \subseteq \sigma_w(M_{C_0})$ has the same proof. Then $\sigma_w(M_C) \cup \partial\mathrm{D} = \sigma_w(M_{C_0}) \cup \partial\mathrm{D}$ is connected for every $C \in B(K, H)$.

(ii) $\sigma(M_C) = \sigma_b(M_C)$ for every $C \in B(K, H)$.

Let $M_C - \lambda_0 I$ is Browder. Then both $A - \lambda_0 I$ and $B - \lambda_0 I$ are Fredholm and $\mathrm{asc}(A - \lambda_0 I) < \infty$, $\mathrm{des}(B - \lambda_0 I) < \infty$. Since $\lambda_0 \notin \sigma_{ab}(M_C)$, $\mathrm{asc}(B - \lambda_0 I) < \infty$, which means that $B - \lambda_0 I$ is Browder. Then $A - \lambda_0 I$ is Browder, and hence $\lambda_0 \notin \sigma_b(M_{C_0})$. But since $\sigma(M_{C_0}) = \sigma_b(M_{C_0})$, it follows that both $A - \lambda_0 I$ and $B - \lambda_0 I$ are invertible. Then $M_C - \lambda_0 I$ is invertible. Therefore $\sigma(M_C) = \sigma_b(M_C)$ for every $C \in B(K, H)$.

(iii) For every $C \in B(K, H)$, $\mathrm{ind}(M_C - \lambda I) \geq$ for each $\lambda \in \rho_{SF}(A)$.

In fact, if $M_C - \lambda_0 I$ is semi-Fredholm with $\mathrm{ind}(M_C - \lambda_0 I) \leq 0$, then $A - \lambda_0 I$ is upper semi-Fredholm with finite ascent. If $d(A - \lambda_0 I) < \infty$, then by [4, Theorem 2.1] $B - \lambda_0 I$ is upper semi-Fredholm. Thus $M_{C_0} - \lambda_0 I$ is semi-Fredholm with

$$\mathrm{ind}(M_{C_0} - \lambda_0 I) = \mathrm{ind}(M_C - \lambda_0 I) < 0.$$

This contradicts the fact that $M_{C_0} \in \overline{HC(H \oplus K)}$. But if $d(A - \lambda_0 I) = \infty$, using [5, Theorem 2.2], there exists $C_1 \in B(K, H)$ such that $\lambda_0 \notin \sigma_{ab}(M_{C_1})$. Then $B - \lambda_0 I$ is upper semi-Fredholm. Therefore $M_{C_0} - \lambda_0 I$ is semi-Fredholm and further $\mathrm{ind}(M_{C_0} - \lambda_0 I) = \mathrm{ind}(M_C - \lambda_0 I) < 0$. This again is a contradiction.

(2) Suppose that $\sigma(A) = \sigma_a(A)$ or $\sigma_d(A) = \sigma(B)$. For every $C \in B(K, H)$, the inclusion $\sigma_{ab}(M_C) \subseteq \sigma_{ab}(A) \cup \sigma_{ab}(B)$ is clear. For the converse inclusion, let $\lambda_0 \notin \sigma_{ab}(M_C)$, then $\lambda_0 \notin \sigma_{ab}(A)$. Therefore $A - \lambda I$ is bounded from below if $0 < |\lambda - \lambda_0|$ is sufficiently small. But since $\sigma_a(A) = \sigma(A)$, it follows that $\lambda_0 \notin \mathrm{acc}\, \sigma(A)$. Then $A - \lambda_0 I$ is Browder [10, Theorem 4.7]. Using the perturbation theory of semi-Fredholm operators and [4, Theorem 2.1], $\lambda_0 \notin \sigma_{ab}(B)$. Then $\lambda_0 \notin \sigma_{ab}(A) \cup \sigma_{ab}(B)$. The proof is complete. □

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COROLLARY 2.7. *If* $\dim K(A) < \infty$ *or* $\dim K(A - \lambda I) < \infty$ *for some* $\lambda \in \mathbb{C}$*, then the result in Theorem 2.6 is true.*

In Lemma 2.5 and Theorem 2.6, we can modify the condition '$K(A)$ is closed' to '$K(A - \lambda I)$ is closed for some $\lambda \in \mathbb{C}$'. It is well known that $K(A - \lambda I) = H$ is closed for any $\lambda \in \rho(A)$, leading to the following corollary.

COROLLARY 2.8. *Suppose that for each eigenspace* $N(A - \lambda I)$ *of finite dimension,* $H_0(A - \lambda I)$ *is closed, then the result in Theorem 2.6 is true.*

One such class which has attracted the attention of a number of authors is the set $H(P)$ of all operators $A \in B(H)$ such that for every complex number λ there exists an integer $d_\lambda \geq 1$ for which

$$H_0(A - \lambda I) = N[(A - \lambda I)^{d_\lambda}].$$

holds. The class $H(P)$ contains the classes of subscalar, algebraically totally paranormal and transaloid operators on a Banach space, $*$-totally paranormal, M-hyponormal, p-hyponormal $(0 < p < 1)$ and log-hyponormal operators on a Hilbert space (see [7, 8, 11]). From Corollary 2.8, we have the following results.

COROLLARY 2.9. *If* $A \in H(P)$*, then the result in Theorem 2.6 is true.*

LEMMA 2.10. *Suppose that* $A^* \in H(P)$. *Then* $\sigma(A) = \sigma_a(A)$ *and* $\sigma_{ab}(M_C) = \sigma_{ab}(A) \cup \sigma_{ab}(B)$ *for every* $B \in B(K)$ *and for every* $C \in B(K, H)$.

PROOF. Let $A - \lambda I$ be bounded form below. Then $A^* - \lambda I$ is surjective. But since A^* has the SVEP, it follows that $A^* - \lambda I$ is invertible. Then $A - \lambda I$ is invertible. This proves that $\sigma(A) = \sigma_a(A)$.

For any $C \in B(K, H)$ and for any $B \in B(K)$, the inclusion

$$\sigma_{ab}(M_C) \subseteq \sigma_{ab}(A) \cup \sigma_{ab}(B)$$

is clear. For the converse inclusion, let $\lambda \notin \sigma_{ab}(M_C)$; then $\lambda \notin \sigma_{ab}(A)$. Since A^* has the SVEP at λ, $A - \lambda I$ is Browder. Then $B - \lambda I$ is upper semi-Fredholm with $\operatorname{asc}(B - \lambda I) < \infty$. This proves that $\sigma_{ab}(M_C) = \sigma_{ab}(A) \cup \sigma_{ab}(B)$. □

Lemma 2.5 and Theorem 2.6 lead to the following result.

COROLLARY 2.11. *Suppose that* $A^* \in H(P)$ *and* $B \in B(K)$*, then the following statements are equivalent:*

(1) $M_0 \in \overline{HC(H \oplus K)}$;
(2) $M_C \in \overline{HC(H \oplus K)}$ *for some* $C \in B(K, H)$;
(3) $M_C \in \overline{HC(H \oplus K)}$ *for every* $C \in B(K, H)$.

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