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ON $\eta^3(a\tau)\eta^3(b\tau)$ WITH $a + b = 8$

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Abstract

We prove an observation associated with $\eta^3(\tau)\eta^3(7\tau)$ which is found on page 54 of Ramanujan's Lost Notebook (S. Ramanujan, *The Lost Notebook and Other Unpublished Papers* (Narosa, New Delhi, 1988)). We then study functions of the type $\eta^3(a\tau)\eta^3(b\tau)$ with $a + b = 8$.

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1. Introduction

Let $q = e^{2\pi i\tau}$ with $\operatorname{Im}\tau > 0$ and set

$$\eta(\tau) = q^{1/24}\prod_{k=1}^{\infty}(1 - q^k).$$

On [11, p. 54], Ramanujan stated that if

$$F(\tau) := \sum_{n=1}^{\infty} a_n q^n = \eta^3(\tau)\eta^3(7\tau), \tag{1.1}$$

then

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s} = \frac{1}{1+7^{1-s}}\prod_{p\equiv 3,5,6 \bmod 7}\frac{1}{1-p^{2(1-s)}}\prod_{p\equiv 1,2,4 \bmod 7}\frac{1}{1+2c_p p^{-s}+p^{2(1-s)}}, \tag{1.2}$$

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where p are primes. Ramanujan also asserted that

$$c_p = 7v^2 - u^2 \tag{1.3}$$

with

$$p = u^2 + 7v^2. \tag{1.4}$$

Equation (1.3) is, in fact, false for the prime $p = 2$, and the correct formula is

$$c_p = \begin{cases} 3/2 & \text{if } p = 2, \\ 7v^2 - u^2 & \text{if } p = u^2 + 7v^2. \end{cases} \tag{1.5}$$

The above assertion of Ramanujan was first studied by Rangachari [12]. Rangachari explained the existence of the Euler product expansion for the Dirichlet series corresponding to $F(\tau)$ but did not determine (1.5) explicitly.

On [11, p. 146], Ramanujan revisited $F(\tau)$ and recorded the Euler product for its corresponding Dirichlet series as

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{1}{1+7^{1-s}} \prod_{p\equiv 3,5,6 \bmod 7} \frac{1}{1-p^{2(1-s)}} \prod_{p\equiv 1,2,4 \bmod 7} \frac{1}{1+C_p p^{-s} + p^{2(1-s)}}, \tag{1.6}$$

where

$$C_p = 2p - a^2 \tag{1.7}$$

with

$$4p = a^2 + 7b^2. \tag{1.8}$$

Note that if p is odd, then $p = u^2 + 7v^2$ implies that $4p = (2u)^2 + 7(2v)^2$. Conversely, if $4p = a^2 + 7b^2$ and p is odd then a and b are even and $p = (a/2)^2 + 7(b/2)^2$. Hence (1.2) is equivalent to (1.6) when p is odd, namely,

$$C_p = 2p - a^2 = 2(p - 2u^2) = 2(u^2 + 7v^2 - 2u^2) = 2(7v^2 - u^2) = 2c_p.$$

When p is even, it is easy to check that C_2 is equal to $2c_2$. This implies that Ramanujan's observations for $F(\tau)$ on pages 54 and 146 of his Lost Notebook are equivalent.

Equations (1.6) and (1.7) were first discussed in a recent paper by Berndt and Ono [2, (8.4)]. They remarked that C_p can be obtained by applying Jacobi's identity [1, p. 500] twice and gave a brief sketch of the proof (see the comments in [2, (8.4)]). As a result, complete proofs of (1.5) and (1.7) are still missing.

In Section 2, we derive (1.2) and (1.5) using an approach similar to that suggested in [2].

In Section 3, we give proofs of (1.2) and (1.5) using Schoeneberg's theta functions (more commonly known as spherical theta functions).

In Section 4, we study functions of the type $\eta^3(a\tau)\eta^3(b\tau)$ with $a + b = 8$ and obtain analogues of (1.2) and (1.5).

2. Proofs of (1.2) and (1.5) using Jacobi's identity

PROOFS OF (1.2) AND (1.5). As indicated in [12] and [2], the function $F(\tau)$ is in $\mathcal{S} := S_3(\Gamma_0(7), (\cdot/7))$, the space of weight 3 cusp forms on $\Gamma_0(7)$ with character $(\cdot/7)$. The space $\mathcal{S}$ is one dimensional [3, Théorème 1] and, hence, $F(\tau)$ is an eigenform. As a result, the corresponding Dirichlet series for $F(\tau)$ has an Euler product expansion [7, p. 163]

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \frac{1}{(1 - a_p p^{-s} + (p/7)p^{2(1-s)})}. \tag{2.1}$$

It remains to determine a_p for all primes p.

When $p = 7$, it follows from the expansion of $F(\tau)$ that $a_7 = -1$ and we obtain the first factor in (1.2). When $p = 2$, the value of a_2 can also obtained directly from the expansion of $F(\tau)$, namely, $a_2 = -3$. This gives the value of c_2 in (1.5).

It remains to determine a_p for other odd primes p. This will complete the proofs of (1.2) and (1.5).

Recall that by Jacobi's identity,

$$\eta^3(\tau) = \sum_{\substack{\alpha\in\mathbf{Z}\\ \alpha\equiv 1 \bmod 4}} \alpha q^{\alpha^2/8}. \tag{2.2}$$

Therefore,

$$\eta^3(\tau)\eta^3(7\tau) = \sum_{\substack{\alpha\equiv 1 \bmod 4\\ \beta\equiv 1 \bmod 4}} \alpha\beta q^{(\alpha^2+7\beta^2)/8}.$$

Note that this means that for all primes p,

$$a_p = \sum_{\substack{(\alpha,\beta)\equiv(1,1) \bmod 4\\ 8p=\alpha^2+7\beta^2}} \alpha\beta.$$

If

$$8p = C^2 + 7D^2 \tag{2.3}$$

with

$$(C, D) \equiv (1, 1) \bmod 4 \tag{2.4}$$

then

$$C = A - 7B \quad \text{and} \quad D = A + B \tag{2.5}$$

for some A, B satisfying $p = A^2 + 7B^2$. Suppose that A and B satisfy (2.5), then

$$A = \frac{C + 7D}{8} \quad \text{and} \quad B = \frac{D - C}{8},$$

and we conclude that A and B are integers since by (2.3) and (2.4),

$$(C, D) \equiv (1, 1) \text{ or } (5, 5) \bmod 8.$$

Note that $p = A^2 + 7B^2$ since

$$8p = C^2 + 7D^2 = (A - 7B)^2 + 7(A + B)^2 = 8(A^2 + 7B^2).$$

This shows that every solution of (2.3) with C and D satisfying (2.4) can be obtained from a solution of $p = A^2 + 7B^2$. In other words, a_p is zero when p is not of the form $A^2 + 7B^2$. This happens when

$$\left(\frac{-7}{p}\right) = \left(\frac{p}{7}\right) = -1.$$

Consequently,

$$a_p = 0 \quad \text{when } p \equiv 3,\, 5,\, 6 \bmod 7.$$

This yields the second product on the right-hand side of (1.2).

We now show that $p = A^2 + 7B^2$ if and only if $p \equiv 1, 2, 4 \bmod 7$. Let

$$\omega = ((1 + \sqrt{-7})/2), \quad \overline{\omega} = ((1 - \sqrt{-7})/2) \quad \text{and} \quad \mathfrak{O} := \mathbf{Z}[(1 + \sqrt{-7})/2].$$

Then the ideal $p\mathfrak{O}$ splits in $\mathfrak{O}$ if and only if $p \equiv 1, 2$ or $4 \bmod 7$. This follows from Kummer's theorem [5, p. 129, Theorem 23 and p. 132, (2.29)], which allows us to say that p splits if and only if

$$x^2 + x + 2 \equiv 0 \bmod p$$

is solvable. The latter condition is equivalent to the condition that

$$\left(\frac{-7}{p}\right) = 1$$

and this happens if and only if $p \equiv 1, 2$ or $4 \bmod 7$.

Suppose that p is an odd prime congruent to 1, 2 or 4 mod 7. Since $\mathfrak{O}$ is a principal ideal domain, every ideal may be written as $(a) = a\mathfrak{O}$ for some $a \in \mathfrak{O}$. Hence, for any prime $p \equiv 1, 2$ or $4 \bmod 7$, we deduce that

$$(p) = (\alpha + \beta\omega)\,(\alpha + \beta\overline{\omega}),$$

for some $\alpha, \beta \in \mathbf{Z}$. Since ± 1 are the only units in $\mathfrak{O}$, we conclude that

$$p = (\alpha + \beta\omega)(\alpha + \beta\overline{\omega}) = \alpha^2 + \alpha\beta + 2\beta^2.$$

The above representation shows that α cannot be even, otherwise p would be even. Hence, α is odd. However, this forces β to be even since p is odd. Therefore, we may write

$$p = \left(\alpha + \frac{\beta}{2}\right)^2 + 7\left(\frac{\beta}{2}\right)^2.$$

Hence, there are integers γ and δ such that

$$p = \gamma^2 + 7\delta^2.$$

This shows that if p is an odd prime, then $p \equiv 1, 2, 4 \bmod 7$ if and only if $p = A^2 + 7B^2$.

We now return to the computation of a_p for $p \equiv 1, 2, 4 \bmod 7$. If $p = A^2 + 7B^2$, then (A, B), $(A, -B)$, $(-A, B)$ and $(-A, -B)$ are all solutions of $p = \gamma^2 + 7\delta^2$ (this follows from the splitting of (p) in $\mathfrak{O}$). Each of these gives rise to a solution (C, D) of (2.3) (see our earlier computations), namely

$$(A - 7B, A + B), \quad (A + 7B, A - B), \quad (-A - 7B, -A + B) \quad \text{and} \quad (-A + 7B, -A - B).$$

Only two, depending on (A, B) mod 4, out of the four give solutions satisfying (2.4). For example, $(A - 7B, A + B)$ and $(A + 7B, A - B)$ could be the desired solutions and in this case

$$(A - 7B)(A + B) + (A + 7B)(A - B) = 2(A^2 - 7B^2).$$

By considering all possible cases for (A, B) mod 4 we conclude that if $p = A^2 + 7B^2$, then

$$a_p = \sum_{\substack{(\alpha,\beta)\equiv(1,1) \bmod 4 \\ 8p=\alpha^2+7\beta^2}} \alpha\beta = 2(A^2 - 7B^2).$$

This completes the proof of (1.5) and the derivation of the third factor in (1.2) for primes $p \neq 2$. $\square$

3. Proofs of (1.2) and (1.5) using Schoeneberg's theta functions

We first show that the following holds.

THEOREM 3.1. *We have*

$$\eta^3(\tau)\eta^3(7\tau) = \frac{1}{2} \sum_{m,n=-\infty}^{\infty} \left(m + n\left\{ \frac{\sqrt{-7}+1}{2} \right\} \right)^2 q^{m^2+mn+2n^2}. \tag{3.1}$$

We recall a class of theta functions studied by Schoeneberg [13].

Let f be an even positive integer and $M = (m_{\mu,\nu})$ be a *symmetric* $f \times f$ matrix such that:

(1) $m_{\mu,\nu} \in \mathbf{Z}$;
(2) $m_{\mu,\mu}$ is even; and
(3) $\mathbf{x}^t M \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbf{R}^f$ such that $\mathbf{x} \neq \mathbf{0}$.

Let N be the smallest positive integer such that NM^{-1} also satisfies conditions 1–3. Let

$$P_k^M(\mathbf{x}) := \sum_{\mathbf{y}} c_{\mathbf{y}}(\mathbf{y}^t M\mathbf{x})^k,$$

where the sum is over finitely many $\mathbf{y} \in \mathbf{C}^f$ with the property $\mathbf{y}^t M\mathbf{y} = 0$, and $c_{\mathbf{y}}$ are arbitrary complex numbers.

When $M\mathbf{h} \equiv \mathbf{0} \bmod N$ and $\operatorname{Im} \tau > 0$, we define

$$\vartheta_{M,\mathbf{h},P_k^M}(\tau) = \sum_{\substack{\mathbf{n}\in\mathbf{Z}^f \\ \mathbf{n}\equiv\mathbf{h} \bmod N}} P_k^M(\mathbf{n}) \exp(((2\pi i\tau)/N)(1/2)((\mathbf{n}^t M\mathbf{n})/N)). \tag{3.2}$$

PROOF OF THEOREM 3.1. Substitute

$$\mathbf{y} = \begin{pmatrix} -1-\sqrt{-7} \\ 2 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, \quad \mathbf{h} = (0,0) \quad \text{and} \quad N = 7$$

in (3.2). Then we conclude that the function

$$A(\tau) = \frac{1}{2}\sum_{m,n=-\infty}^{\infty}\left(m + n\left\{\frac{\sqrt{-7}+1}{2}\right\}\right)^2 q^{m^2+mn+2n^2}$$

is a weight 3 cusp form on $\Gamma_0(7)$ with multiplier system $(\cdot/7)$ (see [13, p. 217, Theorem 4 and p. 218, Theorem 5]), namely,

$$A\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^3\left(\frac{d}{7}\right)A(\tau),$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(7).$$

It can be verified directly that $B(q) = \eta^3(\tau)\eta^3(7\tau)$ is also a form on $\Gamma_0(7)$ with multiplier system $(\cdot/7)$ and that any cusp form of weight 3 on $\Gamma_0(7)$ with multiplier system $(\cdot/7)$ is a constant multiple of $B(q)$ since $B(q)$ is an eigenform (see [7, p. 145, Exercises 12 and 13]). By looking at the expansion of $A(q)$ and $B(q)$, we conclude that the constant is 1 and this completes the proof of the theorem. □

PROOFS OF (1.2) AND (1.5). We first give a formula for a_p. As observed earlier, if p is an odd prime and $p = m^2 + mn + 2n^2$, then $n = 2n'$. Also, if $p = A^2 + 7B^2$, then $m = A - B$ and $n' = B$. Hence, the coefficient of q^p in the expansion of $G(\tau)$ is given by

$$\begin{aligned} a_p = \tfrac{1}{2}\{&(A - B + \sqrt{-7}B)^2 + (A + B + \sqrt{-7}(-B))^2 \\ &+ (-A - B + \sqrt{-7}B)^2 + (-A + B + \sqrt{-7}(-B))^2\} = 2(A^2 - 7B^2). \end{aligned} \tag{3.3}$$

The value of a_2 can be obtained directly from the expansion of $\eta^3(\tau)\eta^3(7\tau)$. Alternatively, it follows from the right-hand side of (3.1) that

$$a_2 = -3.$$

In Section 2, we concluded that the Euler product for the corresponding Dirichlet series for $F(\tau)$ exists because $F(\tau)$ is an eigenform. Alternatively, we may establish this fact using the right-hand side of (3.1) as follows.

Since $\mathfrak{O} := \mathbf{Z}[(1+\sqrt{-7})/2]$ is a principal ideal domain and there are only two units, namely ± 1, and every integral ideal has only two generators. With this observation, we find that the series representation of $\eta^3(\tau)\eta^3(7\tau)$ can be expressed in the form

$$\frac{1}{2}\sum_{\alpha\in\mathfrak{O}} \alpha^2 q^{\mathbf{N}(\alpha)} = \sum_{\mathfrak{a}=(\alpha)\subset\mathfrak{O}} \alpha^2 q^{\mathbf{N}(\mathfrak{a})}.$$

Let $\mathcal{P}$ denote the set of nonzero prime ideals of $\mathfrak{O}$. The corresponding Dirichlet series for $G(\tau)$ is

$$\begin{aligned}\sum_{0\neq(\alpha)\subset\mathfrak{O}} \frac{\alpha^2}{(\mathbf{N}(\alpha))^s} &= \prod_{\substack{\mathfrak{p}=(\alpha)\in\mathcal{P}\\ \alpha^2 \text{ is prime in } \mathbf{Z}}} \left(1+\frac{\alpha^2}{\mathbf{N}(\mathfrak{p})^s}+\frac{\alpha^4}{\mathbf{N}(\mathfrak{p}^2)^s}+\cdots\right)\\ &\quad\times \prod_{\substack{\mathfrak{p}=(\alpha)\in\mathcal{P},\\ \alpha \text{ is prime in } \mathbf{Z}}} \left(1+\frac{\alpha^2}{\mathbf{N}(\mathfrak{p})^s}+\frac{\alpha^4}{\mathbf{N}(\mathfrak{p}^2)^s}+\cdots\right)\\ &\quad\times \prod_{\substack{\mathfrak{p}=(\alpha),\,\mathfrak{p}'=(\alpha'),\,\mathfrak{p}\neq\mathfrak{p}'\\ \alpha\alpha'=p,\text{a prime in } \mathbf{Z}}} \left(1+\frac{\alpha^2}{\mathbf{N}(\mathfrak{p})^s}+\frac{\alpha^4}{\mathbf{N}(\mathfrak{p}^2)^s}+\cdots\right).\end{aligned}$$

There is only one term in the first product and the prime ideal involved is $(\sqrt{-7})$. The first product is then given by

$$1-\frac{7}{7^s}+\frac{7^2}{7^{2s}}-\cdots = \frac{1}{1+7^{1-s}}.$$

The second product is over all integral primes p such that (p) is a prime ideal in $\mathfrak{O}$ (these are primes that are quadratic nonresidues modulo 7). A typical term is given by

$$1+\frac{p^2}{p^{2s}}+\frac{p^4}{p^{4s}}+\cdots = \frac{1}{1-p^{2-2s}}.$$

Finally we can pair up the terms in the third product for each prime p that splits in $\mathfrak{O}$, namely,

$$p = \alpha\alpha' \quad \text{with } \alpha,\, \alpha' \in \mathfrak{O}.$$

A typical term is given by

$$\begin{aligned}\left(1+\frac{\alpha^2}{p^s}+\frac{\alpha^2}{p^{2s}}+\cdots\right)\left(1+\frac{\alpha'^2}{p^s}+\frac{\alpha'^2}{p^{2s}}+\cdots\right) &= \frac{1}{1-\alpha^2p^{-s}}\cdot\frac{1}{1-\alpha'^2p^{-s}} \\ &= \frac{1}{1-(\alpha^2+\alpha'^2)p^{-s}+p^{2-2s}} \\ &= \frac{1}{1-a_pp^{-s}+p^{2-2s}}.\end{aligned}$$

Hence,

$$\sum_{n\geq 1}\frac{a_n}{n^s}=\frac{1}{1+7^{1-s}}\prod_{p\equiv 3,5,6 \bmod 7}\frac{1}{1-p^{2-2s}}\prod_{p\equiv 1,2,4 \bmod 7}\frac{1}{1-a_pp^{-s}+p^{2-2s}}.$$

Comparing this with (1.2), we conclude that

$$a_p=-2c_p. \tag{3.4}$$

Using (3.3), we complete the proof of (1.5). □

We end this section with a proof of a congruence satisfied by Ramanujan's τ function.

COROLLARY 3.2. *Let*

$$\Delta(\tau):=\eta^{24}(\tau)=\sum_{k\geq 1}\tau(k)q^k.$$

Then

$$\tau(p)\equiv\begin{cases}0 \bmod 7 & \text{if } p\equiv 3,5,6 \bmod 7\\ 2u^2 \bmod 7 & \text{if } p\equiv 1,2,4 \bmod 7 \text{ and } p=u^2+7v^2.\end{cases}$$

PROOF. Write

$$\Delta\equiv\eta^3(\tau)\eta^3(7\tau) \bmod 7.$$

We then conclude that

$$a_p\equiv\tau(p) \bmod 7. \tag{3.5}$$

We know that a_p is zero when p is a quadratic nonresidue modulo 7 and, hence, the first part follows.

When p is a quadratic residue, we have $p=u^2+7v^2$ and by (1.3) and (3.4),

$$a_p\equiv -2c_p\equiv 2u^2-14v^2\equiv 2u^2 \bmod 7.$$

Using (3.5), we conclude that if p is a quadratic residue modulo 7, then

$$\tau(p)\equiv 2u^2 \bmod 7. \tag{3.6}$$

□

Note that we may also rewrite (3.6) as

$$\tau(p) \equiv p^4 + p \bmod 7, \tag{3.7}$$

when p is a quadratic residue modulo 7. Congruence (3.7) is due to Ramanathan [10]. For more congruences such as (3.7) satisfied by $\tau(n)$ and the reasons why such congruences exist, see [14].

4. Identities associated with $\eta^3(a\tau)\eta^3(b\tau)$, with $a+b=8$

In our attempt to derive a_p for primes p of the form u^2+7v^2 where a_n is defined as in (1.1), we also discovered similar results for the η-products

$$\eta^3(2\tau)\eta^3(6\tau), \quad \eta^3(3\tau)\eta^3(5\tau) \quad \text{and} \quad \eta^6(4\tau).$$

The proofs of the following identities are similar to the proof of Theorem 3.1.

THEOREM 4.1. *We have*

$$\eta^3(2\tau)\eta^3(6\tau) = \frac{1}{2}\sum_{m,n=-\infty}^{\infty}(m+n\sqrt{-3})^2 q^{m^2+3n^2}, \tag{4.1}$$

$$\eta^3(3\tau)\eta^3(5\tau) = \frac{1}{2}\sum_{m,n=-\infty}^{\infty}\left(m+n\left(\frac{1+\sqrt{-15}}{2}\right)\right)^2 q^{m^2+mn+4n^2}. \tag{4.2}$$

$$\eta^6(4\tau) = \frac{1}{2}\sum_{m,n=-\infty}^{\infty}(m+2n\sqrt{-1})^2 q^{m^2+4n^2}. \tag{4.3}$$

PROOF. Let $S_3(\Gamma_0(N), (\delta/\cdot))$ be the space of cusp forms of weight 3 with multiplier $(\delta/\cdot)$ under the action of $\Gamma_0(N)$.

Let the right-hand side of (4.1) be denoted as $R_1(\tau)$. By [13, p. 217, Theorem 4 and p. 218, Theorem 5],

$$R_1(\tau) \in S_3(\Gamma_0(12), (-6/\cdot)) =: \mathcal{C}_1.$$

The space $\mathcal{C}_1$ is one dimensional [3, Théorème 1] over $\mathbf{C}$ and generated by $L_1(\tau) = \eta^3(2\tau)\eta^3(6\tau)$ (see [6, p. 174]). By comparing the leading coefficients of $R_1(\tau)$ and $L_1(\tau)$, we complete the proof of (4.1).

To prove (4.2), let the right-hand side of (4.2) be $R_2(\tau)$. Then

$$R_2(\tau) \in S_3(\Gamma_0(15), (-15/\cdot)) =: \mathcal{C}_2.$$

The dimension of $\mathcal{C}_2$ over $\mathbf{C}$ is two [3, Théorème 1] and a basis can be taken as

$$\{\eta^3(\tau)\eta^3(15\tau), \eta^3(3\tau)\eta^3(5\tau)\}.$$

Comparing the coefficients of $R_2(\tau)$ and the elements in the basis, we conclude the proof of (4.2).

The proof of (4.3) is similar and follows from the fact that $S_3(\Gamma_0(16), (-4/\cdot))$ is one dimensional [3, Théorème 1] and spanned by $\eta^6(4\tau)$. □

REMARKS. The Euler products exist for the Dirichlet series corresponding to the forms in (4.1) and (4.3) since these are eigenforms [7, p. 163].

By comparing the coefficients of both sides in (4.1) and (4.3), we obtain the following analogues of (1.5).

COROLLARY 4.2. (i) *Let*

$$\eta^3(2\tau)\eta^3(6\tau) = \sum_{n=1}^{\infty} b_n q^n.$$

Then

$$b_p = 2(u^2 - 3v^2) \quad \textit{when } p = u^2 + 3v^2 \textit{ and } p > 3.$$

(ii) *Let*

$$\eta^6(4\tau) = \sum_{n=1}^{\infty} d_n q^n.$$

Then

$$d_p = 2(u^2 - 4v^2) \quad \textit{when } p = u^2 + 4v^2.$$

PROOF. It is known that [4, p. 61] if p can be written as $am^2 + bn^2$ with $\gcd(a, b) = 1$ and $ab > 1$, then there are exactly four ways of writing p in this form. Therefore, the only four solutions to the equation $p = u^2 + 3v^2$ are (u, v), $(u, -v)$, $(-u, -v)$ and $(-u, v)$. Hence,

$$\begin{aligned} b_p &= \tfrac{1}{2}((u + v\sqrt{-3})^2 + (u - v\sqrt{-3})^2 + (-u + v\sqrt{-3})^2 + (-u - v\sqrt{-3})^2) \\ &= 2(u^2 - 3v^2). \end{aligned}$$

The expression for d_p can also be proved in the same way. □

REMARKS. Using Schoeneberg's theta series as we did in the proof of Theorem 3.1, one can also show the following identity:

$$\eta^3(\tau)\eta^3(15\tau) = \frac{-3}{2} \sum_{m,n=-\infty}^{\infty} \left(m + n\left(\frac{3 + \sqrt{-15}}{6}\right)\right)^2 q^{3m^2+3mn+2n^2}. \tag{4.4}$$

The analogue of (1.5) in this case is given by the following result.

COROLLARY 4.3. *Let*

$$E^{\pm}(\tau) = \eta^3(3\tau)\eta^3(5\tau) \pm \eta^3(\tau)\eta^3(15\tau) := \sum_{n=1}^{\infty} e_n^{\pm} q^n.$$

Then $E^{\pm}(\tau)$ are eigenforms. When $p \neq 2, 3, 5$, then

$$e_p^{\pm} = \begin{cases} \mp 2(3u^2 - 5v^2) & \text{if } p = 3u^2 + 5v^2, \\ 2(u^2 - 15v^2) & \text{if } p = u^2 + 15v^2. \end{cases} \tag{4.5}$$

Furthermore,

$$e_2^{\pm} = \pm 1, \quad e_3^{\pm} = \mp 3 \quad \text{and} \quad e_5^{\pm} = \pm 5.$$

PROOF. Let

$$E_1(\tau) = \eta^3(3\tau)\eta^3(5\tau) = \sum_{n=1}^{\infty} \alpha(n) q^n$$

and

$$E_2(\tau) = \eta^3(\tau)\eta^3(15\tau) = \sum_{n=2}^{\infty} \beta(n) q^n.$$

Let

$$E(\tau) = \sum_{n=1}^{\infty} \epsilon(n) q^n$$

be an eigenform in $S_3(\Gamma_0(15), (-15/\cdot))$ with $\epsilon(1) = 1$. Suppose

$$E(\tau) = E_1(\tau) + v E_2(\tau).$$

Applying the Hecke operator T_2 (see [7, p. 161]) to both sides of the above, we find that

$$E(\tau)|_{T_2} = \epsilon(2) E(\tau) = E_1(\tau)|_{T_2} + v E_2(\tau)|_{T_2}. \tag{4.6}$$

Comparing the coefficients of q and q^2 of (4.6), we find that

$$\epsilon(2) = \alpha(2) + v\beta(2) = v\beta(2) = v$$

and

$$\epsilon(2)^2 = \alpha(4) + 4\alpha(1) + v\beta(4) + 4v\beta(1) = 1.$$

Hence, $v = \pm 1$ and $E^{\pm}(\tau)$ are indeed the eigenforms for $S_3(\Gamma_0(15), (-15/\cdot))$.

In order to determine the eigenvalues $e_p^{\pm}$ corresponding to T_p for $E^{\pm}$, we note that if $p = 3m^2 + 3mn + 2n^2$, then $p = 3u^2 + 5v^2$ where

$$u = m + \frac{n}{2} \quad \text{and} \quad v = \frac{n}{2}.$$

By [4, p. 61], we find that there are exactly four solutions to the latter equation and these are

$$S := \{(u, v), (-u, -v), (u, -v), (-u, v)\}.$$

Using the right-hand side of (4.4), together with the substitutions

$$m = U - V, \quad n = 2V \quad \text{with } (U, V) \in S,$$

we deduce immediately the first part of (4.5). The second part of (4.5) follows similarly using the right-hand side of (4.2). The values of $e_p^{\pm}$ for $p = 2$, 3 and 5 follow from the expansion of $E^{\pm}(\tau)$. □

REMARKS. The functions $\eta^3(\tau)\eta^3(7\tau)$, $\eta^3(2\tau)\eta^3(6\tau)$ and $\eta^6(4\tau)$ were studied by Ono in connection with Gaussian hypergeometric series over finite fields. For more details, see [9, pp. 194–195]. Murata also connected the coefficients b_p and d_p with the number of $\mathbf{F}_p$-rational points on the $K3$-surfaces

$$xy(x + y + 1)(x + y + xy) = z^2 \quad \text{and} \quad xy(x - y)(xy - 1) = z^2$$

respectively. Readers who are interested in this connection are encouraged to read [8].

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