

REMARKS ON ℓ_1 AND ℓ_∞ -MAXIMAL REGULARITY FOR POWER-BOUNDED OPERATORS

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Abstract

We discuss ℓ_p -maximal regularity of power-bounded operators and relate the discrete to the continuous time problem for analytic semigroups. We give a complete characterization of operators with ℓ_1 and ℓ_∞ -maximal regularity. We also introduce an unconditional form of Ritt's condition for power-bounded operators, which plays the role of the existence of an H^∞ -calculus, and give a complete characterization of this condition in the case of Banach spaces which are L_1 -spaces, $C(K)$ -spaces or Hilbert spaces.

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1. Introduction

Let T be a power-bounded operator on a Banach space X . In [4] and [5], Blunck studied ℓ_p -maximal regularity for the discrete equation

$$u_n = Tu_{n-1} + x_n \quad \text{for all } n \geq 1,$$

where $u_0 = 0$. See Section 2 for precise definitions. Blunck studied the case $1 < p < \infty$. The cases $p = 1$ and $p = \infty$ are studied in [17] where some discrete analogues of the results of Baillon [2] and Guerre-Delabrière [9] are given. However, these analogues are not completely satisfying and, moreover, the proofs of Theorems 4.4 and 4.5 are rather confused.

In this paper we improve these results and also give a complete description of operators T with ℓ_1 or ℓ_∞ regularity. We then point out that one can obtain the analogous and apparently new descriptions for closed operators A such that $-A$ generates a bounded analytic semigroup and has either L_1 or L_∞ -maximal regularity. We relate our results to classical result of Da Prato and Grisvard on L_∞ -maximal regularity in real interpolation spaces.

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In the final section we introduce and study an unconditional form of Ritt's condition for power-bounded operators. This is analogous to McIntosh's definition of an H^∞ -calculus for sectorial operators [14]. We show that on an L_1 -space the unconditional Ritt condition is equivalent to ℓ_1 -maximal regularity and dually on a $C(K)$ -space it is equivalent to ℓ_∞ -maximal regularity. These results use Grothendieck's theorem. Finally, we give a discrete analogue of a result of Auscher *et al.* [1] characterizing the unconditional Ritt condition on Hilbert spaces.

2. Preliminaries

Suppose $-A$ is the generator of a bounded analytic semigroup on a (complex) Banach space X . We shall say that A has L_p -maximal regularity if the solution of the abstract Cauchy problem

$$\begin{aligned} u' + Au &= f(t) \quad \text{for all } 0 \leq t < \infty, \\ u(0) &= 0 \end{aligned}$$

given by

$$u(t) = \int_0^t e^{-(t-s)A} f(s) \, ds$$

has the property that $u' \in L_p(\mathbb{R}_+, X)$ whenever $f \in L_p(\mathbb{R}_+, X)$. This is equivalent to the requirement that there is a constant C such that

$$\left(\int_0^\infty \left\| \int_0^t A e^{-(t-s)A} f(s) \, ds \right\|^p dt \right)^{1/p} \leq C \|f\|_p.$$

(Note that we have no need of $u \in L_p(\mathbb{R}_+, X)$ which is sometimes additionally required.)

Similarly, suppose that T is a bounded operator. We say that T satisfies *Ritt's condition* (or generates a discrete analytic semigroup) [18] if there is a constant C so that

$$\|(1 - \lambda)R(\lambda, T)\| \leq C \quad \text{for all } |\lambda| \geq 1. \quad (2.1)$$

The following characterization of operators satisfying Ritt's condition is due to Nagy and Zemánek [15] and Lyubich [13].

THEOREM 2.1. *T satisfies Ritt's condition (2.1) if and only if T is power-bounded and*

$$\sup_{n \geq 1} \|n(T^{n-1} - T^n)\| < \infty. \quad (2.2)$$

Note for future reference that (2.2) implies

$$\sup_{n \geq 1} \|n^2(I - T)^2 T^{n-1}\| < \infty. \quad (2.3)$$

More generally, if

$$C = \sup_{n \geq 1} \|n(T^{n-1} - T^n)\|$$

then

$$\|n^r(I - T)^r T^{n-1}\| \leq Cr^r \quad \text{for all } n = 1, 2, \dots, r = 1, 2, \dots \quad (2.4)$$

We say that T has ℓ_p -maximal regularity if the solution of the difference equation

$$\begin{aligned} u_n &= T u_{n-1} + x_n \quad \text{for all } n = 1, 2, \dots, \\ u_0 &= 0 \end{aligned}$$

has the property that $(u_n - u_{n-1})_{n=1}^\infty \in \ell_p(X)$ whenever $(x_n)_{n=1}^\infty \in \ell_p(X)$. This is equivalent to the requirement that there exists a constant C such that

$$\left(\sum_{n=1}^\infty \left\| \sum_{k=1}^n T^{n-k} (T - I)x_k \right\|^p \right)^{1/p} \leq C \left(\sum_{k=1}^\infty \|x_k\|^p \right)^{1/p}. \quad (2.5)$$

This definition was suggested and investigated by Blunck [4] and [5]. It was shown by Blunck [4] that a necessary condition for T to have ℓ_p -maximal regularity for some p is that T satisfies Ritt's condition (2.1).

There is a simple connection between these problems.

PROPOSITION 2.2. *In order that A has L_p -maximal regularity it is necessary and sufficient that the operator $T_h = e^{-hA}$ has ℓ_p -maximal regularity uniformly (that is, with uniform constants) for $0 < h < \infty$.*

PROOF. Suppose that $0 < h < \infty$ and that $(x_n)_{n=1}^\infty \in \ell_p(X)$. Let

$$F(t) = \int_0^t A e^{-(t-s)A} f(s) ds$$

where

$$f = \sum_{k=1}^\infty x_k \chi_{((k-1)h, kh]}.$$

Similarly, let

$$v_n = \sum_{k=1}^n T_h^{n-k} (T_h - I)x_k \quad \text{for all } n = 1, 2, \dots$$

Then

$$F(nh) = -v_n$$

and more generally

$$F((n-1)h + \tau) = -e^{-\tau A} v_{n-1} + (I - e^{-\tau A})x_n \quad \text{for all } 0 < \tau < h.$$

It follows that

$$\|F((n-1)h + \tau)\| \leq M \|v_{n-1}\| + (M+1) \|x_n\| \quad \text{for all } 0 < \tau < h,$$

where $M = \sup_{t>0} \|e^{-tA}\|$.

Now if we assume that T_h has ℓ_p -maximal regularity uniformly in h we obtain a uniform estimate

$$\|F\|_p \leq C\|f\|_p,$$

where C is independent of h and hence A has L_p -maximal regularity.

Conversely, assume that T has L_p -maximal regularity. Then

$$\begin{aligned} v_n &= T_h v_{n-1} + (T_h - 1)x_n \\ &= -e^{-(h-\tau)A}F((n-1)h + \tau) - x_n + e^{-(h-\tau)A}x_n \quad \text{for all } 0 < \tau < h. \end{aligned}$$

Hence

$$\|v_n\| \leq Mh^{-1/p} \left(\int_{(n-1)h}^{nh} \|F(s)\|^p \right)^{1/p} + (M+1)\|x_n\|.$$

Thus

$$\left(\sum_{n=1}^{\infty} \|v_n\|^p \right)^{1/p} \leq CMh^{-1/p}\|f\|_p,$$

which gives a uniform estimate

$$\left(\sum_{n=1}^{\infty} \|v_n\|^p \right)^{1/p} \leq C \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p}. \quad \square$$

The following proposition is essentially contained in [17] but we state the result and give the brief proof here for completeness.

PROPOSITION 2.3. *Let T be a power-bounded operator. Suppose that $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. Then T has ℓ_p -maximal regularity if and only if T^* has ℓ_q -maximal regularity.*

PROOF. Consider the operator $\mathcal{S} : c_{00}(\mathbb{Z}, X) \rightarrow \ell_{\infty}(\mathbb{Z}, X)$ given by

$$(\mathcal{S}(x_j)_{j \in \mathbb{Z}})_n = \sum_{k=-\infty}^n T^{n-k}(T - I)x_k.$$

If $1 \leq p < \infty$, T has ℓ_p -maximal regularity if and only if \mathcal{S} extends to a bounded operator $\mathcal{S} : \ell_p(\mathbb{Z}, X) \rightarrow \ell_p(\mathbb{Z}, X)$. If $p = \infty$ we must consider \mathcal{S} as an operator $\mathcal{S} : c_0(\mathbb{Z}, X) \rightarrow \ell_{\infty}(\mathbb{Z}, X)$.

The formal adjoint $\mathcal{S}^* : c_{00}(\mathbb{Z}, X) \rightarrow \ell_{\infty}(\mathbb{Z}, X^*)$ is given by

$$(\mathcal{S}^*(x_j^*)_{j \in \mathbb{Z}})_n = \sum_{k=n}^{\infty} (T^{k-n}(T - I))^* x_k^*.$$

If we denote by $\mathcal{U} : \ell_{\infty}(\mathbb{Z}, X^*) \rightarrow \ell_{\infty}(\mathbb{Z}, X^*)$ the map

$$\mathcal{U}(x_j^*)_{j \in \mathbb{Z}} = (x_{-j}^*)_{j \in \mathbb{Z}},$$

it is clear that $\mathcal{S}^* = \mathcal{U}\mathcal{S}\mathcal{U}$. From this it is easy to check the result. \square

3. Operators with ℓ_1 or ℓ_∞ -maximal regularity

THEOREM 3.1. *Let T be a power-bounded operator on a Banach space X . Then the following conditions on T are equivalent:*

- (i) T has ℓ_1 -maximal regularity;
- (ii) there is a constant C such that

$$\sum_{k=1}^{\infty} \|(T^k - T^{k-1})x\| \leq C\|x\| \quad \text{for all } x \in X. \quad (3.1)$$

PROOF. Assume that (i) holds. Then (ii) follows trivially from considerations of the sequence $x_1 = x$ and $x_k = 0$ for $k \geq 2$ in (2.5).

Assume that (ii) holds. If $(x_k)_{k=1}^{\infty}$ is any sequence,

$$\sum_{n=1}^{\infty} \sum_{k=1}^n \|T^{n-k}(I - T)x_k\| = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \|T^{j-1}(I - T)x_k\| \leq C \sum_{k=1}^{\infty} \|x_k\|,$$

that is, we have (2.5) for $p = 1$. □

Before proving the corresponding result for ℓ_∞ -maximal regularity, let us record a lemma that we will use several times.

LEMMA 3.2. *Let T be a power-bounded operator on a Banach space X . Suppose that $x \in X$ is such that $\lim_{n \rightarrow \infty} \|T^{n-1}(I - T)x\| = 0$. Then for $x^* \in X^*$,*

$$\begin{aligned} \sum_{k=1}^{\infty} |x^*(T^{k-1}(I - T)x)| &\leq 4 \left(\sum_{k=1}^{\infty} k \|(T^*)^{k-1}(I - T^*)x^*\|^2 \right)^{1/2} \\ &\quad \times \left(\sum_{k=1}^{\infty} k \|T^{k-1}(I - T)x\|^2 \right)^{1/2} \end{aligned} \quad (3.2)$$

and

$$\sum_{k=1}^{\infty} |x^*(T^{k-1}(I - T)x)| \leq 4 \left(\sum_{k=1}^{\infty} \|(T^*)^{k-1}(I - T^*)x^*\| \right) \sup_{k \geq 1} \|kT^{k-1}(I - T)x\|. \quad (3.3)$$

PROOF. Since $\lim_{n \rightarrow \infty} x^*(T^{n-1}(I - T)x) = 0$,

$$x^*(T^{k-1}(I - T)x) = \sum_{j=k}^{\infty} x^*(T^{j-1}(I - T)^2x).$$

Hence

$$\begin{aligned} \sum_{k=1}^{\infty} |x^*(T^{k-1}(I-T)x)| &\leq \sum_{k=1}^{\infty} k|x^*(T^{k-1}(I-T)^2x)| \\ &\leq \sum_{k=1}^{\infty} 2k|x^*(T^{2k-1}(I-T)x)| \\ &\quad + \sum_{k=1}^{\infty} (2k-1)|x^*(T^{k-2}(I-T)x)|. \end{aligned}$$

Now

$$2k|x^*((I-T)^2T^{2k-1}x)| \leq 2k\|((I-T)T^k)^*x^*\|(I-T)T^{k-1}x\|$$

and

$$(2k-1)|x^*((I-T)^2T^{2k-2}x)| \leq (2k-1)\|((I-T)T^{k-1})^*x^*\|(I-T)T^{k-1}x\|.$$

Then (3.2) and (3.3) follow from the Cauchy–Schwarz inequality and the trivial case of Hölder’s inequality. \square

THEOREM 3.3. *Let T be a power-bounded operator. Then the following conditions are equivalent:*

- (i) T has ℓ_∞ -maximal regularity;
- (ii) T satisfies Ritt’s condition (2.1) and there is a constant C so that

$$\|x\| \leq C \left(\sup_{n \geq 1} n\|(T^n - T^{n-1})x\| + \limsup_{n \rightarrow \infty} \|T^n x\| \right). \quad (3.4)$$

PROOF. We prove that (i) implies (ii). Suppose that T has ℓ_∞ -maximal regularity; then T^* has ℓ_1 -maximal regularity and satisfies (3.1) for some constant C . In particular, T^* and T satisfy Ritt’s condition. It follows from (3.3), for any $x^* \in X^*$, $x \in X$ and $N \in \mathbb{N}$, that

$$|x^*(x - T^N x)| \leq C\|x^*\| \sup_{k \geq 1} k\|T^{k-1}(I-T)x\|.$$

Hence

$$\|x - T^N x\| \leq C \sup_{k \geq 1} k\|T^{k-1}(I-T)x\|$$

and (3.4) follows.

Assume that (ii) holds. Suppose that $\|x_k\| \leq 1$ for $1 \leq k \leq n$ and let $y = \sum_{k=1}^n (T^k - T^{k-1})x_k$. For any $m \geq 1$,

$$(T^m - T^{m-1})y = \sum_{k=1}^n (T - I)^2 T^{m+k-2} x_k,$$

so that we have an estimate (using the analyticity of the semigroup and (2.3))

$$\|(T^m - T^{m-1})y\| \leq C_1 \sum_{k=1}^n \frac{1}{(m+k)^2} \leq C_2 m^{-1}$$

for absolute constants C_1, C_2 .

On the other hand,

$$T^m y = (T^m - T^{m-1}) \sum_{k=1}^n T^k x_k$$

so that $\lim_{m \rightarrow \infty} T^m y = 0$. Using (ii) we see that $\|y\| \leq CC_2$ and this proves (i). \square

The continuous analogue of the next theorem is well known (see, for example, [8, Theorem 7.1]).

COROLLARY 3.4. *Suppose that T is an operator that has either ℓ_1 or ℓ_∞ -maximal regularity. Then T has ℓ_p -maximal regularity for every $1 < p < \infty$.*

PROOF. We need only consider the case where T has ℓ_∞ -maximal regularity, since, once this case is done, the other case follows by duality. Suppose that $(x_k)_{k=1}^\infty \in c_{00}(X)$ and

$$y_n = \sum_{k=1}^n T^{n-k} (T - I)x_k \quad \text{for all } 1 \leq n < \infty.$$

Then, for any j ,

$$\begin{aligned} \|jT^{j-1}(I - T)y_n\| &= \left\| \sum_{k=0}^{n-1} jT^{k+j-1}(I - T)^2 x_{n-k} \right\| \\ &\leq C \sum_{k=0}^{n-1} \frac{j}{(k+j)^2} \|x_{n-k}\| \\ &\leq C \max_{1 \leq r \leq n} \frac{1}{r} \sum_{k=n-r+1}^n \|x_k\|. \end{aligned}$$

Now by Theorem 2.1, since $\lim_{j \rightarrow \infty} T^j y_n = 0$,

$$\|y_n\| \leq C \max_{1 \leq r \leq n} \frac{1}{r} \sum_{k=n-r+1}^n \|x_k\|$$

and so

$$\left(\sum_{n=1}^{\infty} \|y_n\|^p \right)^{1/p} \leq C' \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p}$$

by the boundedness of the discrete maximal function on ℓ_p . \square

We next prove the discrete analogues of the results of Baillon [2] and Guerre-Delabrière [9].

THEOREM 3.5. *Suppose that either:*

- (a) *X contains no copy of c_0 and T has ℓ_∞ -maximal regularity; or*
- (b) *X contains no complemented copy of ℓ_1 and T has ℓ_1 -maximal regularity.*

Then X splits as a direct sum $X_1 \oplus X_2$ of T -invariant subspaces such that $T|_{X_1} = I_{X_1}$ and the spectral radius of $T|_{X_2}$ is strictly less than one.

PROOF. (a) We first estimate $\|\sum_{k=1}^n a_k(T^{k-1} - T^k)\|$ if $|a_k| \leq 1$. By Theorem 3.3(ii), for a suitable constant C ,

$$\begin{aligned} & \left\| \sum_{k=1}^n a_k(T^{k-1} - T^k) \right\| \\ & \leq C \left(\sup_{m \geq 1} m \left\| \sum_{k=1}^n a_k T^{k+m-2} (I - T)^2 \right\| + \limsup_{m \rightarrow \infty} \left\| \sum_{k=1}^n a_k (T^{m+k-1} - T^{m+k}) \right\| \right). \end{aligned}$$

The second term reduces to 0 and the first is estimated by

$$C \sup_{m \geq 1} \sup_{k \geq 1} \frac{m}{(m+k-1)^2} \leq C'$$

for some suitable C' . Thus for each $x \in X$ the series $\sum_{k=1}^\infty (T^{k-1} - T^k)x$ is a weakly unconditionally Cauchy series and by the Bessaga-Pełczyński theorem [3] the series converges in norm. Hence $Px = \lim_{n \rightarrow \infty} T^n x$ exists for all $x \in X$ and P is a bounded projection onto the eigenspace $X_1 = \{x \in X \mid Tx = x\}$. Now $(I - T)X$ is dense in the complementary space $X_2 = (I - P)X$ since $T^n x \rightarrow 0$ for $x \in X_2$. We therefore deduce that

$$\lim_{n \rightarrow \infty} n(I - T)T^{n-1}x = 0 \quad \text{for all } x \in X.$$

On X_2 the map $x \rightarrow (n(T^{n-1}x - T^n x))_{n=1}^\infty$ is thus an embedding of X_2 into $c_0(X_2)$. If X_2 contains no copy of c_0 a standard gliding hump argument shows that there exist N and a constant C_1 so that

$$\|x\| \leq C_1 \max_{1 \leq k \leq n} k \|T^{k-1}x - T^k x\| \quad \text{for all } x \in X_2.$$

This implies that

$$\|T^m x\| \leq C_1 \max_{1 \leq k \leq n} k \|T^{m+k-1}x - T^{m+k}x\| \leq C_2 \left(\max_{1 \leq k \leq n} \frac{k}{m+k} \right) \|x\|.$$

Thus $\limsup \|T^m\| < 1$.

(b) If T has ℓ_1 -maximal regularity then $\sum_{k=1}^\infty (T^{k-1} - T^k)x$ converges absolutely for $x \in X$. Thus the projection $Px = \lim_{n \rightarrow \infty} T^n x$ is well defined. We can split

$X = X_1 \oplus X_2$ so that $T|_{X_1} = I_{X_1}$ and X_2 is T -invariant with $\lim_{n \rightarrow \infty} T^n x = 0$ for $x \in X_2$.

To complete the proof, we will reduce to the situation where $\lim_{n \rightarrow \infty} T^n x = 0$ for $x \in X$. If X contains no complemented copy of ℓ_1 then c_0 does not embed into X^* . Since T^* has ℓ_∞ -maximal regularity we can use (a). Suppose $T^* x^* = x^*$; then $x^*(x - Tx) = 0$ for $x \in X$ and this implies that $x^* = 0$. Hence by (a), T^* and hence T has spectral radius less than one. \square

THEOREM 3.6. *Let $-A$ be the generator of a bounded analytic semigroup. The following conditions on A are equivalent:*

- (i) A has L_1 -maximal regularity;
- (ii) there is a constant C so that

$$\int_0^\infty \|Ae^{-tA}x\| dt \leq C\|x\| \quad \text{for all } x \in X. \quad (3.5)$$

PROOF. We prove that (i) implies (ii). If A has maximal regularity then e^{-hA} has ℓ_1 -maximal regularity uniformly for $h > 0$, so that

$$\sum_{k=0}^\infty \|(e^{-khA} - e^{-(k-1)hA})x\| \leq C\|x\| \quad \text{for all } h > 0, x \in X. \quad (3.6)$$

Hence letting $h \rightarrow 0$, we obtain (3.5).

Assume that (ii) holds. Equation (3.5) trivially implies (3.6). \square

THEOREM 3.7. *Let $-A$ be the generator of a bounded analytic semigroup. The following conditions on A are equivalent:*

- (i) A has L_∞ -maximal regularity;
- (ii) there is a constant C so that

$$\|x\| \leq C \sup_{t>0} \|tAe^{-tA}x\| + \limsup_{t \rightarrow \infty} \|e^{-tA}x\| \quad \text{for all } x \in X. \quad (3.7)$$

REMARK. If A has dense range then $\lim_{t \rightarrow \infty} e^{-tA}x = 0$ for every $x \in X$ and we can drop the last term.

PROOF. Assume that (i) holds. Then (ii) is very similar to the preceding theorem case (i).

Assume that (ii) holds. Observe that if $f \in L_\infty(\mathbb{R}_+, X)$ with $\|f\|_\infty \leq 1$, then

$$\begin{aligned} \left\| \int_0^t Ae^{-(t-s)A} f(s) ds \right\| &\leq C \sup_{\tau>0} \left\| \int_0^t \tau A^2 e^{-(t+\tau-s)A} f(s) ds \right\| \\ &\leq C_1 \sup_{\tau>0} \int_0^t \frac{\tau}{(t+\tau-s)^2} ds \\ &\leq C_2, \end{aligned}$$

where C_1, C_2 are suitable constants depending on A . \square

At this point let us remark that it is now easy to recover the results of Da Prato and Grisvard [7] on L_∞ -maximal regularity in real interpolation spaces. Da Prato and Grisvard consider maximal regularity on a finite interval which is equivalent to maximal regularity on the infinite interval for $s + A$ for some $s > 0$. Thus it is enough to consider the case of an invertible operator.

Let us consider the real interpolation space $(X, \text{Dom}(A))_{(\theta, \infty)}$ for $0 < \theta < 1$ which is defined by the norm

$$\|x\|_{(\theta, \infty)} = \sup_{t>0} t^{-\theta} K(t, x)$$

where

$$K(t, x) = K(t, x; X, \text{Dom}(A)) = \inf\{\|y\| + t\|Az\| : y + z = x\}.$$

The space $(X, \text{Dom}(A))_{(\theta, \infty)}$ can be given several equivalent norms, see [7] and [12]; we will need one of these which we now describe for completeness. If $x = y + z$,

$$\|tAe^{-tA}x\| \leq \|tAe^{-tA}\|\|y\| + \|e^{-tA}\|\|tAz\|,$$

so that

$$\|tAe^{-tA}x\| \leq CK(t, x).$$

On the other hand,

$$\begin{aligned} K(t, x) &\leq \|x - e^{-tA}x\| + \|tAe^{-tA}x\| \\ &\leq \int_0^t \|Ae^{-sA}x\| ds + \|tAe^{-tA}x\|. \end{aligned}$$

If $t \geq 1$, then

$$K(t, x) \leq \|x\|,$$

while if $0 < t < 1$, then

$$K(t, x) \leq 2\theta^{-1}t^\theta \sup_{0 < s < 1} s^{1-\theta} \|Ae^{-sA}x\|.$$

We may pick $\tau > 0$ so that $\|e^{-\tau A}\| < 1/2$. Then

$$\|x\| \leq \|e^{-\tau A}x\| + \int_0^\tau \|Ae^{-sA}x\| ds$$

so that

$$\|x\| \leq 2\theta^{-1}\tau^\theta \sup_{0 < s < \tau} s^{1-\theta} \|Ae^{-sA}x\|.$$

Combining these remarks, we see that $\|\cdot\|_{\theta,\infty}$ is equivalent to

$$\|x\|_0 = \sup_{t>0} t^{1-\theta} \|Ae^{-tA}x\|.$$

Now $-A$ generates a bounded analytic semigroup on the space Y_θ which is defined to be the closure of $\text{Dom}(A)$ in $(X, \text{Dom}(A))_{(\theta,\infty)}$.

THEOREM 3.8 (Da Prato and Grisvard [7]). *A has L_∞ -maximal regularity on Y_θ .*

PROOF. For $x \in Y$,

$$\sup_{t>0} \|Ae^{-tA}x\|_0 = \sup_{s,t>0} s^{1-\theta} t \|A^2 e^{-(s+t)A}x\|.$$

Hence for a suitable constant,

$$\sup_{t>0} t^{2-\theta} \|A^2 e^{-tA}x\| \leq C \sup_{t>0} \|Ae^{-tA}x\|_0.$$

Now

$$\|Ae^{-tA}x\| \leq \int_t^\infty \|A^2 e^{-sA}x\| ds \leq (1-\theta)^{-1} t^{\theta-1} \sup_{t>0} t^{2-\theta} \|A^2 e^{-tA}x\|,$$

so that

$$\|x\|_0 \leq C(1-\theta)^{-1} \sup_{t>0} \|Ae^{-tA}x\|_0. \quad \square$$

4. The unconditional Ritt condition

In this section we study the discrete analogue of the H^∞ -calculus for sectorial operators which was introduced by McIntosh [14].

Before proceeding, we develop some basic ideas which will be useful later. Assume that T satisfies the Ritt condition. For any $m \geq 0$ we consider the operator V_m defined by

$$V_m = \sum_{k=0}^{\infty} c_k (T^{km} - T^{(k+1)m}) \quad (4.1)$$

where

$$c_k = \frac{(2k)!}{2^{2k}(k!)^2}.$$

Note that there is a constant M so that $|c_k| \leq M/\sqrt{k}$ for $k \geq 1$, so that it follows from (2.2) that the series in (4.1) converges absolutely. Of course $V_0 = 0$.

LEMMA 4.1. *For $m \geq 1$ we have $V_m^2 = I - T^m$.*

PROOF. Consider the function

$$F_m(t) = \sum_{k=0}^{\infty} c_k (t^{km} T^{km} - t^{(k+1)m} T^{(k+1)m}) \quad \text{for all } 0 \leq t \leq 1.$$

Since

$$\begin{aligned} t^{km} T^{km} - t^{(k+1)m} T^{(k+1)m} &= t^{km} (T^{km} - T^{(k+1)m}) \\ &\quad + (t^{km} - t^{(k+1)m}) T^{(k+1)m} \quad \text{for all } 0 \leq t \leq 1, \end{aligned}$$

it follows that the series on the right converges uniformly to $F_m(t)$ for $0 \leq t \leq 1$. If $0 < t < 1$,

$$F_m(t) = (I - t^m T^m) \sum_{k=0}^{\infty} c_k t^{km} T^{km},$$

and as

$$(1 - z)^{-1/2} = \sum_{k=0}^{\infty} c_k z^k \quad \text{for all } |z| < 1,$$

we deduce that

$$F_m(t)^2 = I - t^m T^m \quad \text{for all } 0 < t < 1.$$

Letting t tend to 1, and using uniform convergence, we deduce that

$$V_m^2 = I - T^m. \quad \square$$

LEMMA 4.2. *Suppose that T satisfies the Ritt condition and define*

$$\rho_k(x) = \max_{\substack{2^k - 1 \leq u \leq 2^{k+1} - 1 \\ 2^k - 1 \leq v \leq 2^{k+2} - 1}} \| (T^u - T^v)x \| \quad \text{for all } x \in X, k = 0, 1, 2, \dots$$

and

$$\sigma_k(x) = \max_{2^k - 1 \leq n < 2^{k+1} - 1} \| T^n - T^{n+1}x \| \quad \text{for all } x \in X, k = 0, 1, 2, \dots$$

Then $\rho_0(x) = \sigma_0(x) = \|(I - T)x\|$ and, in general,

$$\rho_k(x) \leq 2^k \sigma_k(x) + 2^{k+1} \sigma_{k+1}(x) \quad \text{for all } x \in X, k = 0, 1, 2, \dots, \quad (4.2)$$

$$2^k \sigma_k(x) \leq C(\rho_k(x) + \rho_{k-1}(x)) \quad \text{for all } x \in X, k = 1, 2, \dots, \quad (4.3)$$

for a suitable constant C .

PROOF. Inequality (4.2) is trivial. Next, suppose that $2^k - 1 \leq m < n < 2^{k+1} - 1$ where $k \geq 1$. Then pick m' with $m - m' \geq 2^{k-1}$ and $2^{k-1} - 1 \leq m' < 2^k - 1$. Then

$$(T^m - T^{m+1})x - (T^n - T^{n+1})x = T^{m-m'}(I - T)(T^{m'} - T^{m'+n-m})x,$$

so that if $C_1 = \sup_{n \geq 1} n \|T^n - T^{n+1}\|$, then

$$\begin{aligned} \|(T^m - T^{m+1})x - (T^n - T^{n+1})x\| &\leq 2C_1 2^{-k} \|(T^{m'} - T^{m'+n-m})x\| \\ &\leq 2C_1 2^{-k} \rho_{k-1}(x). \end{aligned}$$

Summing gives

$$\|(T^{2^k-1} - T^{2^{k+1}-1})x - 2^k(T^n - T^{n+1})x\| \leq 2C_3 \rho_{k-1}(x),$$

and so

$$2^k \sigma_k(x) \leq 2C_3 \rho_{k-1}(x) + \rho_k(x). \quad \square$$

Let us say that an operator T satisfies the *unconditional Ritt condition* if there is a constant C such that

$$\left\| (I - T) \sum_{k=0}^N a_k T^k \right\| \leq C \max_{0 \leq k \leq N} |a_k| \quad \text{for all } a_0, \dots, a_N \in \mathbb{C}, N = 1, 2, \dots \quad (4.4)$$

This is easily seen to be equivalent to the condition

$$\sum_{k=1}^{\infty} |x^*(T^{k-1}(I - T)x)| \leq C \|x\| \|x^*\| \quad \text{for all } x \in X, x^* \in X^*. \quad (4.5)$$

The unconditional Ritt condition is a discrete analogue of the existence of an H^∞ -calculus with angle less than $\pi/2$ for a sectorial operator (see [14] and [6]). We will discuss the connection at the end of the paper.

PROPOSITION 4.3. *If T satisfies the unconditional Ritt condition (4.4) then T satisfies the Ritt condition (2.1).*

PROOF. From (4.4) we deduce that

$$\|(I - T)(I - \lambda^{-1}T)^{-1}\| \leq C \quad \text{for all } |\lambda| > 1,$$

that is,

$$\|(I - T)R(\lambda, T)\| \leq C |\lambda|^{-1} \quad \text{for all } |\lambda| > 1.$$

Hence

$$\|(1 - \lambda)R(\lambda, T)\| \leq 1 + C |\lambda|^{-1} \quad \text{for all } |\lambda| > 1. \quad \square$$

Now let $(r_k)_{k=0}^\infty$ and $(m_k)_{k=0}^\infty$ be any pair of sequences of integers such that

$$0 \leq m_k < 2^{k+3} - 1, \quad 2^k - 1 \leq r_k < 2^{k+1} - 1 \quad \text{for all } k = 0, 1, \dots \quad (4.6)$$

LEMMA 4.4. *Suppose that T satisfies the unconditional Ritt condition. Then there is a constant C such that for any pair of sequences $(r_k)_{k=0}^\infty$ and $(m_k)_{k=0}^\infty$ satisfying (4.6),*

$$\left\| \sum_{k=0}^N a_k T^{r_k} V_{m_k} \right\| \leq C \max_{0 \leq k \leq N} |a_k|. \quad (4.7)$$

PROOF. Suppose that $\max_{0 \leq k \leq N} |a_k| \leq 1$ and $x \in X$ and $x^* \in X^*$ with $\|x\| = \|x^*\| = 1$. Then

$$\begin{aligned} \left| x^* \left(\sum_{k=0}^N a_k T^{r_k} V_{m_k} x \right) \right| &\leq \sum_{k=0}^N |x^*(T^{r_k} V_{m_k} x)| \\ &\leq \sum_{k=0}^N \sum_{l=0}^\infty c_l |x^*((T^{r_k+m_k l} x - T^{r_k+m_k(l+1)} x))| \\ &= \sum_{l=1}^\infty d_j |x^*((T^j - T^{j+1})x)|, \end{aligned}$$

where

$$d_j = \sum_{r_k \leq j} c_{[j-r_k/m_k]}.$$

Assume that $2^s - 1 \leq j < 2^{s+1} - 1$. Then $r_k \leq j$ implies that $k \leq s$. If $k \leq s-2$, then $[j - r_k/m_k] > 2^{s-1}/m_k \geq 2^{s-4-k}$. Thus,

$$d_j \leq 2 + M \sum_{k=0}^{s-2} 2^{(k-s+4)/2}.$$

Thus we get an estimate

$$d_j \leq C_1 M$$

where C_1 is an absolute constant. Hence

$$\left| x^* \left(\sum_{k=0}^N a_k T^{r_k} V_{m_k} x \right) \right| \leq C_0 C_1 M$$

where C_0 is the constant in (4.4) and thus

$$\left\| \sum_{k=0}^N a_k T^{r_k} V_{m_k} \right\| \leq C_0 C_1 M. \quad \square$$

The following result is the discrete analogue of a similar result for sectorial operators with an H^∞ -calculus proved in [11]. We recall that a Banach space X is called a GT-space (for Grothendieck theorem) if every bounded operator $T : X \rightarrow \ell_2$ is absolutely summing. See [16].

THEOREM 4.5. *Let X be a GT-space (for example, $X = L_1$, ℓ_1 or $X = L_1/H_1$). Let $T : X \rightarrow X$ be any operator. Then T has ℓ_1 -maximal regularity if and only if T satisfies the unconditional Ritt condition.*

PROOF. Assume that T satisfies the unconditional Ritt condition (4.4). Let C_1 be the constant in (4.7).

Suppose that $(u_k)_{k=0}^\infty$ and $(v_k)_{k=0}^\infty$ are two sequences of natural numbers such that

$$2^k - 1 \leq u_k \leq 2^{k+1} - 1 \quad \text{and} \quad 2^k - 1 \leq v_k \leq 2^{k+2} - 1.$$

For $k \geq 0$ we write $u_{k+1} = r_k + s_k$ where $0 \leq r_k - s_k \leq 1$ and $m_k = v_k - u_k$. Thus $2^k + 1 \leq r_k$, $s_k \leq 2^{k+1} - 1$ and $0 \leq m_k < 2^{k+3}$, that is, $(r_k)_{k=0}^\infty$ and $(m_k)_{k=0}^\infty$ satisfy (4.6).

At this point we use the hypothesis that X is a GT-space, which means that there is a constant K so that for any operator $T : X \rightarrow \ell_2$ we have $\pi_1(T) \leq K \|T\|$ where $\pi_1(T)$ is the usual absolutely summing norm. For any $x_0^*, x_1^*, \dots, x_N^* \in X^*$ with $\|x_k^*\| \leq 1$ and any $a_0, \dots, a_N \in \mathbb{C}$ with $\sum_{k=0}^N |a_k|^2 \leq 1$, consider the operator $S : X \rightarrow \ell_2^{N+1}$ defined by

$$Sx = (a_k x_k^*(x))_{k=0}^N.$$

Then $\|S\| \leq 1$ and so $\pi_1(S) \leq K$. Hence for any $x \in X$,

$$\sum_{k=0}^N \|ST^{r_k} V_{m_k} x\| \leq K C_1 \|x\|.$$

In particular,

$$\sum_{k=0}^N |a_k| |x_k^*(T^{r_k} V_{m_k} x)| \leq K C_1 \|x\|.$$

Since this is true for all such choices of $(a_k)_{k=0}^N$ and $(x_k^*)_{k=0}^N$,

$$\left(\sum_{k=0}^N \|T^{r_k} V_{m_k} x\|^2 \right)^{1/2} \leq K C_1 \|x\| \quad \text{for all } x \in X.$$

Now, for any fixed $x_0^*, \dots, x_N^* \in X^*$ with $\|x_k^*\| \leq 1$, consider the operator $R : X \rightarrow \ell_2^{N+1}$ defined by

$$Rx = (x_k^*(T^{s_k} V_{m_k} x))_{k=0}^N$$

and observe that

$$\pi_1(R) \leq K \|R\| \leq K^2 C_1.$$

It thus follows that

$$\sum_{k=0}^N \|RT^{r_k} V_{m_k} x\| \leq K^2 C_1^2 \|x\| \quad \text{for all } x \in X,$$

and, as before,

$$\sum_{k=0}^N |x_k^* (T^{r_k+s_k} V_{m_k}^2 x)| \leq K^2 C_1^2 \|x\| \quad \text{for all } x \in X.$$

Again this implies that

$$\sum_{k=0}^N \|T^{u_k} (I - T^{m_k}) x\| \leq K^2 C_1^2 \|x\| \quad \text{for all } x \in X.$$

We conclude that if $(u_k)_{k=0}^\infty$ and $(v_k)_{k=0}^\infty$ satisfy $2^k \leq u_k < 2^{k+1}$ and $u_k \leq v_k \leq 2^{k+2}$, then

$$\sum_{k=1}^\infty \|(T^{u_k} - T^{v_k})x\| \leq C_2 \|x\| \quad \text{for all } x \in X,$$

where $C_2 = K^2 C_1^2$.

Thus we have an estimate

$$\sum_{k=1}^\infty \rho_k(x) \leq C_2 \|x\| \quad \text{for all } x \in X,$$

which implies by Lemma 4.2 an estimate

$$\sum_{k=0}^\infty 2^k \sigma_k(x) \leq C_3 \|x\| \quad \text{for all } x \in X,$$

and hence that

$$\sum_{k=1}^\infty \|T^{k-1} x - T^k x\| \leq C_3 \|x\|.$$

The result now follows by Theorem 3.1.

The converse direction is trivial. □

The dual result is now easy.

THEOREM 4.6. *Let X be a Banach space such that X^* is a GT-space (for example, X is a $C(K)$ space or the disc algebra). If T is a power-bounded operator on X then T has ℓ_∞ -maximal regularity if and only if T satisfies the unconditional Ritt condition.*

Finally we establish a corresponding result for Hilbert spaces. The continuous analogue which we discuss later is due to McIntosh [14]. See also further discussion in [6] and [1].

THEOREM 4.7. *Let T be a power-bounded operator on a Hilbert space H . Then T satisfies the unconditional Ritt condition if and only if there is a constant C such that*

$$C^{-1}\|x\| \leq \left(\sum_{k=1}^{\infty} k \|T^{k-1}x - T^kx\|^2 \right)^{1/2} + \limsup_{n \rightarrow \infty} \|T^n x\| \leq C\|x\| \quad \text{for all } x \in H. \quad (4.8)$$

PROOF. Suppose that T satisfies the unconditional Ritt condition with constant C_0 . We first observe that for every $x \in H$ the series $\sum_{k=1}^{\infty} (T^{k-1}x - T^kx)$ is weakly unconditionally Cauchy and hence unconditionally convergent to some Px where P is a projection whose kernel is the eigenspace $\{x \mid Tx = x\}$. We may therefore easily reduce to the case where $\lim_{n \rightarrow \infty} T^n x = 0$ for every $x \in H$.

Then, for any pair of sequences $(u_k)_{k=0}^{\infty}, (v_k)_{k=0}^{\infty}$ with $2^k - 1 \leq u_k \leq 2^{k+1} - 1$ and $2^k - 1 \leq v_k \leq 2^{k+2} - 1$, we have an estimate

$$\left\| \sum_{k=0}^N \epsilon_k (T^{u_k} - T^{v_k}) \right\| \leq 2C_0 \quad \text{for all } \epsilon_k = \pm 1, k = 1, 2, \dots, N,$$

and it follows from the generalized parallelogram law that

$$\left(\sum_{k=0}^{\infty} \|(T^{u_k} - T^{v_k})x\|^2 \right)^{1/2} \leq 2C_0\|x\| \quad \text{for all } x \in H. \quad (4.9)$$

Thus

$$\left(\sum_{k=0}^{\infty} \rho_k(x)^2 \right)^{1/2} \leq 2C_0\|x\| \quad \text{for all } x \in H. \quad (4.10)$$

Now by Lemma 4.2 we can deduce an estimate

$$\left(\sum_{k=0}^{\infty} k \|T^{k-1}x - T^kx\|^2 \right)^{1/2} \leq C_1\|x\| \quad \text{for all } x \in H,$$

for a suitable constant C_1 . Thus the right-hand side of (4.8) follows. Note that the same inequality also holds for the adjoint T^* , that is,

$$\left(\sum_{k=0}^{\infty} k \|(T^*)^{k-1}x - (T^*)^kx^*\|^2 \right)^{1/2} \leq C_1\|x^*\| \quad \text{for all } x^* \in H.$$

We now turn to the left-hand estimate. If $x \in H$, pick $x^* \in H^*$ with $\|x^*\| = 1$ and $x^*(x) = \|x\|$. Then, since we assume $\lim_{n \rightarrow \infty} \|T^n x\| = 0$,

$$\begin{aligned} \|x\| &= x^*(x) \\ &\leq \sum_{k=1}^{\infty} |x^*(T^{k-1}(I - T)x)| \\ &\leq 4C_1 \left(\sum_{k=0}^{\infty} k \|T^{k-1}x - T^k x\|^2 \right)^{1/2} \end{aligned}$$

by an application of (3.2) combined with (4.9) and (4.10).

We now turn to the converse. Assuming (4.8), let us first show that T satisfies the Ritt condition. Note that we have $\lim_{n \rightarrow \infty} \|T^{n-1}(I - T)x\| = 0$ for every x . Therefore

$$\begin{aligned} \|nT^{n-1}(I - T)x\|^2 &\leq C^2 \sum_{k=1}^{\infty} n^2 k \|T^{n+k-2}(I - T)^2 x\|^2 \\ &\leq C^2 \sum_{k=1}^{\infty} k^3 \|T^{k-1}(I - T)^2 x\|^2 \\ &\leq 6C^2 \sum_{k=1}^{\infty} \sum_{j=1}^k j(k+1-j) \|T^{k-1}(I - T)^2 x\|^2 \\ &= 6C^2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} jk \|T^{j+k-2}(I - T)^2 x\|^2 \\ &\leq 6C^4 \sum_{j=1}^{\infty} \|T^{j-1}(I - T)x\|^2 \\ &\leq 6C^6 \|x\|^2, \end{aligned}$$

so that

$$\|T^{n-1}(I - T)\| \leq \sqrt{6}C^3/n.$$

Thus T satisfies the Ritt condition (2.1).

Now suppose that $(x_k)_{k=1}^{\infty}$ is any finitely nonzero sequence in H . Let

$$y = \sum_{k=1}^{\infty} k^{1/2} (T^{k-1} - T^k)x_k.$$

Note that $\lim_{n \rightarrow \infty} T^n y = 0$. Then

$$\begin{aligned} \|j^{1/2}(T^{j-1} - T^j)y\| &\leq \sum_{k=1}^{\infty} (jk)^{1/2} \|T^{j+k-2}(I - T)^2 x_k\| \\ &\leq C_0 \sum_{k=1}^{\infty} \frac{(jk)^{1/2}}{(j+k)^2} \|x_k\| \\ &\leq C_0 \sum_{k=1}^{\infty} \frac{1}{j+k} \|x_k\|. \end{aligned}$$

The matrix $a_{jk} = (1/(j+k))_{j,k}$ defines a bounded operator on ℓ_2 by Hilbert's inequality. Thus, for a suitable constant C_1 ,

$$\left(\sum_{j=1}^{\infty} j \| (T^{j-1} - T^j)y \|^2 \right)^{1/2} \leq C_1 \left(\sum_{k=1}^{\infty} \|x_k\|^2 \right)^{1/2}.$$

We conclude from (4.8) that

$$\|y\| \leq C_1 \left(\sum_{k=1}^{\infty} \|x_k\|^2 \right)^{1/2}.$$

Now suppose that $x^* \in H^*$. For $N \in \mathbb{N}$, pick x_1, \dots, x_N with

$$\|x_k\| = 1 \quad \text{and} \quad x^*(T^{k-1}(I - T)x) = \|(T^*)^{k-1}(I - T^*)x^*\|.$$

Then for any scalars a_1, \dots, a_N ,

$$\begin{aligned} \sum_{k=1}^N a_k k^{1/2} \|(T^*)^{k-1}(I - T^*)x^*\| &\leq \|x^*\| \left\| \sum_{k=1}^N a_k k^{1/2} T^{k-1}(I - T)x_k \right\| \\ &\leq C_1 \|x^*\| \left(\sum_{k=1}^N |a_k|^2 \right)^{1/2}. \end{aligned}$$

Hence

$$\left(\sum_{k=1}^{\infty} k \|(T^*)^{k-1}(I - T^*)x^*\|^2 \right)^{1/2} \leq C_1 \|x^*\|.$$

At this point we can appeal to (3.2):

$$\sum_{k=1}^{\infty} |x^*(T^{k-1}(I - T)x)| \leq C_2 \|x\| \|x^*\| \quad \text{for all } x \in H, x^* \in H^*.$$

This implies the unconditional Ritt condition. \square

Now suppose again that $-A$ is the generator of a bounded analytic semigroup on a Banach space X , and suppose also for convenience that A has dense domain and range (that is, A is sectorial). Then $\lim_{t \rightarrow \infty} \|e^{-tA}x\| = 0$ for $x \in X$. The continuous version of the unconditional Ritt condition is

$$\int_0^\infty |x^*(Ae^{-tA}x)| dt \leq C\|x\|\|x^*\| \quad \text{for all } x \in X, x^* \in X^*. \quad (4.11)$$

Let us call this the *continuous unconditional Ritt condition*. If (4.11) holds then e^{-tA} uniformly satisfies the unconditional Ritt condition.

We recall that A has an $H^\infty(\Sigma_\psi)$ -calculus where $\Sigma_\psi = \{z : |\arg z| < \psi\}$ if $f(A)$ is a bounded operator for every $f \in H^\infty(\Sigma_\psi)$; see [6] for more details. Let $\omega(A)$ be the infimum of all ϕ so that we have the resolvent estimates

$$\|\lambda R(\lambda, A)\| \leq C, \quad |\arg z| \geq \phi,$$

and let $\omega_H(A)$ be the infimum of all ϕ so that A has an $H^\infty(\Sigma_\phi)$ -calculus.

It is easy to show that if $\omega_H(A) < \pi/2$ then A satisfies the continuous unconditional Ritt condition.

Conversely, it follows from [6, Theorem 4.5] that if A satisfies the continuous unconditional Ritt condition, then A has an $H^\infty(\Sigma_\psi)$ -calculus as long as $\psi > \pi/2$; thus $\omega_H(A) \leq \pi/2$. If X is a Hilbert space, then results of McIntosh [14] imply that $\omega_H(A) = \omega(A) < \pi/2$. One cannot apply this argument for an arbitrary Banach space [10]. Thus it is open whether the continuous Ritt condition is equivalent to $\omega_H(A) < \pi/2$.

It is easy to prove continuous versions of Theorems 4.5, 4.6 and 4.7 as in Theorem 3.6.

THEOREM 4.8. *Let A be the generator of a bounded analytic semigroup with dense domain and range. Then:*

- (i) *if X is a GT-space then A satisfies the continuous unconditional Ritt condition if and only if there is a constant C so that*

$$C^{-1}\|x\| \leq \int_0^\infty \|Ae^{-tA}x\| dt \leq C\|x\| \quad \text{for all } x \in X;$$

- (ii) *if X^* is a GT-space then A satisfies the continuous unconditional Ritt condition if and only if there is a constant C so that*

$$C^{-1}\|x\| \leq \sup_{t>0} t\|Ae^{-tA}x\| \leq C\|x\| \quad \text{for all } x \in X;$$

- (iii) *if X is a Hilbert space then A satisfies the continuous unconditional Ritt condition if and only if there is a constant C so that*

$$C^{-1}\|x\| \leq \left(\int_0^\infty t\|Ae^{-tA}x\|^2 dt \right)^{1/2} \leq C\|x\| \quad \text{for all } x \in X.$$

In view of our remarks above, (iii) is simply a special case of the result of McIntosh [14] on the equivalence of the H^∞ -calculus with certain quadratic estimates. Similarly, (i) is a close relative of [11, Proposition 7.2].

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