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REGULARITY OF A PARABOLIC EQUATION SOLUTION IN A NONSMOOTH AND UNBOUNDED DOMAIN

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Abstract

This work is concerned with the problem

$$\begin{cases} \partial_t u - c(t)\partial_x^2 u = f \\ u_{|\partial D \setminus \Gamma_T} = 0 \end{cases}$$

posed in the domain

$$D = \{(t, x) \in \mathbb{R}^2 \mid 0 < t < T, \varphi_1(t) < x < \varphi_2(t)\},$$

which is not necessary rectangular, and with

$$\Gamma_T = \{(T, x) \mid \varphi_1(T) < x < \varphi_2(T)\}.$$

Our goal is to find some conditions on the coefficient c and the functions $(\varphi_i)_{i=1,2}$ such that the solution of this problem belongs to the Sobolev space

$$H^{1,2}(D) = \{u \in L^2(D) \mid \partial_t u \in L^2(D),\ \partial_x u \in L^2(D),\ \partial_x^2 u \in L^2(D)\}.$$

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1. Introduction

In the domain

$$D = \{(t, x) \in \mathbb{R}^2 \mid 0 < t < T, \varphi_1(t) < x < \varphi_2(t)\},$$

we consider the problem

$$\begin{cases} \partial_t u - c(t)\partial_x^2 u = f \\ u_{|\partial D \setminus \Gamma_T} = 0, \end{cases} \tag{P_0}$$

where:

(i) $\Gamma_T = \{(T, x) \mid \varphi_1(T) < x < \varphi_2(T)\}$;
(ii) c is a positive coefficient depending on time;
(iii) $(\varphi_i)_{i=1,2}$ and c are differentiable functions on $]0, T[$ satisfying some assumptions to be made precise later on.

The second member f of the equation will be taken in the Lebesgue space $L^2(D)$. We look for a solution u of problem (P_0) in the anisotropic Sobolev space

$$H^{1,2}(D) = \{u \in L^2(D) : \partial_t u \in L^2(D),\ \partial_x u \in L^2(D),\ \partial_x^2 u \in L^2(D)\}.$$

The study of this kind of problems when the coefficient c is constant and $T < +\infty$ has been treated in [19]. In [13], the authors investigated the case when

$$\begin{cases} f \text{ is in a non-Hilbertian Lebesgue space } \mathrm{L}^p(\mathrm{D}) \\ c = 1 \\ T < +\infty \\ \varphi_1 = 0 \text{ and } \varphi_2(t) = t^\alpha, \end{cases}$$

they found some conditions on the exponents α and p assuring the optimal regularity of the solution of problem (P_0). It is possible to consider similar questions with some other operators (see, for example, [11, 12]).

Observe that the case where the domain D is cylindrical and $T < +\infty$ is known, for example, in [15] or [1] when the coefficient c is not regular.

During the last decades numerous authors have been interested in the study of many problems posed in bad domains. Among these we can cite [2, 3, 5–11, 16–18, 20]. For bibliographical references see, for example, those of books by [4–7] and the references therein.

In this paper we are interested in particular in the case $T = +\infty$, $\varphi_1(0) = \varphi_2(0)$ and c depends on the time. Our main result shows that, thanks to some assumptions on the functions $(\varphi_i)_{i=1,2}$ and c, problem (P_0) has a (unique) solution u with optimal regularity, that is $u \in H^{1,2}(D)$ when

$$D = \{(t, x) \in \mathbb{R}^2 \mid 0 < t < +\infty,\ \varphi_1(t) < x < \varphi_2(t)\},$$

and $\varphi_1(0) = \varphi_2(0)$. The proof of this result will be undertaken in four steps:
(1) case of a bounded domain which can be transformed into a rectangle;
(2) case of an unbounded domain which can be transformed into a half strip;
(3) case of a bounded triangular domain;
(4) case of a sectorial domain.

2. The case of a bounded domain which can be transformed into a rectangle

Let us consider the problem

$$\begin{cases} \partial_t u - c(t)\partial_x^2 u = f \in L^2(D_1) \\ u_{|\partial D_1 \setminus \Gamma_T} = 0, \end{cases} \qquad (P_1)$$

where

$$D_1 = \{(t, x) \in \mathbb{R}^2 \mid 0 < t < T,\ \varphi_1(t) < x < \varphi_2(t)\},$$

with the following hypotheses on the functions $(\varphi_i)_{i=1,2}$ and c:

(i) $\begin{cases} (\varphi_i)_{i=1,2} \text{ and } c \text{ are continuous functions on } [0, T], \text{ differentiable on }]0, T[; \\ \text{the derivatives } (\varphi_i')_{i=1,2} \text{ are uniformly bounded;} \end{cases}$

(ii) there exist two constants $\alpha_i > 0$, $i = 1, 2$, such that $\alpha_1 \geq c(t) \geq \alpha_2$, for all $t \in [0, T]$;

(iii) $\varphi_1(t) < \varphi_2(t)$, for all $t \in [0, T]$;

(iv) $T < +\infty$.

Let (H_1) denote these conditions.

The change of variables (t, x) to $(t, (x - \varphi_1(t))/(\varphi_2(t) - \varphi_1(t)))$ transforms D_1 into $R =]0, T[\times]0, 1[$ and problem (P_1) becomes

$$\begin{cases} \partial_t u + a(t, x)\partial_x u - b(t)\partial_x^2 u = f \in L^2(R) \\ u_{|\partial R \setminus \{T\} \times]0,1[} = 0, \end{cases} \qquad (P_1')$$

where

$$a(t, x) = -\frac{x(\varphi_2'(t) - \varphi_1'(t)) + \varphi_1'(t)}{\varphi_2(t) - \varphi_1(t)},$$

and

$$b(t) = \frac{c(t)}{(\varphi_2(t) - \varphi_1(t))^2}.$$

Observe that, thanks to hypothesis (H_1), the coefficient a is bounded. So the operator $a(t, x)\partial_x : H^{1,2}(R) \to L^2(R)$ is compact. Hence, it is sufficient to study the following problem

$$\begin{cases} \partial_t u - b(t)\partial_x^2 u = f \in L^2(R) \\ u_{|\partial R \setminus \{T\} \times]0,1[} = 0. \end{cases} \qquad (P_1'')$$

It is clear that problem (P_1'') admits a (unique) solution $u \in H^{1,2}(R)$ because the coefficient b satisfies the 'uniform parabolicity' condition (see, for example, [1]). On other hand, it is easy to verify that the change of variables (t, x) to $(t, (x - \varphi_1(t))/(\varphi_2(t) - \varphi_1(t)))$ conserves the spaces L^2 and $H^{1,2}$. Consequently, we have the following theorem.

THEOREM 1. *If hypothesis (H_1) is satisfied, problem (P_1) admits a (unique) solution $u \in H^{1,2}(D_1)$ in D_1.*

The uniqueness of the solution may be obtained by developing the scalar product $(\partial_t u - c(t)\partial_x^2 u, u)_{L^2(D_1)}$. Indeed, we prove that the condition $\partial_t u - c(t)\partial_x^2 u = 0$ implies $\partial_x u = 0$. Thus, $\partial_x^2 u = 0$. However, $\partial_t u - c(t)\partial_x^2 u = 0$ leads to $\partial_t u = 0$. So u is constant and the boundary conditions give $u = 0$.

3. The case of an unbounded domain which can be transformed into a half strip

Now, let us consider the problem

$$\begin{cases}\partial_t u - c(t)\partial_x^2 u = f \in L^2(D_2)\\ u_{|\partial D_2} = 0,\end{cases} \tag{P_2}$$

where

$$D_2 = \{(t, x) \in \mathbb{R}^2 \mid 0 < t < +\infty,\ \varphi_1(t) < x < \varphi_2(t)\},$$

and let (H_2) denote the following conditions on the functions $(\varphi_i)_{i=1,2}$ and c:

(i) $\begin{cases}(\varphi_i)_{i=1,2} \text{ and } c \text{ are continuous functions on } [0, +\infty[, \text{ differentiable on}\\ ]0, +\infty[; \text{ the derivatives } (\varphi_i)_{i=1,2} \text{ are uniformly bounded;}\end{cases}$

(ii) there exist $\alpha_i > 0$, $i = 1, 2$ such that $\alpha_1 \geq c(t) \geq \alpha_2 > 0$, for all $t \in [0, +\infty[$;

(iii) $\varphi_2 - \varphi_1$ is increasing in a neighborhood of $+\infty$; or:
there exists $M > 0$ such that $|\varphi_1'(t) - \varphi_2'(t)|(\varphi_2(t) - \varphi_1(t)) \leq M.c(t)$;

(iv) $\varphi_1(0) < \varphi_2(0)$.

The change of variables indicated in the previous section transforms D_2 into the half strip $B =]0, +\infty[\times]0, 1[$. So problem (P_2) can be written as follows

$$\begin{cases}\partial_t u + a(t, x)\partial_x u - b(t)\partial_x^2 u = f \in L^2(B)\\ u_{|\partial B} = 0,\end{cases} \tag{P_2'}$$

keeping in mind that the coefficients a and b are those defined in Section 2. Let f_n be the restriction $f_{|\,]0,n[\,\times\,]0,1[}$ for all $n \in \mathbb{N}$. Then Theorem 1 shows that for all $n \in \mathbb{N}$, there exists a function $u_n \in H^{1,2}(B_n)$ which solves the problem

$$\begin{cases}\partial_t u_n + a(t, x)\partial_x u_n - b(t)\partial_x^2 u_n = f_n \in L^2(B_n),\\ u_{n|\partial B_n \setminus \{n\}\times\,]0,1[} = 0,\end{cases} \tag{P_2''}$$

where $B_n =]0, n[\times]0, 1[$.

Lemma 1. *There exists a constant K independent of n such that*

$$\|u_n\|_{L^2(B_n)} \leq \|\partial_x u_n\|_{L^2(B_n)} \leq K\|f\|_{L^2(B)}.$$

Proof. The Poincaré inequality gives $\|u_n\|_{L^2(B_n)} \leq \|\partial_x u_n\|_{L^2(B_n)}$. Moreover, by developing the scalar product $(\partial_t u_n + a(t, x)\partial_x u_n - b(t)\partial_x^2 u_n, u_n)$ in $L^2(B_n)$ and using condition (iii) in (H_2) we obtain

$$\begin{aligned}(f_n, u_n) &= \int_{B_n} u_n\partial_t u_n\, dt\, dx + \int_{B_n} a(t, x)u_n\partial_x u_n\, dt\, dx - \int_{B_n} b(t)u_n\partial_x^2 u_n\, dt\, dx\\ &= \frac{1}{2}\int_{B_n} \frac{\varphi_1'(t) - \varphi_2'(t)}{\varphi_1(t) - \varphi_2(t)} u_n^2(t, x)\, dt\, dx + \int_{B_n} b(t)\,(\partial_x u_n)^2\, dt\, dx\\ &\geq \int_{B_n} b(t)\,(\partial_x u_n)^2\, dt\, dx \geq \alpha^2\|\partial_x u_n\|^2_{L^2(B_n)}.\end{aligned}$$

Hence, for all $\epsilon > 0$,

$$\begin{aligned}\|\partial_x u_n\|^2_{L^2(B_n)} &\le \frac{1}{\alpha^2}\|u_n\|_{L^2(B_n)}\|f_n\|_{L^2(B_n)}\\ &\le \frac{1}{\alpha^2\epsilon}\|f\|_{L^2(B)} + \frac{\epsilon}{\alpha^2}\|u_n\|_{L^2(B_n)}.\end{aligned}$$

By choosing ϵ small enough, we prove the existence of a constant K such that $\|\partial_x u_n\|_{L^2(B_n)} \le K\|f\|_{L^2(B)}$. □

REMARK 1. Similar computations show that the same result holds true when we substitute the condition that $\varphi_2 - \varphi_1$ increases in a neighborhood of $+\infty$ by the following

$$|\varphi_1'(t) - \varphi_2'(t)|(\varphi_2(t) - \varphi_1(t)) \le Mc(t).$$

PROPOSITION 1. *There exists a constant K independent of n such that*

$$\|u_n\|_{H^{1,2}(B_n)} \le K\|f\|_{L^2(B)}.$$

PROOF. We have

$$\begin{aligned}\|f_n\|^2_{L^2(B)} &= (\partial_t u_n + a(t,x)\partial_x u_n - b(t)\partial_x^2 u_n,\ \partial_t u_n + a(t,x)\partial_x u_n - b(t)\partial_x^2 u_n)_{L^2(B_n)}\\ &= \|\partial_t u_n\|^2_{L^2(B_n)} + \|a.\partial_x u_n\|^2_{L^2(B_n)} + \|b.\partial_x^2 u_n\|^2_{L^2(B_n)}\\ &\quad + 2\int_{B_n} a\partial_t u_n.\partial_x u_n\,dt\,dx - 2\int_{B_n} ab\partial_x u_n.\partial_x^2 u_n\,dt\,dx\\ &\quad - 2\int_{B_n} b\partial_t u_n.\partial_x^2 u_n\,dt\,dx.\end{aligned}$$

Observe that the conditions (i), (iii) and (iv) of (H_2) show that the coefficients a and b are bounded. So, thanks to Lemma 1, for all $\epsilon > 0$ we obtain

$$\begin{aligned}&\|\partial_t u_n\|^2_{L^2(B_n)} + \|b.\partial_x^2 u_n\|^2_{L^2(B_n)} - 2\int_{B_n} b\partial_t u_n.\partial_x^2 u_n\,dt\,dx\\ &\le \|f\|^2_{L^2(B)} + \|a.\partial_x u_n\|^2_{L^2(B_n)} + 2\|\partial_t u_n\|_{L^2(B_n)}\|a.\partial_x u_n\|_{L^2(B_n)}\\ &\quad + 2\|\partial_x^2 u_n\|_{L^2(B_n)}\|ab.\partial_x u_n\|_{L^2(B_n)}\\ &\le \|f\|^2_{L^2(B)} + K_1\left(1 + \frac{2}{\epsilon}\right)\|\partial_x u_n\|^2_{L^2(B_n)} + \epsilon\|\partial_t u_n\|^2_{L^2(B_n)} + \epsilon\|\partial_x^2 u_n\|^2_{L^2(B_n)}\\ &\le K_\epsilon\|f\|^2_{L^2(B)} + \epsilon\|\partial_t u_n\|^2_{L^2(B_n)} + \epsilon\|\partial_x^2 u_n\|^2_{L^2(B_n)},\end{aligned}$$

where K_1 and K_ϵ are constants independent of n. Consequently,

$$(1-\epsilon)(\|\partial_t u_n\|^2_{L^2(B_n)} + \|b.\partial_x^2 u_n\|^2_{L^2(B_n)}) \le 2\int_{B_n} b\partial_t u_n.\partial_x^2 u_n\,dt\,dx + K_\epsilon\|f\|^2_{L^2(B)}. \tag{3.1}$$

Let us now consider the term $2\int_{B_n} b\partial_t u_n.\partial_x^2 u_n\,dt\,dx$. We have

$$\begin{aligned}2\int_{B_n} b\partial_t u_n.\partial_x^2 u_n\,dt\,dx &= 2\int_{B_n}(b\partial_x(\partial_t u_n.\partial_x u_n)\,dt\,dx + b\partial_t(\partial_x u_n)^2)\,dt\,dx\\ &= -\int_0^1 b(\partial_x u_n(n,x))^2 dx + 2\int_{B_n} b'(\partial_x u_n)^2 dt.dx.\end{aligned}$$

Note that the functions b (which is positive) and b', defined by

$$b'(t) = \frac{c'(t)}{(\varphi_2(t)-\varphi_1(t))^2} - \frac{2c(t)\,(\varphi_2'(t)-\varphi_1'(t))}{(\varphi_2(t)-\varphi_1(t))^3},$$

are bounded by virtue of hypothesis (H_2). Using Lemma 1, this yields

$$\begin{aligned}2\int_{B_n} b\partial_t u_n.\partial_x^2 u_n\,dt\,dx &\le 2\int_{B_n} b'(\partial_x u_n)^2 dt.dx\\ &\le K_2\|\partial_x u_n\|^2\\ &\le K_3\|f\|^2,\end{aligned}$$

where $(K_i)_{i=1,2}$ stand for constants independent of n. Consequently, choosing $\epsilon = 1/2$ in the relationship (3.1) we obtain, thanks to condition (ii) of (H_2),

$$\|\partial_t u_n\|^2 + \|\partial_x^2 u_n\|^2 \le K\|f\|^2. \qquad \square$$

THEOREM 2. *Suppose that the conditions (H_2) are satisfied. Then, problem (P_2) admits a (unique) solution $u \in H^{1,2}(D_2)$.*

PROOF. We obtain the solution u by letting n go to infinity in the previous proposition. The uniqueness can be proven as in Theorem 1. $\square$

4. The case of a bounded triangular domain

Let us consider the problem

$$\begin{cases}\partial_t u - c(t)\partial_x^2 u = f \in L^2(D_3)\\ u_{|\partial D_3\setminus\{T\}\times\,]\varphi_1(T),\varphi_2(T)[} = 0,\end{cases} \qquad (P_3)$$

where

$$D_3 = \{(t,x)\in\mathbb{R}^2 \mid 0<t<T,\ \varphi_1(t)<x<\varphi_2(t)\},$$

and let (H_3) denote the following conditions on the functions $(\varphi_i)_{i=1,2}$ and c:

(i) $(\varphi_i)_{i=1,2}$ and c are continuous functions on $[0, T]$, differentiable on $]0, T[$ such that $|\varphi_i'|(\varphi_2 - \varphi_1) \le \epsilon$ where ϵ is small enough;
(ii) $c(t) > 0$, for all $t \in [0, T]$;
(iii) $\varphi_1(0) = \varphi_2(0)$;
(iv) $T < +\infty$, and T is small enough.

Set

$$\Omega_n = \left\{(t, x) \in D_3 \middle| \frac{1}{n} < t < T, \varphi_1(t) < x < \varphi_2(t)\right\}.$$

Let f be an element of $L^2(D_3)$. For all $n \in \mathbb{N}$, we set $f_n = f_{|\Omega_n}$. Theorem 1 gives the existence of a function $u_n \in H^{1,2}(\Omega_n)$ which is a solution of the problem

$$\begin{cases} \partial_t u_n - c(t)\partial_x^2 u_n = f_n \in L^2(\Omega_n) \\ u_{n|\partial\Omega_n \setminus \{T\}\times]\varphi_1(T),\varphi_2(T)[} = 0. \end{cases} \tag{P_3'}$$

LEMMA 2. *There exists a constant K independent of n such that for all $t \in]0, T[$:*

(1) $\|u_n\|_{L^2(\Omega_n)} \le K\|(\varphi_2 - \varphi_1)\partial_x u_n\|_{L^2(\Omega_n)}$;
(2) $\int_{\varphi_1(t)}^{\varphi_2(t)} u_n^2(t, x)\, dx \le K(\varphi_2 - \varphi_1)^4 \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x^2 u_n)^2(t, x)\, dx$;
(3) $\int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x u_n)^2(t, x)\, dx \le K(\varphi_2 - \varphi_1)^2 \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x^2 u_n)^2(t, x)\, dx$;
(4) $\|\partial_x u_n\|_{L^2(\Omega_n)} \le K\|f\|_{L^2(D_3)}$.

PROOF. (1) Inequality is a consequence of the Poincaré inequality.

The operator

$$\begin{aligned} H^2(0, 1) \cap H_0^1(0, 1) &\to L^2(0, 1) \\ v &\to v'', \end{aligned}$$

is an isomorphism. So, there exists a constant K such that

$$\begin{cases} \|v\|_{L^2(0,1)} \le K\|v''\|_{L^2(0,1)} \\ \|v'\|_{L^2(0,1)} \le K\|v''\|_{L^2(0,1)}. \end{cases}$$

The change of variables (for fixed t) x in $y = (1 - x)\varphi_1(t) + x\varphi_2(t)$ transforming the interval $(0, 1)$ into the interval $(\varphi_1(t), \varphi_2(t))$ leads to the estimates (2) and (3).

To prove (4), it is sufficient to expand the scalar product (f_n, u_n) and use the inequality (1) Indeed, we deduce, for all $\epsilon > 0$,

$$\begin{aligned} \int_{B_n} c(t)\, (\partial_x u_n)^2(t, x) &\le |(f_n, u_n)| \\ &\le \frac{1}{\epsilon}\|f_n\|^2 + \epsilon\|u_n\|^2 \\ &\le \frac{1}{\epsilon}\|f\|_{L^2(D_3)}^2 + \epsilon K\|(\varphi_2 - \varphi_1)\partial_x u_n\|_{L^2(\Omega_n)}^2. \end{aligned}$$

However, $\varphi_2 - \varphi_1$ is bounded and $c > \alpha$ according to the condition (ii) of (H_3). Choosing ϵ small enough yields the desired result. □

PROPOSITION 2. *There exists a constant K independent of n such that*

$$\|u_n\|_{H^{1,2}(\Omega_n)} \le K\|f\|_{L^2(D_3)}.$$

PROOF. We have

$$\|\partial_t u_n\|^2_{L^2(\Omega_n)} + \|c\partial_x^2 u_n\|^2_{L^2(\Omega_n)} - 2\int_{\Omega_n} c(t)\partial_t u_n.\partial_x^2 u_n \, dt\, dx = \|f_n\|^2_{L^2(\Omega_n)},$$

and, thanks to the relationship $\partial_t u_n + \varphi_i'(t)(\partial_x u_n) = 0$ on the boundary $\partial\Omega_n$, we show that

$$\begin{aligned}
-2&\int_{\Omega_n} c(t)\partial_t u_n.\partial_x^2 u_n \, dt\, dx\\
&= 2\int_{\partial\Omega_n} c(t)\partial_t u_n.\partial_x u_n dt + \int_{\partial\Omega_n} c(t)\,(\partial_x u_n)^2\, dx\\
&\quad - \int_{\Omega_n} c'(t)\,(\partial_x u_n)^2\, dt\, dx\\
&= -\int_{1/n}^{T} 2c(t)\varphi_1'(t)\,(\partial_x u_n)^2 dt + \int_{1/n}^{T} 2c(t)\varphi_2'(t)\,(\partial_x u_n)^2 dt\\
&\quad + \int_{1/n}^{T} c(t)\varphi_1'(t)\,(\partial_x u_n)^2\, dt - \int_{1/n}^{T} c(t)\varphi_2'(t)\,(\partial_x u_n)^2\, dt\\
&\quad - \int_{\Omega_n} c'(t)\,(\partial_x u_n)^2\, dt\, dx\\
&= -\int_{1/n}^{T} c(t)\varphi_1'(t)\,(\partial_x u_n)^2\, dt + \int_{1/n}^{T} c(t)\varphi_2'(t)\,(\partial_x u_n)^2\, dt\\
&\quad - \int_{\Omega_n} c'(t)\,(\partial_x u_n)^2\, dt\, dx.
\end{aligned}$$

So, since c' is bounded, Assertion (4) of Lemma 2 yields

$$\begin{aligned}
&\left|-2\int_{\Omega_n} c(t)\partial_t u_n.\partial_x^2 u_n\, dt\, dx\right|\\
&\quad \le \left|\int_{1/n}^{T} c(t)\varphi_1'(t)\,(\partial_x u_n)^2\, dt\right| + \left|\int_{1/n}^{T} c(t)\varphi_2'(t)\,(\partial_x u_n)^2\, dt\right| + K\|f\|^2_{L^2(D_3)}.
\end{aligned}$$

Now, we estimate the term $I = |\int_{1/n}^{T} c(t)\varphi_1'(t)(\partial_x u_n)^2\, dt|$. For this purpose, we set

$$\psi(t, x) = \frac{\varphi_2(t) - x}{\varphi_2(t) - \varphi_1(t)}.$$

Hence,

$$
\begin{aligned}
I &= \int_{1/n}^{T} c(t)\varphi_1'(t)\left\{\int_{\varphi_1(t)}^{\varphi_2(t)} \partial_x[\psi(t,x)\,(\partial_x u_n(t,x))^2]\,dx\right\}dt \\
&= \int_{\Omega_n} c(t)\varphi_1'(t)\partial_x[\psi(t,x)\,(\partial_x u_n(t,x))^2]\,dx\,dt \\
&= \int_{\Omega_n} \frac{c(t)\varphi_1'(t)}{\varphi_2(t)-\varphi_1(t)}(\partial_x u_n(t,x))^2\,dx\,dt \\
&\quad + 2\int_{\Omega_n} c(t)\varphi_1'(t)\psi(t,x))\partial_x u_n(t,x)\partial_x^2 u_n(t,x)\,dx\,dt.
\end{aligned}
$$

Note that there exists a constant K such that

$$
\begin{aligned}
&\left|2\int_{\Omega_n} c(t)\varphi_1'(t)\psi(t,x)\partial_x u_n(t,x)\partial_x^2 u(t,x)\,dx\,dt\right| \\
&\qquad \le K\|\partial_x^2 u_n\|\|\varphi_1'\partial_x u_n\| \\
&\qquad \le K\epsilon\|\partial_x^2 u_n\|.
\end{aligned}
$$

(where $\epsilon = \sup \varphi_1'(\varphi_2-\varphi_1)$). Furthermore,

$$
\begin{aligned}
&\left|\int_{\Omega_n} \frac{c(t)\varphi_1'(t)}{\varphi_2(t)-\varphi_1(t)}(\partial_x u_n(t,x))^2 dx\,dt\right| \\
&\quad \le K\int_{1/n}^{T} \frac{c(t)\varphi_1'(t)}{\varphi_2(t)-\varphi_1(t)}(\varphi_2(t)-\varphi_1(t))^2 \int_{\varphi_1(t)}^{\varphi_2(t)} (\partial_x^2 u_n(t,x))^2\,dx\,dt \\
&\quad \le K\int_{\Omega_n} c(t)\varphi_1'(t)\,(\varphi_2(t)-\varphi_1(t))\,(\partial_x^2 u_n(t,x))^2\,dx\,dt \\
&\quad \le K\epsilon\|\partial_x^2 u_n\|^2.
\end{aligned}
$$

Then, there exists a constant K' such that

$$\|\partial_t u_n\| + \|\partial_x^2 u_n\| \le K'\|f\|.$$

Consequently,

$$\|u_n\|_{H^{1,2}(\Omega_n)} \le K'\|f\|. \qquad \square$$

THEOREM 3. *Suppose that conditions* (H_3) *are satisfied. Then, problem* (P_3) *admits a (unique) solution* $u \in H^{1,2}(D_3)$.

PROOF. Thanks to Proposition 2, the solution u can be obtained by letting n go to infinity. $\square$

5. The case of a sectorial domain

In this section, we consider the problem

$$\begin{cases} \partial_t u - c(t)\partial_x^2 u = f \in L^2(D_4) \\ u_{|\partial D_4} = 0, \end{cases} \tag{P_4}$$

where

$$D_4 = \{(t, x) \in \mathbb{R}^2 \mid 0 < t < +\infty,\ \varphi_1(t) < x < \varphi_2(t)\},$$

under the hypotheses (H_4) on the functions $(\varphi_i)_{i=1,2}$ and c:

(i) $\begin{cases} (\varphi_i)_{i=1,2} \text{ and } c \text{ are continuous functions on } [0, +\infty[, \text{ differentiable on} \\ ]0, +\infty[; \text{ here } |\varphi_i'|(\varphi_2\text{-}\ \varphi_1) \text{ is small enough in a neighborhood of 0 and} \\ (\varphi_i')_{i=1,2} \text{ is bounded in a neighborhood of } +\infty. \end{cases}$

(ii) $\varphi_2 - \varphi_1$ is increasing a neighborhood of $+\infty$ or

$$\text{there exists } M > 0, \quad |\varphi_1'(t) - \varphi_2'(t)|(\varphi_2(t) - \varphi_1(t)) \le M.c(t);$$

(iii) there exist $\alpha_i > 0$, $i = 1, 2$ such that $\alpha_1 \ge c(t) \ge \alpha_2 > 0$, for all $t \in [0, +\infty[$;

(iv) $\varphi_1(0) = \varphi_2(0)$;

(v) $T = +\infty$.

In order to prove our main result, we need the following trace theorem [15, Theorem 2.1, Chapter 4]:

THEOREM 4.

(i) *If* $u \in H^{1,2}(]0, T[\times]0, 1[)$, *then*

$$u_{|\{0\}\times]0,1[} \in H_0^1(0, 1) = \{u \in H^1(0, 1) \mid u(0) = u(1) = 0\}.$$

(ii) *If* $\varphi \in H_0^1(0, 1)$, *there exists* $u \in H^{1,2}(]0, T[\times]0, 1[)$ *such that* $u_{|\{0\}\times]0,1[} = \varphi$ *and* $u_{|\,]0,T[\,\times\{0\}\cup]0,T[\,\times\{1\}} = 0$.

COROLLARY 1. *Let* φ *be an element of* $H_0^1(0, 1)$. *If hypotheses* (H_1) *are fulfilled, then the problem*

$$\begin{cases} \partial_t u - c(t)\partial_x^2 u = f \in L^2(D_1) \\ u_{|\{0\}\times]\varphi_1(0),\varphi_2(0)[} = \varphi \\ u_{|\partial D_1 \setminus \{0\}\times]\varphi_1(0),\varphi_2(0)[\, \cup \{T\}\times]\varphi_1(T),\varphi_2(T)[} = 0, \end{cases}$$

admits a solution $u \in H^{1,2}(D_1)$.

THEOREM 5. *Suppose that the conditions* (H_4) *are satisfied. Then, problem* (P_4) *admits a (unique) solution* $u \in H^{1,2}(D_4)$.

PROOF. The proof of this result can be obtained by 'subdividing' the domain D_4 in three open subdomains D_1, D_2 and D_3 which respectively verify the hypotheses (H_1), (H_2) and (H_3). Furthermore, we impose $\overline{D_4} = \bigcup_{i=1,2,3} \overline{D_i}$. This is possible thanks to (H_4).

Corollary 1 allows us to solve the problem posed in every subdomain $(D_i)_{i=1,2,3}$, and obtain solutions u_1, u_2 and u_3 respectively in D_1, D_2 and D_3 which coincide on the common segments of $(\overline{D_i})_{i=1,2,3}$, that is, $u_1 = u_2$ on $\overline{D_1} \cap \overline{D_2}$ and $u_2 = u_3$ on $\overline{D_2} \cap \overline{D_3}$. The solution u in D_4 is then defined by $u_{|D_i} = u_i$ for all $i = 1, 2, 3$. □

REMARK 2.

(1) In the case where $\varphi_1 = 0$ and $\varphi_2(t) = t^\alpha$, it is easy to see that the condition $\alpha > 1/2$ satisfies hypothesis (H_4).
(2) This work may be extended to other operators (with constant or variable coefficients). Moreover, we can consider the case where the second member is more regular or lies in non-Hilbertian Sobolev spaces (built on Lebesgue spaces L^p).
(3) Instead of looking for the boundary conditions assuring the existence of the solution in the natural space, we can choose a 'bad' domain which generates some singularities in the solution. Then, the following two questions arise.
    (a) What is the optimal regularity of this singular part?
    (b) What is the number of the singularities which generate the singular part?

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