

POSITIVE SOLUTIONS TO A NON-RADIAL SUPERCRITICAL KLEIN–GORDON-TYPE EQUATION

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(Received 9 November 2004)

Abstract We find positive solutions to a nonlinear equation of Klein–Gordon type. Our analysis is carried out by truncating the related functional and estimating mountain pass solutions by Moser’s iterative scheme.

Keywords: Klein–Gordon equation; supercritical; quasilinear; m -Laplacian; mountain pass

2000 Mathematics subject classification: Primary 35B25; 35B34; 35J20; 35J50; 35J60

1. Introduction

In this paper we deal with the equation

$$-\Delta_m u + |u|^{m-2}u - a(x)(\lambda|u|^{p-1}u - |u|^{q-1}u) = 0 \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

where $N \geq 3$, $1 < m < N$, $m - 1 < p < m^* - 1 \leq q$, $m^* = mN/(N - m)$ and λ is a positive parameter. We assume throughout this note that

$$0 \leq a(x) < K \quad \text{for } x \in \mathbb{R}^N \text{ and } K > 0 \text{ a constant}, \quad (1.2)$$

$$a(x) > 0 \quad \text{for } x \in B \subset \mathbb{R}^N, \text{ where } B \text{ is a ball} \quad (1.3)$$

and

$$\lim_{|x| \rightarrow \infty} a(x) = 0. \quad (1.4)$$

A similar problem was treated in [2] where $m = 2$ and $a \equiv 1$ for the equation

$$-\Delta u - g(u) = 0 \quad \text{in } \mathbb{R}^N.$$

The nonlinearity g is subcritical, for instance, $g(u) = u^p - u^q - u$ with $1 < p, q < (N + 2)/(N - 2)$. The authors were able to handle more general nonlinearities $g(u)$ under suitable assumptions.

Also, in [1] the authors solve (1.1) with $m = 2$, $a \equiv 1$, $1 < p < (N + 2)/(N - 2)$ and $q > p$. This supercritical problem is treated in radial coordinates by minimizing $\int \frac{1}{2} |\nabla u|^2$ subject to the constraint $\int G(u) = 1$, where G is the primitive of g with $G(0) = 0$. They recover compactness by working in $H_{\text{rad}}^1(\mathbb{R}^N)$ and using the Strauss lemma [7]. The Lagrange multiplier is ruled out by rescaling the equation, since $a \equiv 1$. Their result is sharp in the sense that they determine a constant $\lambda_0 > 0$ such that (1.1) has a positive (radial) solution if and only if $\lambda > \lambda_0$. Their approach also applies if one considers the m -Laplacian instead of the Laplacian operator. But if one incorporates a weight function $a(x)$ as in (1.1), even a radial one, then their techniques are not applicable, because there is no way to rescale the problem when one finds a constrained minimizer.

The functional setting to attack the supercritical problem (1.1) is by considering

$$J : W^{1,m}(\mathbb{R}^N) \cap L^{1+q}(\mathbb{R}^N) \rightarrow \mathbb{R}$$

defined by

$$J(u) = \frac{1}{m} \int (|\nabla u|^m + |u|^m) - \int a(x) \left(\frac{\lambda}{p+1} |u|^{p+1} - \frac{1}{q+1} |u|^{q+1} \right).$$

For ease of notation, all integrals throughout the paper are computed in \mathbb{R}^N , unless otherwise mentioned. The functional J has no clear geometry, for instance, it does not satisfy the ‘mountain pass theorem’ assumptions in $W^{1,m}(\mathbb{R}^N)$. Note that zero is not a local minimizer in the case $q > m^* - 1$. And, even if $q = m^* - 1$, there is a lack of compactness. Unlike [1, 2], we cannot work with radial functions, since $a(x)$ may be non-radial. We then truncate the nonlinearity $g(x, u) = a(x)(\lambda|u|^{p-1}u - |u|^{q-1}u)$ in order to make the problem appropriately subcritical. The principal part, $-\Delta_m u$, of the equation makes difficult the obtainment of estimates, since we cannot bootstrap using the classical L^p theory involving linear elliptic operators. The new (truncated) functional \hat{J} satisfies the mountain pass geometry. In this way, there is a Palais–Smale (PS) sequence. Since $a(x)$ decays to zero, this allows us to show that the sequence has non-trivial limit u , by [5]. u is then a critical point of \hat{J} . To conclude that u is indeed a solution of the original problem (1.1), we follow a Moser iterative scheme to find an L^∞ bound for u (see [6]). Actually, one concludes that $\|u\|_{L^\infty(\mathbb{R}^N)}$ is small for sufficiently large λ .

There are works related to ours where, say, non-autonomous subcritical problems are addressed. In [3] the authors found exponentially decaying solutions by a weighted-space approach to solving the equation. The equations considered in [4, 8] are more closely related to ours, except for the fact that, in their context, $a(x)$ does not tend to zero at infinity.

2. Statements and proofs

We begin with a non-existence result. Henceforth, we assume that (1.2)–(1.4) hold.

Theorem 2.1. *There is a constant $\tilde{\lambda} > 0$ such that, for $\lambda \leq \tilde{\lambda}$, there is no non-trivial solution.*

Proof of Theorem 2.1. Multiply Equation (1.1) by a possible positive solution u . Then

$$\begin{aligned} \int |\nabla u|^m &= \int a(x)(\lambda|u|^{p+1} - |u|^{q+1}) - |u|^m \\ &= \int_{u \leq \lambda^{1/(q-p)}} a(x)(\lambda|u|^{p+1} - |u|^{q+1}) \\ &\quad - \int |u|^m + \int_{u > \lambda^{1/(q-p)}} a(x)(\lambda|u|^{p+1} - |u|^{q+1}) \\ &\leq \int_{u \leq \lambda^{1/(q-p)}} a(x)(\lambda|u|^{p+1} - |u|^{q+1}) - \int |u|^m \\ &\leq \int_{u \leq \lambda^{1/(q-p)}} K(\lambda|u|^{p+1} - |u|^{q+1}) - |u|^m. \end{aligned}$$

Let $h(\lambda, s) = \lambda K s^{p+1} - K s^{q+1} - s^m$ (see (1.2)). A simple calculation shows that there exists a $\tilde{\lambda}$ such that, for $0 \leq \lambda \leq \tilde{\lambda}$, $h(\lambda, s) \leq 0$ for every $s \geq 0$. Consequently,

$$\int |\nabla u|^m \leq \int_{u \leq \lambda^{1/(q-p)}} h(\lambda, u) \, dx \leq 0.$$

Hence, $u \equiv 0$ for $\lambda \leq \tilde{\lambda}$. □

Remark 2.2. Note that the previous theorem is true for $1 < m < \infty$.

The existence result reads as follows.

Theorem 2.3. *There is a constant $\lambda^* > 0$ such that for $\lambda > \lambda^*$ there is a weak solution $u > 0$ belonging to $W^{1,m}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.*

Proof of Theorem 2.3. Define

$$f(\lambda, u) = \begin{cases} 0 & \text{if } u \leq 0, \\ \left(1 - \frac{u^{q-p}}{\lambda}\right) u^p & \text{if } 0 < u \leq \varepsilon, \\ \left(1 - \frac{\varepsilon^{q-p}}{\lambda}\right) u^p & \text{if } u > \varepsilon. \end{cases} \quad (2.1)$$

We also assume that $\lambda > 1$, because this allows us to take ε in the interval $(0, 1)$ uniformly in λ . Here ε is chosen in such a way that

$$(1 - \varepsilon^{q-p}) u^p \leq f(\lambda, u) \leq u^p \quad (2.2)$$

and

$$\theta F(\lambda, u) \leq f(\lambda, u) u \quad \text{for } u > 0 \text{ for some constant } \theta > m, \quad (2.3)$$

where F is the primitive of f with $F(\lambda, 0) = 0$. We now study the modified equation

$$-\Delta_m u + |u|^{m-2} u - \lambda a(x) f(\lambda, u) = 0. \quad (2.4)$$

The corresponding functional $\hat{J} : W^{1,m}(\mathbb{R}^N) \rightarrow \mathbb{R}$ is defined by

$$\hat{J}(u) = \frac{1}{m} \int |\nabla u|^m + |u|^m - \int \lambda a(x) F(\lambda, u).$$

The functional \hat{J} satisfies the mountain pass geometry. Hence there is a PS sequence $u_n \in W^{1,m}(\mathbb{R}^N)$, that is, $\hat{J}(u_n) \rightarrow c$ and $\hat{J}'(u_n) \rightarrow 0$. Our aim is to prove that a subsequence of u_n converges to some u , thus finding a critical point of \hat{J} . We need also to verify that $u \not\equiv 0$.

By multiplying Equation (2.4) by u_n , using the fact that $\hat{J}(u_n) \rightarrow c$ and by (2.3), we can conclude that $u_n \rightharpoonup u$ in $W^{1,m}(\mathbb{R}^N)$. Since $\hat{J}'(u_n) \rightarrow 0$, by (2.2) one has

$$\|u_n\|_{W^{1,m}(\mathbb{R}^N)}^m \leq C \|u_n\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} + \delta_n \|u_n\|_{W^{1,m}(\mathbb{R}^N)},$$

where $\delta_n \rightarrow 0$. By a variant of the result of [5], there is a sequence x_n such that

$$\int_{B_1(x_n)} |u_n|^m > \bar{c} > 0$$

for some constant \bar{c} . Passing to a subsequence if necessary, there are only two cases to analyse. If there is a constant $K > 0$ such that $|x_n| \leq K$, then there is a sufficiently large R such that

$$\int_{B_R(0)} |u_n|^m > \bar{c} > 0,$$

implying that $u \not\equiv 0$. On the other hand, we may have $|x_n| \rightarrow \infty$. In this case, let $v_n(x) = u_n(x + x_n)$. Hence, $v_n \rightharpoonup v$ and

$$\int_{B_1(0)} |v_n|^m > \bar{c} > 0;$$

thus, $v \not\equiv 0$. By (1.4), v satisfies

$$-\Delta_m v + |v|^{m-2} v = 0 \text{ in } \mathbb{R}^N \quad \text{and} \quad v \in W^{1,m}(\mathbb{R}^N), \quad (2.5)$$

implying that $v \equiv 0$ in \mathbb{R}^N , which constitutes a contradiction.

Moreover, since $u \not\equiv 0$ by (2.1) and (2.4), we conclude that $u > 0$ in \mathbb{R}^N .

By Lemma 2.5 below, the norm $\|u\|_{L^\infty(\mathbb{R}^N)}$ is small for sufficiently large λ . Thus, u is indeed a positive solution of the original problem (1.1). \square

Remark 2.4. It is an open question as to whether $\tilde{\lambda} = \lambda^*$. In general $\tilde{\lambda}$ depends on m ; for $m = 2$ and $a \equiv 1$ the equality $\tilde{\lambda} = \lambda^*$ is valid (see [1]).

Lemma 2.5. *Let u be a mountain pass solution of (1.1). Then*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq C \lambda^{-1/(p+1-m)}$$

for some constant $C > 0$ independent of λ and u .

Proof of Lemma 2.5. The critical value corresponding to a mountain pass solution u is given by

$$c = \min_{\gamma \in \Gamma} \max_{t \in [0,1]} \hat{J}(\gamma(t)) = \hat{J}(u),$$

$$\Gamma = \{\gamma \in C^0([0,1], W^{1,m}(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = e\},$$

where $\hat{J}(0) = 0$ and $\hat{J}(e) \leq 0$. In order to choose some value e , let $\varphi \in W^{1,m}(\mathbb{R}^N)$ with $\text{supp}(\varphi) \subset \{a > 0\}$ and $\varphi \geq 0$. Hence,

$$\begin{aligned} \hat{J}(t\varphi) &= t^m \|\varphi\|_{W^{1,m}(\mathbb{R}^N)}^m - \lambda \int_{\Omega} a(x) F(\lambda, t\varphi) \\ &\leq t^m \|\varphi\|_{W^{1,m}(\mathbb{R}^N)}^m - \frac{t^{p+1}}{p+1} \lambda (1 - \varepsilon^{q-p}) \int_{\Omega} a(x) \varphi^{p+1}. \end{aligned}$$

We choose $t = t_0$, in such a way that $\hat{J}(t_0\varphi) \leq 0$, so that

$$t_0^{p+1-m} = \frac{(p+1) \|\varphi\|_{W^{1,m}(\mathbb{R}^N)}^m}{\lambda (1 - \varepsilon^{q-p}) \int_{\Omega} a(x) \varphi^{p+1}}.$$

Let $e = t_0\varphi$. Then

$$\|e\|_{W^{1,m}(\mathbb{R}^N)} = t_0 \|\varphi\|_{W^{1,m}(\mathbb{R}^N)} \leq \frac{k}{\lambda^{1/(p+1-m)}},$$

where $k > 0$ is a constant independent of λ . Hence,

$$\begin{aligned} c &\leq \max_{t>0} \hat{J}(t\varphi) \\ &\leq \max_{t>0} \left[t^m \|e\|_{W^{1,m}(\mathbb{R}^N)}^m - \frac{t^{p+1}}{p+1} \lambda (1 - \varepsilon^{q-p}) \int_{\Omega} a(x) |e|^{p+1} \right] \\ &\leq t_0^m \|e\|_{W^{1,m}(\mathbb{R}^N)}^m - \frac{t_0^{p+1}}{p+1} \lambda (1 - \varepsilon^{q-p}) \int_{\Omega} a(x) |e|^{p+1} \\ &\leq t_0^m \|e\|_{W^{1,m}(\mathbb{R}^N)}^m \leq \left(\frac{k_1}{\lambda^{1/(p+1-m)}} \right)^m, \end{aligned}$$

where k_1 is independent of λ .

Since u is a solution,

$$\left(\frac{1}{m} - \frac{1}{\theta} \right) \|u\|_{W^{1,m}(\mathbb{R}^N)}^m \leq \hat{J}(u) = c < \left(\frac{k_1}{\lambda^{1/(p+1-m)}} \right)^m$$

and

$$\|u\|_{L^{m^*}(\mathbb{R}^N)} \leq k \|u\|_{W^{1,m}(\mathbb{R}^N)} \leq \frac{k_1}{\lambda^{1/(p+1-m)}}. \quad (2.6)$$

We are going to perform a Moser iterative scheme (see, for example, [6]). Here we need to keep track of the dependence on λ .

Note that $m < p + 1 < m^*$ and, by (1.3) and (2.2),

$$-\Delta_m u + u^{m-1} \leq \lambda u^p \quad \text{in } \mathbb{R}^N. \quad (2.7)$$

For $M > 0$ and $k > 0$, define $v_M(y) = \inf\{u(y), M\}$ and $v(y) = (v_M(y))^{km+1}$. Using v as a test function in (2.7), we obtain

$$\begin{aligned} (km+1) \int v_M^{km} |\nabla v_M|^m + \int v_M^{m+km} &\leq \lambda \int v_M^{p+km+1}, \\ \frac{km+1}{(k+1)^m} \int |\nabla (v_M)^{k+1}|^m &\leq \lambda \int v_M^{(k+1)m+p+1-m}, \\ \frac{1}{c_1} \frac{km+1}{(k+1)^m} \left(\int |(v_M)^{(k+1)m^*}| \right)^{m/m^*} &\leq \lambda \left(\int v_M^{(k+1)ml} \right)^{1/l} \left(\int v_M^{m^*} \right)^{(p+1-m)/m^*}. \end{aligned}$$

From now on c_i , $i = 1, 2, 3, 4$, denote positive constants independent of λ . Note that

$$l = \frac{m^*}{m^* - (p+1-m)}$$

and observe that $m < p + 1 < m^*$, so that $0 < p + 1 - m < m^* - m$. Then $m < m^* - (p + 1 - m) < m^*$. From now on the norm spaces correspond to functions defined in \mathbb{R}^N . Thus,

$$\begin{aligned} \|v_M\|_{L^{m^*(k+1)}} &\leq c_2^{1/(k+1)} \left(\frac{k+1}{(km+1)^{1/m}} \right)^{1/(k+1)} \lambda^{1/((1+k)m)} \|v_M\|_{L^{(k+1)ml}} \|v_M\|_{L^{m^*}}^{(p+1-m)/(m(k+1))}. \end{aligned}$$

Letting $M \rightarrow \infty$ yields

$$\|u\|_{L^{m^*(k+1)}} \leq (c_2 \lambda^{1/m} \|u\|_{L^{m^*}}^{(p+1-m)/m})^{1/(k+1)} \|u\|_{L^{(k+1)ml}} \left(\frac{k+1}{(km+1)^{1/m}} \right)^{1/(k+1)}.$$

Define k_1 as $(k_1 + 1)ml = m^*$. Note that $k_1 + 1 = m^*/ml > 1$ and therefore

$$\|u\|_{L^{m^*(k_1+1)}} \leq (c_2 \lambda^{1/m} \|u\|_{L^{m^*}}^{(p+1-m)/m})^{1/(k_1+1)} \|u\|_{L^{m^*}} \left(\frac{k+1}{(km+1)^{1/m}} \right)^{1/(k_1+1)}.$$

By induction, we define $(k_n + 1)ml = m^*(k_{n-1} + 1)$. Then $k_n + 1 = (m^*/ml)^n$ and

$$\begin{aligned} \|u\|_{L^{m^*(k_n+1)}} &\leq (c_2 \lambda^{1/m} \|u\|_{L^{m^*}}^{(p+1-m)/m})^{\sum_{j=1}^n 1/(k_j+1)} \prod_{j=1}^n \left(\left(\frac{k_j+1}{(k_j m+1)^{1/m}} \right)^{1/\sqrt{k_j+1}} \right)^{1/\sqrt{k_j+1}} \|u\|_{L^{m^*}} \end{aligned}$$

and hence

$$\|u\|_{L^{m^*(k_n+1)}} \leq (c_2 \lambda^{1/m} \|u\|_{L^{m^*}}^{(p+1-m)/m})^{\sum_{j=1}^n 1/(k_j+1)} c_3^{\sum_{j=1}^n 1/\sqrt{k_j+1}} \|u\|_{L^{m^*}}.$$

Since

$$\sum_{j=1}^{\infty} \frac{1}{k_j + 1} = \sum_{j=1}^{\infty} \left(\frac{ml}{m^*} \right)^j = \frac{1}{1 - (ml/m^*)} - 1 = \frac{m^*}{m^* - ml} - 1 = \frac{m}{m^* - (p+1)},$$

we obtain

$$\begin{aligned} \|u\|_{L^\infty} &\leq c_4 \lambda^{1/(m^* - (p+1))} \|u\|_{L^{m^*}}^{(p+1-m)/(m^* - (p+1)) + 1} \\ &= c_4 \lambda^{1/(m^* - (p+1))} \|u\|_{L^{m^*}}^{(m^* - m)/(m^* - (p+1))}. \end{aligned}$$

By (2.6), we obtain the desired estimate. \square

Acknowledgements. The authors have been supported by the UTA Chilean grant no. 4732-04 and the Brazilian scientific organizations CNPq grant no. 478896/2003-4 and FAPESP grant no. 04/06678-0.

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