Proceedings of the Edinburgh Mathematical Society (2006) **49**, 291–308 ©
DOI:10.1017/S0013091504000689 Printed in the United Kingdom

PRESENTATION BY BOREL SUBALGEBRAS AND CHEVALLEY GENERATORS FOR QUANTUM ENVELOPING ALGEBRAS

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(Received 19 July 2004)

Abstract We provide an alternative approach to the Faddeev–Reshetikhin–Takhtajan presentation of the quantum group $U_q(\mathfrak{g})$, with L-operators as generators and relations ruled by an R-matrix. We look at $U_q(\mathfrak{g})$ as being generated by the quantum Borel subalgebras $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$, and use the standard presentation of the latter as quantum function algebras. When $\mathfrak{g} = \mathfrak{gl}_n$, these Borel quantum function algebras are generated by the entries of a triangular q-matrix. Thus, eventually, $U_q(\mathfrak{gl}_n)$ is generated by the entries of an upper triangular and a lower triangular q-matrix, which share the same diagonal. The same elements generate over $\Bbbk[q, q^{-1}]$ the unrestricted $\Bbbk[q, q^{-1}]$-integral form of $U_q(\mathfrak{gl}_n)$ of De Concini and Procesi, which we present explicitly, together with a neat description of the associated quantum Frobenius morphisms at roots of 1. All this holds, *mutatis mutandis*, for $\mathfrak{g} = \mathfrak{sl}_n$ too.

Keywords: quantum groups; L-operators; quantum root vectors

2000 *Mathematics subject classification:* Primary 17B37; 20G42
Secondary 81R50

1. Introduction

Let $\mathfrak{g}$ be a semi-simple Lie algebra over a field $\Bbbk$. Classically, it has two standard presentations: Serre's, which uses a minimal set of generators, and Chevalley's, using a linear basis as generating set. If $\mathfrak{g}$ instead is reductive, a presentation is obtained by that of its semi-simple quotient by adding the centre. When $\mathfrak{g} = \mathfrak{gl}_n$, Chevalley's generators are the elementary matrices, and Serre's form a distinguished subset of them; the general case of any classical matrix Lie algebra $\mathfrak{g}$ is a slight variation on this theme. Finally, both presentations also yield presentations of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$.

At the quantum level, one has correspondingly a Serre-like and a Chevalley-like presentation of $U_q(\mathfrak{g})$, the quantized universal enveloping algebra associated with $\mathfrak{g}$ after Jimbo and Lusztig (i.e. defined over the field $\Bbbk(q)$, where q is an indeterminate). The first presentation is used by Jimbo [**10**] and Lusztig [**13**] and, *mutatis mutandis*, by Drinfeld too; in this case the generators are q-analogues of the Serre generators, and starting from them one builds quantum root vectors via two different methods: iterated quantum brackets, as in [**11**] (and maybe others), or braid group action, as in [**13**] (see [**6**] for a comparison between these two methods). The second presentation was introduced by Faddeev, Reshetikhin and Takhtajan (FRT) [**4**]: the generators in this case, called L-operators, are q-analogues of the classical Chevalley generators; in particular, they are quantum root vectors themselves. An explicit comparison between quantum Serre-like generators and L-operators appears in [**4**, § 2] for the cases of *classical* $\mathfrak{g}$; on the other hand, in [**15**, § 1.2], a similar comparison is made for $\mathfrak{g} = \mathfrak{gl}_n$ between L-operators and quantum root vectors (for *any* root) built out of Serre's generators.

The first purpose of this note is to provide an alternative approach to the FRT presentation of $U_q(\mathfrak{g})$: it amounts to a series of elementary steps, yet the final outcome seems noteworthy. As a second, deeper result, we give an explicit presentation of the $\Bbbk[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by L-operators; call it $\tilde{U}_q(\mathfrak{g})$. By its very construction, this is merely the *unrestricted* $\Bbbk[q, q^{-1}]$-integral form of $U_q(\mathfrak{g})$, defined by De Concini and Procesi (see [**3**]), whose semi-classical limit is $\tilde{U}_q(\mathfrak{g})/(q-1)\tilde{U}_q(\mathfrak{g}) \cong F[G^*]$, where G^* is a connected Poisson algebraic group dual to $\mathfrak{g}$ (see [**3, 5**] and [**7**, §§ 7.3 and 7.9]): our explicit presentation of $\tilde{U}_q(\mathfrak{g})$ yields another, independent (and much easier) proof of this fact. Third, by [**3**] we know that *quantum Frobenius morphisms* exist, which embed $F[G^*]$ into the specializations of $\tilde{U}_q(\mathfrak{g})$ at roots of 1: our presentation of $\tilde{U}_q(\mathfrak{g})$ provides an explicit description of them.

This analysis shows that the two presentations of $U_q(\mathfrak{g})$ correspond to different behaviours with respect to specializations. Indeed, let $\hat{U}_q(\mathfrak{g})$ be the $\Bbbk[q, q^{-1}]$-algebra given by the Jimbo–Lusztig presentation *over* $\Bbbk[q, q^{-1}]$. Its specialization at $q = 1$ is

$$\hat{U}_q(\mathfrak{g})/(q-1)\hat{U}_q(\mathfrak{g}) \cong U(\mathfrak{g})$$

(up to technicalities), with $\mathfrak{g}$ inheriting a Lie bialgebra structure (see [**2, 10, 13**]). On the other hand, the integral form $\tilde{U}_q(\mathfrak{g})$ mentioned above specializes to $F[G^*]$, the Poisson structure on G^* being exactly the one dual to the Lie bialgebra structure on $\mathfrak{g}$. So the existence of two different presentations of $U_q(\mathfrak{g})$ reflects the deep fact that, taking suitable integral forms, $U_q(\mathfrak{g})$ provides quantizations of two different semi-classical objects (this is a general fact; see [**7, 8**]). To the author's knowledge, this was not previously known, as the FRT presentation of $U_q(\mathfrak{g})$ has never been used to study the integral form $\tilde{U}_q(\mathfrak{g})$.

Let us sketch in short the path we follow. First, we note that $U_q(\mathfrak{g})$ is generated by the quantum Borel subgroups $U_q(\mathfrak{b}_-)$ and $U_q(\mathfrak{b}_+)$ (where $\mathfrak{b}_-$ and $\mathfrak{b}_+$ are opposite Borel subalgebras of $\mathfrak{g}$), which share a common copy of the quantum Cartan subgroup $U_q(\mathfrak{t})$. Second, there exist Hopf algebra isomorphisms $U_q(\mathfrak{b}_-) \cong F_q[B_-]$ and $U_q(\mathfrak{b}_+) \cong F_q[B_+]$, where $F_q[B_-]$ and $F_q[B_+]$ are the quantum function algebras associated with $\mathfrak{b}_-$ and $\mathfrak{b}_+$, respectively. Third, when $\mathfrak{g}$ is classical we resume the explicit presentation by generators and relations of $F_q[B_-]$ and $F_q[B_+]$, as given in [**4**, § 1]. Fourth, from the above we argue a presentation of $U_q(\mathfrak{g})$ where the generators are those of $F_q[B_-]$ and $F_q[B_+]$, the toral generators being taken only once, and relations are those of these quantum function algebras plus some additional relations between generators of opposite quantum Borel subgroups. We perform this last step with all details for $\mathfrak{g} = \mathfrak{gl}_n$ and, with slight changes, for $\mathfrak{g} = \mathfrak{sl}_n$ as well. Finally, we refine the last step to provide a presentation of $\tilde{U}_q(\mathfrak{g})$.

https://doi.org/10.1017/S0013091504000689

As an application, our results apply also (with few changes) to the Drinfeld-like quantum groups $U_\hbar(\mathfrak{g})$: in particular we get a presentation of an $\hbar$-deformation of $F[G^*]$, say $\tilde{U}_\hbar(\mathfrak{g}) =: F_\hbar[G^*]$.

2. The general case

2.1. Quantized universal enveloping algebras

Let $\Bbbk$ be a fixed field of zero characteristic, let q be an indeterminate, and let $\mathfrak{g}$ be a semi-simple Lie algebra over $\Bbbk$. Let $U_q(\mathfrak{g})$ be the quantum group à la Jimbo and Lusztig defined over $\Bbbk(q)$: we define it after the conventions in [**3**], [**2**] or [**5**] (for $\varphi = 0$). Actually, we can define a quantum group like that for each lattice M between the root lattice Q and the weight lattice P of $\mathfrak{g}$; thus, we shall write $U_q^M(\mathfrak{g})$. Roughly, $U_q^M(\mathfrak{g})$ is the unital $\Bbbk(q)$-algebra with generators F_i, $\Lambda_i^{\pm 1}$, E_i for $i = 1, \dots, r =: \operatorname{rank}(\mathfrak{g})$ and relations as in [**3**, **5**], which depend on the Cartan datum of $\mathfrak{g}$ and on the choice of the lattice M; in particular, the Λ_i are 'toral' generators, roughly q-exponentials of the elements of a $\mathbb{Z}$-basis of M. Here we recall only the relation

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \quad \forall i, j = 1, \dots, r, \tag{2.1}$$

where K_i is a q-analogue of the coroot corresponding to the ith node of the Dynkin diagram of $\mathfrak{g}$ (in fact, it is a suitable product of the $\Lambda_k^{\pm 1}$). Also, we consider on $U_q^M(\mathfrak{g})$ the Hopf algebra structure given in [**3**, **5**].

The quantum Borel subalgebra $U_q^M(\mathfrak{b}_+)$ is simply the unital $\Bbbk(q)$-subalgebra of $U_q^M(\mathfrak{g})$ generated by $\Lambda_1^{\pm 1}, \dots, \Lambda_r^{\pm 1}$, $E_1, \dots, E_r$, and $U_q^M(\mathfrak{b}_-)$ the subalgebra generated by $F_1, \dots, F_r$, $\Lambda_1^{\pm 1}, \dots, \Lambda_r^{\pm 1}$. In fact, both of these are Hopf $\Bbbk(q)$-subalgebras of $U_q^M(\mathfrak{g})$. It follows that $U_q^M(\mathfrak{g})$ is generated by $U_q^M(\mathfrak{b}_+)$ and $U_q^M(\mathfrak{b}_-)$, and every possible commutation relation between these two subalgebras is a consequence of (2.1) and the commutation relations between the $\Lambda_i^{\pm 1}$ and the F_j or the E_j. Finally, we call $U_q^M(\mathfrak{t})$ the unital $\Bbbk(q)$-subalgebra of $U_q^M(\mathfrak{g})$ (and of $U_q^M(\mathfrak{b}_\pm)$) generated by all the Λ_i ($i = 1, \dots, n$), which also is a Hopf subalgebra.

Mapping $F_i \mapsto E_i$, $\Lambda_i^{\pm 1} \mapsto \Lambda_i^{\mp 1}$ and $E_i \mapsto F_i$ (for all $i = 1, \dots, n$) uniquely defines an algebra automorphism and coalgebra anti-automorphism of $U_q^M(\mathfrak{g})$, that is a Hopf algebra isomorphism

$$\Theta : U_q^M(\mathfrak{g}) \xhookrightarrow{\cong} U_q^M(\mathfrak{g})^{\mathrm{op}},$$

where hereafter, given any Hopf algebra H, we denote by H^{op} the same Hopf algebra as H but for the fact that we take the opposite coproduct. Restricting Θ to quantum Borel subalgebras gives Hopf algebra isomorphisms $U_q^M(\mathfrak{b}_\pm) \cong U_q^M(\mathfrak{b}_\mp)^{\mathrm{op}}$.

2.2. Quantum function algebras

Let M be a lattice between Q and P as in § 2.1, and define $M' := \{\psi \in P \mid \langle \psi, \mu \rangle \in \mathbb{Z},\ \forall \mu \in M\}$, where $\langle \cdot\,, \cdot \rangle$ is the $\mathbb{Q}$-valued scalar product on P induced by scalar extension

https://doi.org/10.1017/S0013091504000689

from the natural $\mathbb{Z}$-valued pairing between Q and P. Such an M' is again a lattice, said to be *dual* to M. Conversely, M is dual to M', i.e. $M = M''$.

We define quantum function algebras after Lusztig. To start with, letting M and M' be mutually dual lattices as above, we define $F_q^{M'}[G]$ as the unital $\Bbbk(q)$-algebra of all matrix coefficients of finite-dimensional $U_q^M(\mathfrak{g})$-modules which have a basis of eigenvectors for all the Λ_i ($i = 1, \dots, n$) with eigenvalue powers of q. Starting from $U_q^M(\mathfrak{b}_+)$ or $U_q^M(\mathfrak{b}_-)$ instead of $U_q^M(\mathfrak{g})$, the same recipe defines the Borel quantum function algebras $F_q^{M'}[B_+]$ and $F_q^{M'}[B_-]$, respectively. All these quantum function algebras are in fact also Hopf algebras.

Finally, the Hopf algebra monomorphisms $j_\pm : U_q^M(\mathfrak{b}_\pm) \hookrightarrow U_q^M(\mathfrak{g})$ induce Hopf algebra epimorphisms $\pi_\pm : F_q^{M'}[G] \twoheadrightarrow F_q^{M'}[B_\pm]$ (see [**2**, **5**] for details).

2.3. Isomorphisms between quantum universal enveloping algebras and quantum function algebras over Borel subgroups

Let M and M' be mutually dual lattices as in § 2.2. According to Tanisaki [**17**], there exist perfect (i.e. non-degenerate) Hopf pairings

$$U_q^M(\mathfrak{b}_+)^{\mathrm{op}} \otimes U_q^{M'}(\mathfrak{b}_-) \to \Bbbk(q), \qquad U_q^M(\mathfrak{b}_-)^{\mathrm{op}} \otimes U_q^{M'}(\mathfrak{b}_+) \to \Bbbk(q);$$

this implies that $U_q^M(\mathfrak{b}_+)^{\mathrm{op}} \cong F_q^M[B_-]$ and $U_q^M(\mathfrak{b}_-)^{\mathrm{op}} \cong F_q^M[B_+]$. By composition of the latter with the isomorphisms $U_q^M(\mathfrak{b}_+) \cong U_q^M(\mathfrak{b}_-)^{\mathrm{op}}$ and $U_q^M(\mathfrak{b}_-) \cong U_q^M(\mathfrak{b}_+)^{\mathrm{op}}$ in § 2.1, it follows that $U_q^M(\mathfrak{b}_+) \cong F_q^M[B_+]$ and $U_q^M(\mathfrak{b}_-) \cong F_q^M[B_-]$ as Hopf $\Bbbk(q)$-algebras.

2.4. Generation of $U_q^M(\mathfrak{g})$ by quantum function algebras

We stated in § 2.1 that $U_q^M(\mathfrak{g})$ is generated by $U_q^M(\mathfrak{b}_-)$ and $U_q^M(\mathfrak{b}_+)$, whose mutual commutation is a consequence of (2.1). In particular, we have a $\Bbbk(q)$-vector space isomorphism

$$U_q^M(\mathfrak{g}) = (U_q^M(\mathfrak{b}_-) \otimes U_q^M(\mathfrak{b}_+))/J,$$

where J is the two-sided ideal of $U_q^M(\mathfrak{b}_-) \otimes U_q^M(\mathfrak{b}_+)$, with the standard tensor product structure, generated by $(\{K_\mu \otimes 1 - 1 \otimes K_\mu\}_{\mu \in M})$, while the multiplication is a consequence of the internal commutation rules of $U_q^M(\mathfrak{b}_\pm)$ and of (2.1). Now, thanks to the isomorphisms in § 2.3, we describe $U_q^M(\mathfrak{g})$ as being generated by $F_q^M[B_-]$ and $F_q^M[B_+]$, with mutual commutation being a consequence of the commutation formulae corresponding to (2.1) under those isomorphisms. So we have a $\Bbbk(q)$-vector space isomorphism

$$U_q^M(\mathfrak{g}) \cong (F_q^M[B_-] \otimes F_q^M[B_+])/I,$$

where I is the ideal of $F_q^M[B_-] \otimes F_q^M[B_+]$ corresponding to J, while commutation rules are the internal rules of $F_q^M[B_\pm]$ and those corresponding to (2.1).

2.5. Relation to L-operators

Tracking carefully the construction of $U_q^M(\mathfrak{g})$ proposed in § 2.4, one realizes that this is just an alternative way to introduce $U_q^M(\mathfrak{g})$ via L-operators as in [**4**]. Such a comparison is essentially the meaning (or a possible interpretation) of the analysis carried

https://doi.org/10.1017/S0013091504000689

out in [**14**]. Moreover, the latter analysis also shows that the L-operators in [**4**] do correspond to suitable matrix coefficients in $F_q^M[B_-]$ and $F_q^M[B_+]$ (embedded inside $F_q^M[G]$); such matrix coefficients then correspond to quantum root vectors in $U_q^M(\mathfrak{b}_+)^{\mathrm{op}}$ and $U_q^M(\mathfrak{b}_-)^{\mathrm{op}}$ via the isomorphisms $F_q^M[B_-] \cong U_q^M(\mathfrak{b}_+)^{\mathrm{op}}$ and $F_q^M[B_+] \cong U_q^M(\mathfrak{b}_-)^{\mathrm{op}}$ in § 2.3, and finally to quantum root vectors in $U_q^M(\mathfrak{b}_-)$ and $U_q^M(\mathfrak{b}_+)$ via the isomorphisms $U_q^M(\mathfrak{b}_+)^{\mathrm{op}} \cong U_q^M(\mathfrak{b}_-)$ and $U_q^M(\mathfrak{b}_-)^{\mathrm{op}} \cong U_q^M(\mathfrak{b}_+)$ in § 2.1.

2.6. Integral $\Bbbk[q, q^{-1}]$-forms, specializations and quantum Frobenius morphisms

In order to look at specializations of a quantum group at special values of the parameter q, one needs the given quantum group to be defined over a subring of $\Bbbk(q)$ whose elements are regular, i.e. have no poles, at such special values. As it is typical, we choose as the ground ring the Laurent polynomial ring $\Bbbk[q, q^{-1}]$. Then, instead of $U_q^M(\mathfrak{g})$, we must consider integral forms of $U_q^M(\mathfrak{g})$ over $\Bbbk[q, q^{-1}]$, i.e. Hopf $\Bbbk[q, q^{-1}]$-subalgebras of $U_q^M(\mathfrak{g})$ which give all of $U_q^M(\mathfrak{g})$ by scalar extension from $\Bbbk[q, q^{-1}]$ to $\Bbbk(q)$: if $\bar{U}_q^M(\mathfrak{g})$ is such a $\Bbbk[q, q^{-1}]$-form, its *specialization* at $q = c \in \Bbbk$ is the quotient Hopf $\Bbbk$-algebra

$$\bar{U}_c^M(\mathfrak{g}) := \bar{U}_q^M(\mathfrak{g}) / (q - c)\bar{U}_q^M(\mathfrak{g}).$$

There are essentially two main types of $\Bbbk[q, q^{-1}]$-integral form: $\hat{U}_q^M(\mathfrak{g})$ (the quantum analogue of Kostant's $\mathbb{Z}$-integral form of $\mathfrak{g}$) introduced by Lusztig [**12**], generated by q-binomial coefficients and q-divided powers; and $\tilde{U}_q^M(\mathfrak{g})$, introduced by De Concini and Procesi [**3**], generated by rescaled quantum root vectors (see [**5**] for details). When q is specialized to any value in $\Bbbk$ *which is not a root of* 1, the choice of either of these two integral forms is irrelevant, because the corresponding specialized Hopf $\Bbbk$-algebras are mutually isomorphic. If, instead, q is specialized to $\varepsilon \in \Bbbk$ *which is a root of* 1, then the specialized algebra changes according to the choice of integral form.

Indeed, the behaviour of $\hat{U}_q^M(\mathfrak{g})$ and $\tilde{U}_q^M(\mathfrak{g})$ with respect to specializations at roots of 1 is quite different, even opposite. In particular, one has semi-classical limits $\hat{U}_1^M(\mathfrak{g}) \cong U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, and $\tilde{U}_1^M(\mathfrak{g}) \cong F[G_M^*]$, the regular function algebra of G_M^*, where G_M^* is a connected Poisson algebraic group with fundamental group isomorphic to P/M and dual to $\mathfrak{g}$, the latter endowed with a structure of Lie bialgebra, inherited from $\hat{U}_q^M(\mathfrak{g})$. Moreover, specializations of an integral form of either type at a root of 1, say $\varepsilon \in \Bbbk$, are linked to its semi-classical limit by the so-called *quantum Frobenius morphisms*

$$\hat{U}_\varepsilon^M(\mathfrak{g}) \twoheadrightarrow \hat{U}_1^M(\mathfrak{g}) \cong U(\mathfrak{g}), \qquad F[G_M^*] \cong \tilde{U}_1^M(\mathfrak{g}) \hookrightarrow \tilde{U}_\varepsilon^M(\mathfrak{g}). \tag{2.2}$$

Such a situation occurs in exactly the same way (*mutatis mutandis*) for the quantum Borel subalgebras $U_q^M(\mathfrak{b}_-)$ and $U_q^M(\mathfrak{b}_+)$. In short, one has two types of $\Bbbk[q, q^{-1}]$-integral forms, $\hat{U}_q^M(\mathfrak{b}_\pm)$ and $\tilde{U}_q^M(\mathfrak{b}_\pm)$, and quantum Frobenius morphisms:

$$\hat{U}_\varepsilon^M(\mathfrak{b}_\pm) \twoheadrightarrow \hat{U}_1^M(\mathfrak{b}_\pm) \cong U(\mathfrak{b}_\pm), \qquad F[B_\pm^*] \cong \tilde{U}_1^M(\mathfrak{b}_\pm) \hookrightarrow \tilde{U}_\varepsilon^M(\mathfrak{b}_\pm). \tag{2.3}$$

https://doi.org/10.1017/S0013091504000689

By construction, $\hat{U}_q^M(\mathfrak{g})$ is generated by $\hat{U}_q^M(\mathfrak{b}_+)$ and $\hat{U}_q^M(\mathfrak{b}_-)$ and, similarly, $\tilde{U}_q^M(\mathfrak{g})$ is generated by $\tilde{U}_q^M(\mathfrak{b}_+)$ and $\tilde{U}_q^M(\mathfrak{b}_-)$. It follows that the morphisms in (2.3) can also be obtained from (2.2) by restriction to quantum Borel subalgebras; conversely, the quantum Frobenius morphisms in (2.2) are uniquely determined, and described, by those in (2.3).

By duality, the same happens also for quantum function algebras: in particular, there exist two $\Bbbk[q,q^{-1}]$-integral forms $\hat{F}_q^M[G]$ and $\tilde{F}_q^M[G]$ of $F_q^M[G]$, which are respectively dual to $\hat{U}_q^M(\mathfrak{g})$ and $\tilde{U}_q^M(\mathfrak{g})$ in the Hopf theoretical sense, for which the dual of (2.2) holds, namely

$$F[G] \cong \hat{F}_1^M[G] \hookrightarrow \hat{F}_\varepsilon^M[G], \qquad \tilde{F}_\varepsilon^M[G] \twoheadrightarrow \tilde{F}_1^M[G] \cong U(\mathfrak{g}^*). \tag{2.4}$$

Similarly, the dual of (2.3) holds for quantum function algebras of Borel subgroups, namely

$$F[B_\pm] \cong \hat{F}_1^M[B_\pm] \hookrightarrow \hat{F}_\varepsilon^M[B_\pm], \qquad \tilde{F}_\varepsilon^M[B_\pm] \twoheadrightarrow \tilde{F}_1^M[B_\pm] \cong U(\mathfrak{b}_\pm^*), \tag{2.5}$$

which follow from (2.4) via the maps $F_q^M[G] \xrightarrow{\pi_\pm} F_q^M[B_\pm]$ in § 2.2 (see [**5**] for details).

We now stress the relation between the isomorphisms of Hopf $\Bbbk(q)$-algebras $U_q^M(\mathfrak{b}_+) \cong F_q^M[B_+]$ and $U_q^M(\mathfrak{b}_-) \cong F_q^M[B_-]$ in § 2.3 and the $\Bbbk[q,q^{-1}]$-integral forms on both sides. The key fact is that the previous isomorphisms restrict to isomorphisms of Hopf $\Bbbk[q,q^{-1}]$-algebras

$$\hat{U}_q^M(\mathfrak{b}_\pm) \cong \tilde{F}_q^M[B_\pm] \quad \text{and} \quad \tilde{U}_q^M(\mathfrak{b}_\pm) \cong \hat{F}_q^M[B_\pm].$$

Therefore, looking at $U_q^M(\mathfrak{g})$, as generated by $F_q^M[B_-]$ and $F_q^M[B_+]$ as explained in § 2.4, one argues that the *first* and *second* quantum Frobenius morphisms in (2.2) are uniquely determined (and described) by the *second* and *first* morphisms, respectively, in (2.5).

3. The case of $\mathfrak{gl}_n$

3.1. q-matrices

Let $\{t_{ij} \mid i,j = 1,\dots,n\}$ be a set of elements in any $\Bbbk(q)$-algebra A, ideally displayed inside an $(n \times n)$-matrix of which they are the entries. We will say that $T := (t_{ij})_{i,j=1,\dots,n}$ *is a q-matrix* if the t_{ij} satistfy the following relations in the algebra A:

$$\begin{aligned} t_{ij}t_{ik} &= qt_{ik}t_{ij}, & t_{ik}t_{hk} &= qt_{hk}t_{ik}, & &\forall j<k,\ i<h, \\ t_{il}t_{jk} &= t_{jk}t_{il}, & t_{ik}t_{jl} - t_{jl}t_{ik} &= (q-q^{-1})t_{il}t_{jk}, & &\forall i<j,\ k<l. \end{aligned}$$

In this case, the so-called 'quantum determinant', defined as

$$\det_q((t_{k,\ell})_{k,\ell=1,\dots,n}) := \sum_{\sigma \in \mathcal{S}_n} (-q)^{l(\sigma)} t_{1,\sigma(1)} t_{2,\sigma(2)} \cdots t_{n,\sigma(n)},$$

commutes with all the $t_{i,j}$. If, in addition, A is a $\Bbbk(q)$-bialgebra, we shall also require that

$$\Delta(t_{ij}) = \sum_{k=1}^n t_{ik} \otimes t_{kj}, \quad \epsilon(t_{ij}) = \delta_{ij}, \quad \forall i,j = 1,\dots,n.$$

https://doi.org/10.1017/S0013091504000689

In this case, the quantum determinant is group-like, that is $\Delta(\det_q) = \det_q \otimes \det_q$ and $\epsilon(\det_q) = 1$. Finally, if A is a Hopf algebra, we call any q-matrix as above whose entries are such that $\det_q$ is invertible in A a *Hopf q-matrix*; then $S(\det_q^{\pm 1}) = \det_q^{\mp 1}$.

For later use, we also recall the following compact notation. Let

$$T_1 := T \otimes I, \qquad T_2 := I \otimes T \in A \otimes \mathrm{Mat}_n(\Bbbk(q))^{\otimes 2} \cong A \otimes \mathrm{Mat}_{n^2}(\Bbbk(q)),$$

where I is the identity matrix, and $T := (t_{ij})_{i,j=1,\dots,n}$ is thought of as an element of $\mathrm{Mat}_n(A) \cong A \otimes \mathrm{Mat}_n(\Bbbk(q))$; consider

$$R := \sum_{i,j=1}^{n} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{1 \leqslant i < j \leqslant n} e_{ij} \otimes e_{ji} \in \mathrm{Mat}_{n^2}(\Bbbk(q)),$$

where $e_{ij} := (\delta_{ih}\delta_{jk})_{h,k=1}^{n}$ is the (i,j)th elementary matrix. Then T is a q-matrix if and only if the identity $RT_2T_1 = T_1T_2R$ holds true in $A \otimes \mathrm{Mat}_{n^2}(\Bbbk(q))$; in detail, for the matrix entry in position $((i,j),(kl))$ this reads

$$\sum_{m,p=1}^{n} R_{ij,mp} t_{pk} t_{ml} = \sum_{m,p=1}^{n} t_{im} t_{jp} R_{mp,kl}.$$

In the bialgebra case, T is a q-matrix if, in addition, $\Delta(T) = T \dot{\otimes} T$, $\epsilon(T) = I$, and in the Hopf algebra case also $TS(T) = I = S(T)T$, i.e. $S(T) = T^{-1}$; see [**4**, **15**] for notation (we use assumptions and normalizations of the latter) and further details.

3.2. Presentation of $F_q^P[G]$, $F_q^P[B_-]$ and $F_q^P[B_+]$ for $G = GL_n$

Let us look now at $G = GL_n$. After [**1**, Appendix], we know that $F_q^P[GL_n]$ has the following presentation: it is the unital associative $\Bbbk(q)$-algebra with generators the elements of $\{t_{ij} \mid i,j = 1,\dots,n\} \cup \{\det_q^{-1}\}$ and relations encoded by the requirement that $(t_{i,j})_{i,j=1,\dots,n}$ be a q-matrix; in particular, $\det_q^{\pm 1}$ belongs to the centre of $F_q^P[GL_n]$. Moreover, $F_q^P[GL_n]$ has the unique Hopf algebra structure such that $(t_{i,j})_{i,j=1,\dots,n}$ is a Hopf q-matrix.

Similarly, $F_q^P[B_-]$ and $F_q^P[B_+]$ are defined in the same way, *but* with the additional relations $t_{i,j} = 0 (i,j = 1,\dots,n; i > j)$ for $F_q^P[B_-]$ and $t_{i,j} = 0$ $(i,j = 1,\dots,n;\ i < j)$ for $F_q^P[B_+]$. Otherwise, we can say that $F_q^P[B_-]$ and $F_q^P[B_+]$ are generated by the entries of the q-matrices

$$\begin{pmatrix} t_{1,1} & 0 & \cdots & 0 & 0 \\ t_{2,1} & t_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{n-1,1} & t_{n-1,2} & \cdots & t_{n-1,n-1} & 0 \\ t_{n,1} & t_{n,2} & \cdots & t_{n,n-1} & t_{n,n} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t_{1,1} & t_{1,2} & \cdots & t_{1,n-1} & t_{1,n} \\ 0 & t_{2,2} & \cdots & t_{2,n-1} & t_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & t_{n-1,n-1} & t_{n-1,n} \\ 0 & 0 & \cdots & 0 & t_{n,n} \end{pmatrix},$$

respectively, and by the additional element $(t_{1,1}t_{2,2}\dots t_{n,n})^{-1}$. Moreover, both $F_q^P[B_-]$ and $F_q^P[B_+]$ are Hopf algebras, the Hopf structure being given by the assumption that their generating matrices be *Hopf* q-matrices (see also [**16**] for all these definitions).

https://doi.org/10.1017/S0013091504000689

By their very definitions, the Hopf algebra epimorphisms $\pi_+ : F_q^P[GL_n] \twoheadrightarrow F_q^P[B_+]$ and $\pi_- : F_q^P[GL_n] \twoheadrightarrow F_q^P[B_-]$ mentioned in § 2.2 are given by $\pi_+ : t_{ij} \mapsto t_{ij} (i \leqslant j)$, $t_{ij} \mapsto 0 (i > j)$ and $\pi_- : t_{ij} \mapsto t_{ij} (i \geqslant j)$, $t_{ij} \mapsto 0 (i < j)$, respectively.

3.3. The quantum algebras $U_q^M(\mathfrak{g})$, $U_q^M(\mathfrak{b}_-)$ and $U_q^M(\mathfrak{b}_+)$ for $\mathfrak{g} = \mathfrak{gl}_n$, $M \in \{P, Q\}$

We recall (see, for example, [**9**]) the definition of the quantized universal enveloping algebra $U_q^P(\mathfrak{gl}_n)$: it is the associative algebra with 1 over $\Bbbk(q)$ with generators

$$F_1, F_2, \dots, F_{n-1}, \qquad G_1^{\pm 1}, G_2^{\pm 1}, \dots, G_{n-1}^{\pm 1}, G_n^{\pm 1}, \qquad E_1, E_2, \dots, E_{n-1}$$

and relations

$$\begin{gathered} G_i G_i^{-1} = 1 = G_i^{-1} G_i, \qquad G_i^{\pm 1} G_j^{\pm 1} = G_j^{\pm 1} G_i^{\pm 1}, \qquad \forall i, j, \\ G_i F_j G_i^{-1} = q^{\delta_{i,j+1} - \delta_{i,j}} F_j, \qquad G_i E_j G_i^{-1} = q^{\delta_{i,j} - \delta_{i,j+1}} E_j, \qquad \forall i, j, \\ E_i F_j - F_j E_i = \delta_{i,j} \frac{G_i G_{i+1}^{-1} - G_i^{-1} G_{i+1}}{q - q^{-1}}, \quad \forall i, j, \\ E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i, \quad \forall i, j : |i - j| > 1, \\ E_i^2 E_j - [2]_q E_i E_j E_i + E_j E_i^2 = 0, \quad F_i^2 F_j - [2]_q F_i F_j F_i + F_j F_i^2 = 0, \quad \forall i, j : |i - j| = 1, \end{gathered}$$

with $[2]_q := q + q^{-1}$. Moreover, $U_q^P(\mathfrak{gl}_n)$ has a Hopf algebra structure given by

$$\begin{aligned} \Delta(F_i) &= F_i \otimes G_i^{-1} G_{i+1} + 1 \otimes F_i, & S(F_i) &= -F_i G_i G_{i+1}^{-1}, & \epsilon(F_i) &= 0, \quad \forall i, \\ \Delta(G_i^{\pm 1}) &= G_i^{\pm 1} \otimes G_i^{\pm 1}, & S(G_i^{\pm 1}) &= G_i^{\mp 1}, & \epsilon(G_i^{\pm 1}) &= 1, \quad \forall i \\ \Delta(E_i) &= E_i \otimes 1 + G_i G_{i+1}^{-1} \otimes E_i, & S(E_i) &= -G_i^{-1} G_{i+1} E_i, & \epsilon(E_i) &= 0, \quad \forall i. \end{aligned}$$

The algebra $U_q^Q(\mathfrak{gl}_n)$ (defined as in [**5**, § 3]) can be realized as a Hopf subalgebra. Namely, define $L_i := G_1 G_2 \cdots G_i$, $K_j := G_j G_{j+1}^{-1}$ for all $i = 1, \dots, n$, $j = 1, \dots, n-1$. Then $U_q^Q(\mathfrak{gl}_n)$ is the $\Bbbk(q)$-subalgebra of $U_q^P(\mathfrak{gl}_n)$ generated by

$$\{F_1, \dots, F_{n-1},\ K_1^{\pm 1}, \dots, K_{n-1}^{\pm 1},\ L_n^{\pm 1},\ E_1, \dots, E_{n-1}\}.$$

The quantum Borel subalgebras $U_q^P(\mathfrak{b}_+)$ and $U_q^P(\mathfrak{b}_-)$ are the subalgebras of $U_q^P(\mathfrak{gl}_n)$ generated by

$$\{G_1^{\pm 1}, \dots, G_n^{\pm 1}\} \cup \{E_1, \dots, E_{n-1}\} \quad \text{and} \quad \{G_1^{\pm 1}, \dots, G_n^{\pm 1}\} \cup \{F_1, \dots, F_{n-1}\},$$

respectively. Similar definitions hold for $U_q^Q(\mathfrak{b}_\pm)$, but with the set $\{K_1^{\pm 1}, \dots, K_{n-1}^{\pm 1}, L_n^{\pm 1}\}$ instead of $\{G_1^{\pm 1}, \dots, G_n^{\pm 1}\}$. All these are in fact *Hopf* subalgebras.

3.4. The Hopf isomorphisms $\zeta_- : U_q^P(\mathfrak{b}_-) \cong F_q^P[B_-]$, $\zeta_+ : U_q^P(\mathfrak{b}_+) \cong F_q^P[B_+]$

The Hopf algebra isomorphisms of § 2.3 are given explicitly by ($i = 1, \dots, n$; $j = 1, \dots, n-1$)

$$\begin{aligned} \zeta_- &: U_q^P(\mathfrak{b}_-) \xrightarrow{\cong} F_q^P[B_-], \qquad G_i^{\pm 1} \mapsto t_{i,i}^{\mp 1}, \quad F_j \mapsto +(q - q^{-1})^{-1} t_{j+1,j+1}^{-1} t_{j+1,j}, \\ \zeta_+ &: U_q^P(\mathfrak{b}_+) \xrightarrow{\cong} F_q^P[B_+], \qquad G_i^{\pm 1} \mapsto t_{i,i}^{\pm 1}, \quad E_j \mapsto -(q - q^{-1})^{-1} t_{j,j+1} t_{j+1,j+1}^{-1}, \end{aligned}$$

https://doi.org/10.1017/S0013091504000689

and their inverse are uniquely determined by

$$\zeta_-^{-1} : F_q^P[B_-] \xrightarrow{\cong} U_q^P(\mathfrak{b}_-), \qquad t_{i,i}^{\pm 1} \mapsto G_i^{\mp 1}, \quad t_{j+1,j} \mapsto +(q-q^{-1})G_{j+1}^{-1}F_j,$$
$$\zeta_+^{-1} : F_q^P[B_+] \xrightarrow{\cong} U_q^P(\mathfrak{b}_+), \qquad t_{i,i}^{\pm 1} \mapsto G_i^{\pm 1}, \quad t_{j,j+1} \mapsto -(q-q^{-1})E_jG_{j+1}^{+1}.$$

A straightforward computation shows that all the above are isomorphisms as claimed.

Theorem 3.1 ('short' FRT-like presentation of $U_q^P(\mathfrak{gl}_n)$). *$U_q^P(\mathfrak{gl}_n)$ is the unital associative $\Bbbk(q)$-algebra with generators the elements of the set $\{\beta_{i,j}\}_{1\leqslant i\leqslant j\leqslant n} \cup \{\gamma_{j,i}\}_{1\leqslant i\leqslant j\leqslant n}$ and relations*

$$\beta_{i,i+1}\gamma_{j+1,j} - \gamma_{j+1,j}\beta_{i,i+1} = (\delta_{i,j+1}(1-q^{-1}) + \delta_{i,j-1}(1-q))\beta_{i,i+1}\gamma_{j+1,j} - \delta_{ij}(q-q^{-1})(\alpha_i\alpha_{i+1}^{-1} - \alpha_i^{-1}\alpha_{i+1}), \quad (3.1)$$

$$\beta_{k,k}\gamma_{k,k} = 1 \quad (3.2)$$

(for all $i,j = 1,\dots,n-1$, $k = 1,\dots,n$) plus the relations encoded in the requirement that the triangular matrices $B := (\beta_{ij})_{i,j=1}^n$ and $\Gamma := (\gamma_{ij})_{i,j=1}^n$ be q-matrices. Moreover, this algebra has the unique Hopf algebra structure such that these are Hopf q-matrices.

Proof. This follows directly from § 2.4 and the isomorphisms in § 3.4. Indeed, in the given presentation, the $\beta_{h,k}$ generate a copy of $F_q^P[B_+]$, with $\beta_{h,k} \cong t_{h,k}$, isomorphic to $U_q^P(\mathfrak{b}_+)$ via § 3.4; similarly, the $\gamma_{r,s}$ generate a copy of $F_q^P[B_-]$, with $\gamma_{r,s} \cong t_{r,s}$, isomorphic to $U_q^P(\mathfrak{b}_-)$. The additional set of 'mixed' relations (3.1) simultaneously involving the $\beta_{i,i+1}$ and the $\gamma_{j+1,j}$ then corresponds to the set of relations (2.1),or to the third line of the set of relations in § 3.3, via the isomorphisms $\zeta_\pm$ of § 3.4; indeed, these isomorphisms give

$$\beta_{i,i+1} \cong -(q-q^{-1})E_iG_{i+1}^{+1}, \qquad \beta_{k,k} \cong G_k$$

and

$$\gamma_{j+1,j} \cong +(q-q^{-1})G_{j+1}^{-1}F_j, \qquad \gamma_{k,k} \cong G_k^{-1},$$

from which, computing $-(q-q^{-1})^2[E_iG_{i+1}^{+1}, G_{j+1}^{-1}F_j]$ in $U_q^P(\mathfrak{gl}_n)$, we obtain formula (3.1). As to the Hopf structure, it is determined by that of the Hopf subalgebras $U_q^P(\mathfrak{b}_+)$ and $U_q^P(\mathfrak{b}_-)$: thus, the claim follows from the previous discussion. □

Remark 3.2. Note that any other commutation relation between a generator $\beta_{h,k}$ ($h<k$) and a generator $\gamma_{r,s}$ ($r>s$) can be deduced from the ones between the $\beta_{i,i+1}$ and the $\gamma_{j+1,j}$ by repeatedly using the relations

$$\beta_{i,j} = (q-q^{-1})^{-1}(\beta_{i,k}\beta_{k,j} - \beta_{k,j}\beta_{i,k})\beta_{k,k}^{-1}, \quad \forall i<k<j,$$

which arise from the relations $\beta_{i,k}\beta_{k,j} - \beta_{k,j}\beta_{i,k} = (q-q^{-1})\beta_{k,k}\beta_{i,j}$ for the q-matrix B, and the relations

$$\gamma_{j,i} = (q-q^{-1})^{-1}(\gamma_{k,i}\gamma_{j,k} - \gamma_{j,k}\gamma_{k,i})\gamma_{k,k}^{+1}, \quad \forall j>k>i,$$

which arise from the relations $\gamma_{k,i}\gamma_{j,k} - \gamma_{j,k}\gamma_{k,i} = (q-q^{-1})\gamma_{k,k}\gamma_{j,i}$ for the q-matrix Γ.

https://doi.org/10.1017/S0013091504000689

3.5. Quantum root vectors and L-operators

In this subsection we describe the generators of $U_q^P(\mathfrak{gl}_n)$ considered in Theorem 3.1 in terms of generators of the FRT presentation, the so-called L-operators, in [**4**].

Our comparison 'passes through' that with quantum root vectors built on the Jimbo–Lusztig generators given in § 3.3. For any x, y, a, let $[x,y]_a := xy - ayx$. Define

$$E^{\pm}_{i,i+1} := E_i, \quad E^{\pm}_{i,j} := [E^{\pm}_{i,k}, E^{\pm}_{k,j}]_{q^{\pm 1}}, \quad \forall i<k<j,$$
$$F^{\pm}_{i+1,i} := F_i, \quad F^{\pm}_{j,i} := [F^{\pm}_{j,k}, F^{\pm}_{k,i}]_{q^{\mp 1}}, \quad \forall j>k>i,$$

as in [**11**]: all these are quantum root vectors, in that, in the semi-classical limit at $q=1$, they specialize to root vectors for $\mathfrak{gl}_n$, namely the elementary matrices e_{ij} with $i \neq j$. As a matter of notation, we also set $\dot{E}^{\pm}_{i,j} := (q-q^{-1})E^{\pm}_{i,j}$ and $\dot{F}^{\pm}_{j,i} := (q-q^{-1})F^{\pm}_{j,i}$ for all $i<j$.

For the L-operators, introduced in [**4**], we recall from [**15**, § 1.2] the formulae

$$\left.\begin{array}{llll} L^+_{ii} := G_i^{+1}, & L^+_{ij} := +G_i^{+1}\dot{F}^+_{j,i}, & L^+_{j,i} := 0, & \forall i<j, \\ L^-_{ii} := G_i^{-1}, & L^-_{ji} := -\dot{E}^+_{i,j}G_i^{-1}, & L^-_{i,j} := 0, & \forall i<j \end{array}\right\} \tag{3.3}$$

to define them; setting $L^+ := (L^+_{ij})_{i,j=1}^n$ and $L^- := (L^-_{ij})_{i,j=1}^n$, the relations

$$RL_1^+L_2^+ = L_2^+L_1^+R, \qquad RL_1^-L_2^- = L_2^-L_1^-R, \qquad RL_1^+L_2^- = L_2^-L_1^+R \tag{3.4}$$

express in compact form their mutual commutation properties (with notation as in § 3.1). Indeed, the FRT presentation amounts exactly to claiming that $U_q^P(\mathfrak{gl}_n)$ is the unital associative $\Bbbk(q)$-algebra with generators $L^{\pm}_{i,j}$ (for all $i,j=1,\dots,n$) and relations (3.4) and

$$L^+_{k,k}L^-_{k,k} = 1 = L^-_{k,k}L^+_{k,k}, \quad \forall k=1,\dots,n, \tag{3.5}$$

and it has the unique Hopf algebra structure such that

$$\Delta(L^\varepsilon) = L^\varepsilon \dot{\otimes} L^\varepsilon, \quad \epsilon(L^\varepsilon) = I, \quad S(L^\varepsilon) = (L^\varepsilon)^{-1}, \quad \forall \varepsilon \in \{+,-\}, \tag{3.6}$$

where L^+ and L^- are the upper and lower triangular matrices whose non-zero entries are the $L^+_{i,j}$ and $L^-_{j,i}$, respectively, I is the $(n\times n)$-identity matrix and we use standard compact notation as in [**4**].

Now, using the identifications $\zeta_+^{\pm 1}$, we get the identities

$$\beta_{i,i} = G_i^{+1}, \quad \beta_{i,j} = +(-q)^{j-i}G_j^{+1}\dot{E}^-_{i,j}, \quad \forall i<j. \tag{3.7}$$

Indeed, the identities $\beta_{ii} = G_i^{+1}$ and $\beta_{i,j} = -qG_j^{+1}\dot{E}^-_{i,j} = -\dot{E}^-_{i,j}G_j^{+1}$ for $j=i+1$ follow directly from the description of ζ_+^{-1} and the identifications $\beta_{i,i} \cong t_{i,i}$, $\beta_{i,i+1} \cong t_{i,i+1}$. In the other cases the result follows easily by induction on $j-i$, using the relations

$$\beta_{i,j} = (q-q^{-1})^{-1}(\beta_{i,k}\beta_{k,j} - \beta_{k,j}\beta_{i,k})\beta_{k,k}^{-1}, \quad \text{for } i<k<j,$$

given in Remark 3.2.

https://doi.org/10.1017/S0013091504000689

Formulae (3.7) show that the $\beta_{i,j}$ are also quantum root vectors, for positive roots. Similarly, for negative roots the $\gamma_{j,i}$ are involved. Namely, the identifications $\zeta_-^{\pm 1}$ yield

$$\gamma_{i,i} = G_i^{-1}, \quad \gamma_{j,i} = -(-q)^{i-j} \dot{F}_{j,i}^{-} G_j^{-1}, \quad \forall i < j, \tag{3.8}$$

which are the analogues of (3.7). Again this is proved by induction on $j-i$: the cases $j-i \leqslant 1$ are a direct consequence of the description of ζ_-^{-1} and the identifications $\gamma_{i,i} \cong t_{i,i}$, $\gamma_{i+1,i} \cong t_{i+1,i}$, while the inductive step follows easily by means of the relations

$$\gamma_{j,i} = (q-q^{-1})^{-1}(\gamma_{k,i}\gamma_{j,k} - \gamma_{j,k}\gamma_{k,i})\gamma_{k,k}^{+1}, \quad \text{for } j > k > i,$$

given in Remark 3.2.

In order to compare (3.3) with (3.7) and (3.8) we must be able to compare quantum root vectors with opposite superscripts. The tool is the unique $\Bbbk(q)$-algebra anti-automorphism

$$\Psi : U_q^P(\mathfrak{gl}_n) \xrightarrow{\cong} U_q^P(\mathfrak{gl}_n), \quad E_i \mapsto E_i, \quad F_i \mapsto F_i, \quad G_j^{\pm 1} \mapsto G_j^{\mp 1}, \quad \forall i, j,$$

which is clearly an involution; a straightforward computation shows that

$$\Psi(E_{i,j}^{\pm}) = (-q)^{\mp(i-j+1)} E_{i,j}^{\mp}, \qquad \Psi(F_{j,i}^{\pm}) = (-q)^{\pm(i-j+1)} F_{j,i}^{\mp}, \quad \forall i < j. \tag{3.9}$$

Now, comparing (3.3) with (3.7) and (3.8) by using (3.9), we get

$$L_{ij}^{+} = \Psi(\gamma_{j,j}^{-1}\gamma_{j,i}\gamma_{i,i}^{+1}), \qquad L_{ji}^{-} = \Psi(\beta_{i,i}^{+1}\beta_{i,j}\beta_{j,j}^{-1}), \qquad \forall i \leqslant j, \tag{3.10}$$

$$\gamma_{j,i} = \Psi((L_{ii}^{+})^{-1}L_{ij}^{+}L_{jj}^{+}), \quad \beta_{i,j} = \Psi(L_{jj}^{-}L_{ji}^{-}(L_{ii}^{-})^{-1}), \quad \forall i \leqslant j. \tag{3.11}$$

3.6. Presentation of $\tilde{U}_q{}^P(\mathfrak{g})$

Again let $G := GL_n$. It is well known that the $\Bbbk[q,q^{-1}]$-integral form $\hat{F}_q^P[G]$ has the same presentation as $F_q^P[G]$, but over $\Bbbk[q,q^{-1}]$ instead of $\Bbbk(q)$. The same holds for $\hat{F}_q^P[B_+]$ and $\hat{F}_q^P[B_-]$. In addition, $\hat{F}_q^P[B_\pm] \cong \tilde{U}_q^P(\mathfrak{b}_\pm)$ and $\tilde{U}_q^P(\mathfrak{g})$ is generated by $\tilde{U}_q^P(\mathfrak{b}_+)$ and $\tilde{U}_q^P(\mathfrak{b}_-)$. Therefore, the previous analysis implies that $\tilde{U}_q^P(\mathfrak{g})$ as a $\Bbbk[q,q^{-1}]$-algebra is generated by the entries of the q-matrices B and Γ of Theorem 3.1. The latter provides explicitly some relations (over $\Bbbk[q,q^{-1}]$, that is, inside $\tilde{U}_q^P(\mathfrak{g})$ itself) among such generators, but these do *not* form a *complete* set of relations: the general mixed relations among the $\beta_{i,j}$ and the $\gamma_{r,s}$ are missing, as those in Remark 3.2 do not make sense inside $\tilde{U}_q^P(\mathfrak{g})$. However, since we know the relationship between these generators and L-operators and we know all relations among the latter, we can eventually write down a complete set of relations for the given generators! This leads to the following presentation.

Theorem 3.3 (FRT-like presentation of $\tilde{U}_q{}^P(\mathfrak{gl}_n)$). *$\tilde{U}_q{}^P(\mathfrak{gl}_n)$ is the unital $\Bbbk[q,q^{-1}]$-algebra with generators the entries of the triangular matrices $B := (\beta_{ij})_{i,j=1}^n$ and $\Gamma := (\gamma_{ij})_{i,j=1}^n$ and relations*

$$RB_2B_1 = B_1B_2R, \qquad R\Gamma_2\Gamma_1 = \Gamma_1\Gamma_2R, \tag{3.12}$$

$$R^{\mathrm{op}}\Gamma_1^D B_2^D = B_2^D \Gamma_1^D R^{\mathrm{op}}, \qquad D_\beta D_\gamma = I = D_\gamma D_\beta, \tag{3.13}$$

https://doi.org/10.1017/S0013091504000689

where

$$R := \sum_{i,j=1}^{n} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{1 \leqslant i < j \leqslant n} e_{ij} \otimes e_{ji},$$

$X_1 := X \otimes I$, $X_2 := I \otimes X$ (*as in* § *3.1*),

$$R^{\mathrm{op}} := \sum_{i,j=1}^{n} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{1 \leqslant i < j \leqslant n} e_{ji} \otimes e_{ij}$$

and $D_\beta := \mathrm{diag}(\beta_{1,1}, \dots, \beta_{n,n})$, $D_\gamma := \mathrm{diag}(\gamma_{1,1}, \dots, \gamma_{n,n})$, $B^D := D_\beta^{+1} B D_\beta^{-1}$, $\Gamma^D := D_\gamma^{-1} \Gamma D_\gamma^{+1}$.

The first (*compact*) *relation in* (*3.13*) *above is also equivalent to*

$$\sum_{i,k=1}^{n} q^{\delta_{i,k}} (e_{i,i} \otimes I)(R^{\mathrm{op}} \Gamma_1^- B_2^+)(I \otimes e_{k,k}) = \sum_{j,s=1}^{n} q^{\delta_{j,s}} (e_{j,j} \otimes I)(B_2^- \Gamma_1^+ R^{\mathrm{op}})(I \otimes e_{s,s}), \quad (3.14)$$

where $X^\pm := (q^{\pm \delta_{h,k}} \chi_{h,k})$ *for all* $X \in \{B, \Gamma\}$ (*and* $\chi \in \{\beta, \gamma\}$) *and, in explicit, expanded form, it is equivalent to the set of relations* (*for all* $i, k, j, s = 1, \dots, n$)

$$\begin{aligned} q^{\delta_{i,j}} \gamma_{i,k} \beta_{j,s} + \delta_{i>j} (q - q^{-1}) q^{\delta_{i,s} - \delta_{jk}} \gamma_{j,k} \beta_{i,s} \\ = q^{\delta_{k,s}} \beta_{j,s} \gamma_{i,k} + \delta_{s>k} (q - q^{-1}) q^{\delta_{i,s} - \delta_{jk}} \beta_{j,k} \gamma_{i,s}, \end{aligned} \quad (3.15)$$

where obviously $\delta_{h>k} := 1$ *if* $h > k$ *and* $\delta_{h>k} := 0$ *if* $h \not> k$.

Furthermore, $\tilde{U}_q^P(\mathfrak{gl}_n)$ *has the unique Hopf algebra structure given by*

$$\Delta(X) = X \dot{\otimes} X, \quad \epsilon(X) = I, \quad S(X) = X^{-1}, \quad \forall X \in \{B, \Gamma\}. \quad (3.16)$$

Proof. The commutation formulae in (3.12) and the Hopf formulae in (3.16) are merely a compact way of saying that B and Γ are Hopf q-matrices. The second equality of (3.13) is merely another way of writing (3.2).

Moreover, the first equality of (3.13) arises from the similar compact relation for L-operators and the link between the latter and the present generators. Indeed, substituting (3.10) in the last identity in (3.4) we obtain

$$R \Psi (D_\gamma^{-1} \Gamma^{\mathrm{T}} D_\gamma^{+1})_1 \Psi (D_\beta^{+1} B^{\mathrm{T}} D_\beta^{-1})_2 = \Psi (D_\beta^{+1} B^{\mathrm{T}} D_\beta^{-1})_2 \Psi (D_\gamma^{-1} \Gamma^{\mathrm{T}} D_\gamma^{+1})_1 R$$

(where a superscript 'T' denotes 'transpose'). Using the fact that Ψ is an algebra anti-automorphism and extending its action to $\Psi(R) = R$, we then argue that

$$\Psi((D_\beta^{+1} B D_\beta^{-1})_2 (D_\gamma^{-1} \Gamma D_\gamma^{+1})_1 R^{\mathrm{op}}) = \Psi(R^{\mathrm{op}} (D_\gamma^{-1} \Gamma D_\gamma^{+1})_1 (D_\beta^{+1} B D_\beta^{-1})_2),$$

from which (3.13) eventually follows because $\Psi^2 = \mathrm{id}$.

Finally, on expanding (3.13), one finds explicitly (for all $i, k, j, s = 1, \dots, n$) that

$$\begin{aligned} q^{\delta_{i,j}} \gamma_{i,i}^{-1} \gamma_{i,k} \gamma_{k,k}^{+1} \beta_{j,j}^{+1} \beta_{j,s} \beta_{s,s}^{-1} + \delta_{i>j} (q - q^{-1}) \gamma_{j,j}^{-1} \gamma_{j,k} \gamma_{k,k}^{+1} \beta_{i,i}^{+1} \beta_{i,s} \beta_{s,s}^{-1} \\ = q^{\delta_{k,s}} \beta_{j,j}^{+1} \beta_{j,s} \beta_{s,s}^{-1} \gamma_{i,i}^{-1} \gamma_{i,k} \gamma_{k,k}^{+1} + \delta_{s>k} (q - q^{-1}) \beta_{j,j}^{+1} \beta_{j,k} \beta_{k,k}^{-1} \gamma_{i,i}^{-1} \gamma_{i,s} \gamma_{s,s}^{+1}. \end{aligned}$$

https://doi.org/10.1017/S0013091504000689

From this, making repeated use of all the relations encoded in (3.12) and in the second equality of (3.13) one can cancel out all 'diagonal' factors, i.e. those of type $\beta_{\ell,\ell}$ or $\gamma_{\ell,\ell}$. The outcome is (for all $i,k,j,s=1,\dots,n$) given by

$$q^{\delta_{i,j}}\gamma_{i,k}\beta_{j,s}+\delta_{i>j}(q-q^{-1})q^{\delta_{i,s}-\delta_{jk}}\gamma_{j,k}\beta_{i,s}=q^{\delta_{k,s}}\beta_{j,s}\gamma_{i,k}+\delta_{s>k}(q-q^{-1})q^{\delta_{i,s}-\delta_{jk}}\beta_{j,k}\gamma_{i,s};$$

that is, exactly the set of relations (3.15). As a last step, manipulating the exponents of q a little, one finds (for $i,k,j,s=1,\dots,n$) that

$$\begin{aligned} q^{2\delta_{i,k}}&(q^{\delta_{i,j}}(q^{-\delta_{i,k}}\gamma_{i,k})(q^{+\delta_{j,s}}\beta_{j,s})+\delta_{i>j}(q-q^{-1})(q^{-\delta_{j,k}}\gamma_{j,k})(q^{+\delta_{i,s}}\beta_{i,s}))\\ &=q^{2\delta_{j,s}}(q^{\delta_{k,s}}(q^{-\delta_{j,s}}\beta_{j,s})(q^{+\delta_{i,k}}\gamma_{i,k})+\delta_{s>k}(q-q^{-1})(q^{-\delta_{j,k}}\beta_{j,k})(q^{+\delta_{i,s}}\gamma_{i,s})), \quad (3.17)\end{aligned}$$

which, when written in compact form, yields exactly (3.14). □

Remark 3.4. The argument used to obtain formulae (3.13) from the last identity in (3.4) may be also applied to the first two identities therein. This yields relations among the β_{ij} and among the γ_{ji} which are different from, but equivalent to, formulae (3.12).

Corollary 3.5. *The Poisson–Hopf $\Bbbk$-algebra $\tilde{U}_1^P(\mathfrak{gl}_n)$ is the polynomial, Laurent-polynomial algebra in the variables*

$$\{\bar{\beta}_{i,j}\}_{1\leqslant i\leqslant j\leqslant n}\cup\{\bar{\gamma}_{j,i}\}_{1\leqslant i\leqslant j\leqslant n},$$

the $\beta_{\ell\ell}$ and the γ_{ii} being invertible, with relations $\beta_{ii}^{\pm1}=\gamma_{ii}^{\mp1}$, $\forall i$, whose Hopf structure is given (in compact notation) by

$$\Delta(\bar{X})=\bar{X}\dot{\otimes}\bar{X},\quad \epsilon(\bar{X})=I,\quad S(\bar{X})=\bar{X}^{-1},\quad \forall X\in\{B,\Gamma\}$$

(with B and Γ as in Theorem 3.3) and with the unique Poisson structure such that

$$\left.\begin{aligned} \{\bar{x}_{i,h},\bar{x}_{i,\ell}\}=\bar{x}_{i,h}\bar{x}_{i,\ell},\quad \{\bar{x}_{h,j},\bar{x}_{\ell,j}\}=\bar{x}_{h,j}\bar{x}_{\ell,j},\quad \{\bar{x}_{h,h},\bar{x}_{\ell,\ell}\}=0\quad (h<\ell)\\ \{\bar{x}_{i,j},\bar{x}_{h,k}\}=0\quad (i<h,\ j>k),\qquad \{\bar{x}_{i,j},\bar{x}_{h,k}\}=2\bar{x}_{i,k}\bar{x}_{h,j}\quad (i<h,\ j<k),\end{aligned}\right\}\quad (3.18)$$

with either all x_{pq} being β_{pq} (and $\beta_{pq}:=0$ for all $p>q$) or all x_{pq} being γ_{pq} (and $\gamma_{pq}:=0$ for all $p<q$), and

$$\{\bar{\beta}_{j,s},\bar{\gamma}_{i,k}\}=(\delta_{i,j}-\delta_{k,s})\bar{\beta}_{j,s}\bar{\gamma}_{i,k}+2\delta_{i>j}\bar{\gamma}_{j,k}\bar{\beta}_{i,s}-2\delta_{s>k}\bar{\beta}_{j,k}\bar{\gamma}_{i,s}. \quad (3.19)$$

In particular $\tilde{U}_1^P(\mathfrak{gl}_n)\cong F[(GL_n)_P^*]$ as Poisson Hopf algebras, where $(GL_n)_P^*$ is the algebraic group of pairs of matrices (Γ,B) where Γ and B are lower triangular and upper triangular invertible matrices, respectively, and the diagonals of Γ and B are inverse to each other, with the Poisson structure dual to the Lie bialgebra structure of $\mathfrak{gl}_n$.

Proof. If we write $\bar{x}:=x \mod (q-1)\tilde{U}_q^P(\mathfrak{gl}_n)$ for every $x\in\tilde{U}_q^P(\mathfrak{gl}_n)$, then setting $q=1$ in the presentation of $\tilde{U}_q^P(\mathfrak{gl}_n)$ of Theorem 3.3 yields a presentation for $\tilde{U}_1^P(\mathfrak{gl}_n)$. The latter is a commutative, polynomial Laurent-polynomial algebra as claimed, whence

$$\tilde{U}_1^P(\mathfrak{gl}_n)\cong F[(GL_n)_P^*]$$

https://doi.org/10.1017/S0013091504000689

as algebras, via an isomorphism which for all $i \leqslant j$ maps

$$\overline{\beta_{ij}} := \beta_{ij} \mod (q-1)\tilde{U}_q^P(\mathfrak{gl}_n)$$

to the matrix coefficient corresponding to the (i,j)th entry of the matrix B in a pair (Γ, B) as in the claim, and maps

$$\overline{\gamma_{ji}} := \gamma_{ji} \mod (q-1)\tilde{U}_q^P(\mathfrak{gl}_n)$$

to the matrix coefficient corresponding to the (j,i)th entry of the matrix Γ in a pair (Γ, B). The formulae for the Hopf structure in $\tilde{U}_q^P(\mathfrak{gl}_n)$ imply that this is also an isomorphism of Hopf algebras, for the Hopf structure on the right-hand side induced by the group structure of $(GL_n)_P^*$.

Since $\tilde{U}_1^P(\mathfrak{gl}_n)$ is commutative, it inherits from $\tilde{U}_q^P(\mathfrak{gl}_n)$ the unique Poisson bracket given by the rule

$$\{\bar{x}, \bar{y}\} := \frac{xy - yx}{q-1} \mod (q-1)\tilde{U}_q^P(\mathfrak{gl}_n),$$

for all $x, y \in \tilde{U}_q^P(\mathfrak{gl}_n)$. Then the Poisson brackets in (3.19) come directly from (3.15), while all those in (3.18) follow from the commutation formulae among the β_{ij} and the γ_{ji} in (3.11).

Finally, checking that this Poisson structure on the algebraic group $(GL_n)_P^*$ is exactly the one dual to the Lie bialgebra structure of $\mathfrak{gl}_n$ is just a matter of bookkeeping. □

3.7. The quantum Frobenius morphisms $F[(GL_n)_P^*] \cong \tilde{U}_1^P(\mathfrak{gl}_n) \hookrightarrow \tilde{U}_\varepsilon^P(\mathfrak{gl}_n)$

Let $\Bbbk_\varepsilon$ be the extension of $\Bbbk$ by a primitive ℓth root of 1, say ε. Since $\tilde{U}_q^P(\mathfrak{gl}_n)$ is generated by copies of

$$\tilde{U}_q^P(\mathfrak{b}_+) \cong \hat{F}_q^P[B_+] \quad \text{and} \quad \tilde{U}_q^P(\mathfrak{b}_-) \cong \hat{F}_q^P[B_-],$$

taking specializations the same is true for $\tilde{U}_\varepsilon^P(\mathfrak{gl}_n)$; in particular the latter is presented like in Theorem 3.3 but with $q = \varepsilon$.

In addition, the quantum Frobenius morphisms

$$F[GL_n] \cong \hat{F}_1^P[GL_n] \hookrightarrow \hat{F}_\varepsilon^P[GL_n] \quad \text{and} \quad F[B_\pm] \cong \hat{F}_1^P[B_\pm] \hookrightarrow \hat{F}_\varepsilon^P[B_\pm]$$

have a pretty neat description, as they are given by $t_{i,j} \mapsto t_{i,j}^\ell$. Hereafter, we denote by the same symbol an element in a quantum algebra and its corresponding coset after any specialization (see, for example, [**16**] for details). As mentioned in § 2.6, the morphism $F[(GL_n)_P^*] \cong \tilde{U}_1^P(\mathfrak{gl}_n) \hookrightarrow \tilde{U}_\varepsilon^P(\mathfrak{gl}_n)$ is determined by its restriction to the quantum Borel subalgebras, hence to the copies of $\hat{F}_1^P[B_+]$ and $\hat{F}_1^P[B_-]$ which generate $\tilde{U}_1^P(\mathfrak{gl}_n)$. When reformulated in light of Corollary 3.5, this implies the following theorem.

Theorem 3.6. *The quantum Frobenius morphism $F[(GL_n)_P^*] \cong \tilde{U}_1^P(\mathfrak{gl}_n) \hookrightarrow \tilde{U}_\varepsilon^P(\mathfrak{gl}_n)$ is given by $\beta_{i,j} \mapsto \beta_{i,j}^\ell$, $\gamma_{j,i} \mapsto \gamma_{j,i}^\ell$, for all $i \leqslant j$.*

https://doi.org/10.1017/S0013091504000689

4. The case of $\mathfrak{sl}_n$

4.1. From $\mathfrak{gl}_n$ to $\mathfrak{sl}_n$

In this section, we consider $\mathfrak{g} = \mathfrak{sl}_n$ and $G = SL_n$. The constructions and results of § 3 about $\mathfrak{gl}_n$ essentially give the same for $\mathfrak{sl}_n$, up to minor details. In this section we shall draw on these results, briefly explaining the changes in order.

First, the ideal generated by $(L_n - 1)$ in $U_q^P(\mathfrak{gl}_n)$ is a *Hopf ideal*. We then define $U_q^P(\mathfrak{sl}_n)$ as the quotient Hopf $\Bbbk(q)$-algebra $U_q^P(\mathfrak{sl}_n) := U_q^P(\mathfrak{gl}_n)/(L_n - 1)$. With similar notation (see § 3.3) to that for generators of $U_q^P(\mathfrak{gl}_n)$ and their images in $U_q^P(\mathfrak{sl}_n)$, we define $U_q^Q(\mathfrak{sl}_n)$ as the $\Bbbk(q)$-subalgebra of $U_q^P(\mathfrak{sl}_n)$ generated by $\{F_i, K_i^{\pm 1}, E_i\}_{i=1,\dots,n-1}$; this is also the image of $U_q^Q(\mathfrak{gl}_n)$ when mapping $U_q^P(\mathfrak{gl}_n)$ onto $U_q^P(\mathfrak{sl}_n)$. In this setting, $U_q^P(\mathfrak{b}_+)$ and $U_q^P(\mathfrak{b}_-)$ are the $\Bbbk(q)$-subalgebras of $U_q^P(\mathfrak{sl}_n)$ generated by $\{L_i^{\pm 1}, E_i\}_{i=1,\dots,n-1}$ and by $\{F_i, L_i^{\pm 1}\}_{i=1,\dots,n-1}$, respectively, whereas $U_q^Q(\mathfrak{b}_+)$ and $U_q^Q(\mathfrak{b}_-)$ alternatively, are the $\Bbbk(q)$-subalgebras of $U_q^Q(\mathfrak{sl}_n)$ generated by $\{K_i^{\pm 1}, E_i\}_{i=1,\dots,n-1}$ and $\{F_i, K_i^{\pm 1}\}_{i=1,\dots,n-1}$, respectively. These are all *Hopf* subalgebras of $U_q^P(\mathfrak{sl}_n)$ and $U_q^Q(\mathfrak{sl}_n)$, and Hopf algebra quotients of the similar quantum Borel subalgebras for $\mathfrak{gl}_n$.

In this context, we can repeat step by step the construction made for $\mathfrak{gl}_n$, up to minimal details (namely, taking into account everywhere the relation $L_n = 1$); in particular, in quantum function algebras the additional relation $t_{1,1}t_{2,2}\cdots t_{n,n} = 1$ has to be taken into account. Otherwise, the results for the $\mathfrak{sl}_n$ case can be immediately argued from the corresponding results for $\mathfrak{gl}_n$. The first of these results, analogous to Theorem 3.1, follows.

Theorem 4.1 ('short' FRT-like presentation of $U_q^P(\mathfrak{sl}_n)$). *$U_q^P(\mathfrak{sl}_n)$ is the quotient algebra of $U_q^P(\mathfrak{gl}_n)$ modulo the two-sided ideal I generated by*

$$\Bigl(\prod_{i=1}^{n} \beta_{ii} - 1\Bigr), \quad \textit{or by} \quad \Bigl(\prod_{j=1}^{n} \gamma_{jj} - 1\Bigr),$$

which gives the same result. Moreover, I is a Hopf ideal of $U_q^P(\mathfrak{gl}_n)$, therefore $U_q^P(\mathfrak{sl}_n)$ inherits from $U_q^P(\mathfrak{gl}_n)$ a structure of quotient Hopf algebra, given by formulae like those in Theorem 3.1 (with the obvious, additional simplifications). In particular, $U_q^P(\mathfrak{sl}_n)$ has the same presentation as $U_q^P(\mathfrak{gl}_n)$ in Theorem 3.1 plus the additional relation $\beta_{1,1}\beta_{2,2}\cdots\beta_{n,n} = 1$, or $\gamma_{1,1}\gamma_{2,2}\cdots\gamma_{n,n} = 1$.

4.2. Quantum root vectors, L-operators and new generators for $\tilde{U}_q{}^P(\mathfrak{sl}_n)$

Definitions imply that the Hopf algebra epimorphism $U_q^P(\mathfrak{gl}_n) \twoheadrightarrow U_q^P(\mathfrak{sl}_n)$ maps any quantum root vector, say $E_{i,j}$ or $F_{j,i}$, in $U_q^P(\mathfrak{gl}_n)$ onto a corresponding quantum root vector in $U_q^P(\mathfrak{sl}_n)$, for which we use similar notation. A similar result clearly also holds for each L-operator (in $U_q^P(\mathfrak{gl}_n)$), whose image in $U_q^P(\mathfrak{sl}_n)$ we denote by the same symbol. The discussion in § §3.5 and 3.6 can then be repeated verbatim; in particular, formulae (3.3)–(3.11) also hold true within $U_q^P(\mathfrak{sl}_n)$. The outcome then is the analogue of Theorem 3.3 (and can also be deduced directly from it since $\tilde{U}_q^P(\mathfrak{gl}_n)$ maps onto $\tilde{U}_q^P(\mathfrak{sl}_n)$) and its immediate corollary.

https://doi.org/10.1017/S0013091504000689

Theorem 4.2 (FRT-like presentation of $\tilde{U}_q{}^P(\mathfrak{sl}_n)$). *$\tilde{U}_q{}^P(\mathfrak{sl}_n)$ is the unital $\Bbbk[q,q^{-1}]$-algebra with generators given by the entries of the triangular matrices $B := (\beta_{ij})_{i,j=1}^n$ and $\Gamma := (\gamma_{ij})_{i,j=1}^n$ and relations (notation of Theorem 3.3)*

$$RB_2B_1 = B_1B_2R, \qquad R\Gamma_2\Gamma_1 = \Gamma_1\Gamma_2R, \tag{4.1}$$
$$R^{\mathrm{op}}\Gamma_1^D B_2^D = B_2^D\Gamma_1^D R^{\mathrm{op}}, \qquad D_\beta D_\gamma = I = D_\gamma D_\beta, \tag{4.2}$$
$$\det(D_\beta) = 1 = \det(D_\gamma). \tag{4.3}$$

The first (compact) relation in (3.13) above is equivalent to

$$\sum_{i,k=1}^{n} q^{\delta_{i,k}}(e_{i,i}\otimes I)(R^{\mathrm{op}}\Gamma_1^- B_2^+)(I\otimes e_{k,k}) = \sum_{j,s=1}^{n} q^{\delta_{j,s}}(e_{j,j}\otimes I)(B_2^-\Gamma_1^+ R^{\mathrm{op}})(I\otimes e_{s,s}), \tag{4.4}$$

and in expanded form it is equivalent to the set of relations (for all $i,k,j,s = 1,\dots,n$)

$$\begin{aligned} q^{\delta_{i,j}}\gamma_{i,k}\beta_{j,s} + \delta_{i>j}(q-q^{-1})q^{\delta_{i,s}-\delta_{jk}}\gamma_{j,k}\beta_{i,s} \\ = q^{\delta_{k,s}}\beta_{j,s}\gamma_{i,k} + \delta_{s>k}(q-q^{-1})q^{\delta_{i,s}-\delta_{jk}}\beta_{j,k}\gamma_{i,s}. \end{aligned} \tag{4.5}$$

Furthermore, $\tilde{U}_q^P(\mathfrak{sl}_n)$ has the unique Hopf algebra structure given by

$$\Delta(X) = X \dot{\otimes} X, \quad \epsilon(X) = I, \quad S(X) = X^{-1}, \quad \forall X \in \{B, \Gamma\}. \tag{4.6}$$

Corollary 4.3. *The Poisson–Hopf $\Bbbk$-algebra $\tilde{U}_1^P(\mathfrak{sl}_n)$ is the polynomial algebra in the variables*

$$\{\bar\beta_{i,j}\}_{1\leqslant i\leqslant j\leqslant n} \cup \{\bar\gamma_{j,i}\}_{1\leqslant i\leqslant j\leqslant n}$$

modulo the relations $\bar\beta_{1,1}\bar\beta_{2,2}\cdots\bar\beta_{n,n} = 1$, $\bar\gamma_{1,1}\bar\gamma_{2,2}\cdots\bar\gamma_{n,n} = 1$, $\bar\beta_{i,i}\bar\gamma_{i,i} = 1$ (for all $i = 1,\dots n$), with the Hopf structure given by

$$\Delta(\bar X) = \bar X \dot{\otimes} \bar X, \quad \epsilon(\bar X) = I, \quad S(\bar X) = \bar X^{-1}, \quad \forall X \in \{B, \Gamma\}$$

(with B and Γ as in Theorem 4.2) and with the unique Poisson structure such that

$$\left.\begin{aligned} \{\bar x_{i,h}, \bar x_{i,\ell}\} = \bar x_{i,h}\bar x_{i,\ell}, \quad \{\bar x_{h,j}, \bar x_{\ell,j}\} = \bar x_{h,j}\bar x_{\ell,j}, \quad \{\bar x_{h,h}, \bar x_{\ell,\ell}\} = 0 \quad (h<\ell) \\ \{\bar x_{i,j}, \bar x_{h,k}\} = 0 \quad (i<h, j>k), \qquad \{\bar x_{i,j}, \bar x_{h,k}\} = 2\bar x_{i,k}\bar x_{h,j} \quad (i<h,\ j<k), \end{aligned}\right\} \tag{4.7}$$

with either all x_{pq} being β_{pq} (and $\beta_{pq} := 0$ for all $p>q$) or all x_{pq} being γ_{pq} (and $\gamma_{pq} := 0$ for all $p<q$), and

$$\{\bar\beta_{j,s}, \bar\gamma_{i,k}\} = (\delta_{i,j} - \delta_{k,s})\bar\beta_{j,s}\bar\gamma_{i,k} + 2\delta_{i>j}\bar\gamma_{j,k}\bar\beta_{i,s} - 2\delta_{s>k}\bar\beta_{j,k}\bar\gamma_{i,s}. \tag{4.8}$$

In particular $\tilde{U}_1^P(\mathfrak{sl}_n) \cong F[(SL_n)_P^*]$ as Poisson Hopf algebras, where $(SL_n)_P^*$ is the algebraic group of pairs of matrices (Γ, B), where Γ and B are lower and upper triangular matrices, respectively, with determinant equal to 1, and the diagonals of Γ and B are inverse to each other, with the Poisson structure dual to the Lie bialgebra structure of $\mathfrak{sl}_n$.

4.3. The quantum Frobenius morphisms $F[(SL_n)^*_P] \cong \tilde{U}^P_1(\mathfrak{sl}_n) \hookrightarrow \tilde{U}^P_\varepsilon(\mathfrak{sl}_n)$

Once again, for quantum Frobenius morphisms one can repeat verbatim the discussion made for $U^P_q(\mathfrak{gl}_n)$ for the case of $U^P_q(\mathfrak{sl}_n)$, via minimal changes where needed. Otherwise, the results in the $\mathfrak{gl}_n$ case induce similar results in the $\mathfrak{sl}_n$ case via the defining epimorphism $U^P_q(\mathfrak{gl}_n) \twoheadrightarrow U^P_q(\mathfrak{sl}_n)$. Indeed, the latter is clearly compatible (in the obvious sense) with specializations at roots of 1; therefore, the specializations of the epimorphism itself yield the following commutative diagram:

$$\begin{tikzcd} F[(GL_n)^*_P] \cong \tilde{U}^P_1(\mathfrak{gl}_n) \arrow[r] \arrow[d] & \tilde{U}^P_\varepsilon(\mathfrak{gl}_n) \arrow[d] \\ F[(SL_n)^*_P] \cong \tilde{U}^P_1(\mathfrak{sl}_n) \arrow[r] & \tilde{U}^P_\varepsilon(\mathfrak{sl}_n) \end{tikzcd}$$

(for ε any root of 1) in which the vertical arrows are the above mentioned specialized epimorphisms and the horizontal ones are the quantum Frobenius (mono)morphisms.

This yields at once the following analogue of Theorem 3.6.

Theorem 4.4. *The quantum Frobenius morphism $F[(SL_n)^*_P] \cong \tilde{U}^P_1(\mathfrak{sl}_n) \hookrightarrow \tilde{U}^P_\varepsilon(\mathfrak{sl}_n)$ is given by $\beta_{i,j} \mapsto \beta^\ell_{i,j}$, $\gamma_{j,i} \mapsto \gamma^\ell_{j,i}$, for all $i \leqslant j$.*

Acknowledgements. The author thanks P. Möseneder Frajria and D. Parashar for helpful conversations.

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