

AN EXISTENCE THEOREM FOR QUASILINEAR SYSTEMS

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Abstract This paper deals with the existence of positive radial solutions for the quasilinear system $\operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i) + \lambda f^i(u_1, \dots, u_n) = 0$, $|x| < 1$, $u_i(x) = 0$, on $|x| = 1$, $i = 1, \dots, n$, $p > 1$, $\lambda > 0$, $x \in \mathbb{R}^N$. The f^i , $i = 1, \dots, n$, are continuous and non-negative functions. Let $\mathbf{u} = (u_1, \dots, u_n)$, $\|\mathbf{u}\| = \sum_{i=1}^n |u_i|$,

$$f_0^i = \lim_{\|\mathbf{u}\| \rightarrow 0} \frac{f^i(\mathbf{u})}{\|\mathbf{u}\|^{p-1}},$$

$i = 1, \dots, n$, $\mathbf{f} = (f^1, \dots, f^n)$, $\mathbf{f}_0 = \sum_{i=1}^n f_0^i$. We prove that the problem has a positive solution for sufficiently small $\lambda > 0$ if $\mathbf{f}_0 = \infty$. Our methods employ a fixed-point theorem in a cone.

Keywords: p -Laplacian; elliptic system; existence; fixed-point theorem; cone

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1. Introduction

In this paper we consider the existence and non-existence of positive radial solutions for the quasilinear elliptic system

$$\left. \begin{aligned} \operatorname{div}(|\nabla u_1|^{p-2}\nabla u_1) + \lambda f^1(u_1, \dots, u_n) &= 0 \text{ in } B, \\ &\vdots \\ \operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) + \lambda f^n(u_1, \dots, u_n) &= 0 \text{ in } B, \\ u_i &= 0 \text{ on } \partial B, \quad i = 1, \dots, n, \end{aligned} \right\} \quad (1.1)$$

where $p > 1$, $B = \{x \in \mathbb{R}^N : |x| < 1, N \geq 2\}$ and $\lambda > 0$ is a parameter.

When $p = 2$, (1.1) becomes

$$\left. \begin{aligned} \Delta u_1 + \lambda f^1(u_1, \dots, u_n) &= 0 \text{ in } B, \\ &\vdots \\ \Delta u_n + \lambda f^n(u_1, \dots, u_n) &= 0 \text{ in } B, \\ u_i &= 0 \text{ on } \partial B, \quad i = 1, \dots, n. \end{aligned} \right\} \quad (1.2)$$

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When $n = 1$ and $p = 2$, (1.1) becomes

$$\left. \begin{aligned} \Delta u + \lambda f(u) &= 0 \text{ in } B, \\ u &= 0 \text{ on } \partial B. \end{aligned} \right\} \quad (1.3)$$

System (1.3) has been the subject of extensive investigation over the past several decades. Lions [5] discussed the existence and non-existence of positive solutions of (1.3) in a general bounded regular domain in \mathbb{R}^N . The results of [5] are also interpreted in terms of bifurcation diagrams.

Joseph and Lundgren [4] determined the number of solutions for (1.3) in the case $f(u) = e^u$ and $f(u) = (1 + \alpha u)^\beta$ for $\alpha, \beta > 0$. If $0 < \beta < 1$, it is understood that $f(u) = (1 + \alpha u)^\beta$ (or u^β) is sublinear. If we define

$$f_0 = \lim_{u \rightarrow 0^+} \frac{f(u)}{u},$$

then $f_0 = \infty$ for $f(u) = (1 + \alpha u)^\beta$ (or u^β), $0 < \beta < 1$. Note that

$$f_0 = \lim_{u \rightarrow 0^+} \frac{f(u)}{u} = \infty$$

for $f(u) = e^u$. Also $f_0 = \infty$ can apply to the case in which $f(0) = 0$ (for instance, $f(u) = \sqrt{u}$). For n -dimensional system (1.1), we define \mathbf{f}_0 in (1.4), which is a natural extension of f_0 . As in the scalar case, $\mathbf{f}_0 = \infty$ can also apply to $\mathbf{f}(0) = 0$, and thus zero is a trivial solution in this case. We shall prove that (1.1) has a positive solution for sufficiently small $\lambda > 0$ if $\mathbf{f}_0 = \infty$, regardless of the behaviour of \mathbf{f} at ∞ .

Our arguments are based on the fixed-point index. Many authors have used the fixed-point index to prove the existence of positive solutions of differential equations (see, for example, [1, 3, 6–8]). Variational methods have frequently been used for Hamiltonian systems and gradient systems. However, there is apparently no possibility of using variational methods for the n -dimensional quasilinear elliptic system (1.1), and one has to use topological methods.

We now turn to general assumptions made in this paper. Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$ and

$$\mathbb{R}_+^n = \underbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}_n.$$

Also, for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n$, let $\|\mathbf{u}\| = \sum_{i=1}^n |u_i|$. We make the following assumption.

(H1) $f^i : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is continuous, $i = 1, \dots, n$.

In order to state our results we introduce the notation

$$\mathbf{f}(\mathbf{u}) = (f^1(\mathbf{u}), \dots, f^n(\mathbf{u})) = (f^1(u_1, \dots, u_n), \dots, f^n(u_1, \dots, u_n)),$$

$$f_0^i = \lim_{\|\mathbf{u}\| \rightarrow 0} \frac{f^i(\mathbf{u})}{\|\mathbf{u}\|^{p-1}},$$

where $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n$,

$$\mathbf{f}_0 = \sum_{i=1}^n f_0^i. \quad (1.4)$$

Our main result is the following theorem.

Theorem 1.1. *Assume that (H1) holds. If $\mathbf{f}_0 = \infty$, then (1.1) has a positive radial solution for sufficient small $\lambda > 0$.*

For the ordinary differential equation case ($N = 1$), Wang [8] proved that the existence, multiplicity and non-existence of positive solutions of (1.1) can be determined by appropriate combinations of superlinearity and sublinearity of $\mathbf{f}(\mathbf{u})$ at zero and infinity.

2. Preliminaries

Let $\varphi(t) = |t|^{p-2}t$; then, for $t > 0$, $\varphi(t) = t^{p-1}$ and $\varphi^{-1}(t) = t^{1/(p-1)}$. It is easy to see that $\varphi^{-1}(\sigma\varphi(t)) = \varphi^{-1}(\sigma)t$ for $t > 0$ and $\sigma > 0$.

A radial solution of (1.1) can be considered as a solution of the system

$$\left. \begin{aligned} (r^{N-1}\varphi(u_1'(r)))' + \lambda r^{N-1}f^1(\mathbf{u}) &= 0, & 0 < r < 1, \\ &\vdots \\ (r^{N-1}\varphi(u_n'(r)))' + \lambda r^{N-1}f^n(\mathbf{u}) &= 0, & 0 < r < 1, \\ \mathbf{u}'(0) = \mathbf{u}(1) &= 0, & i = 1, \dots, n. \end{aligned} \right\} \quad (2.1)$$

We will deal with classical solutions of (2.1), namely a vector-valued function $\mathbf{u} = (u_1(r), \dots, u_n(r))$ with $u_i \in C^1[0, 1]$, and $\varphi(u_i') \in C^1(0, 1)$, $i = 1, \dots, n$, which satisfies (2.1). A solution $\mathbf{u}(r) = (u_1(r), \dots, u_n(r))$ is positive if $u_i(r) \geq 0$, $i = 1, \dots, n$, for all $r \in (0, 1)$ and there is at least one non-trivial component of \mathbf{u} . In fact, it is easy to prove that such a non-trivial component of \mathbf{u} is positive on $(0, 1)$.

The following well-known result of the fixed-point index is crucial in our arguments.

Lemma 2.1 (see [2, 3]). *Let E be a Banach space equipped with a norm $\|\cdot\|_*$ and let K be a cone in E . For $r > 0$, define $K_r = \{u \in K : \|u\|_* < r\}$, and $\partial K_r = \{u \in K : \|u\|_* = r\}$, which is the relative boundary of K_r with respect to K . Assume that $T : \bar{K}_r \rightarrow K$ is completely continuous.*

(i) *If there exists a $x_0 \in K \setminus \{0\}$ such that*

$$x - Tx \neq tx_0, \quad \text{for all } x \in \partial K_r \text{ and } t \geq 0,$$

then

$$i(T, K_r, K) = 0.$$

(ii) *If $\|Tx\|_* \leq \|x\|_*$ for $x \in \partial K_r$ and $Tx \neq x$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 1.$$

In order to apply Lemma 2.1 to (2.1), let X be the Banach space

$$\underbrace{C[0, 1] \times \cdots \times C[0, 1]}_n$$

and, for $\mathbf{u} = (u_1, \dots, u_n) \in X$, define its norm by

$$\|\mathbf{u}\|_* = \sum_{i=1}^n \sup_{t \in [0, 1]} |u_i(t)|.$$

Define K to be a cone in X by

$$K = \{(u_1, \dots, u_n) \in X : u_i(t) \geq 0, t \in [0, 1], i = 1, \dots, n\}.$$

Also, define Ω_r , for r a positive number, by

$$\Omega_r = \{\mathbf{u} \in K : \|\mathbf{u}\|_* < r\}.$$

Note that $\partial\Omega_r = \{\mathbf{u} \in K : \|\mathbf{u}\|_* = r\}$.

Let $\mathbf{T}_\lambda : K \rightarrow X$ be a map with components $(T_\lambda^1, \dots, T_\lambda^n)$. We define T_λ^i , $i = 1, \dots, n$, by

$$T_\lambda^i \mathbf{u}(r) = \int_r^1 \varphi^{-1} \left(\frac{1}{s^{N-1}} \int_0^s \tau^{N-1} \lambda f^i(\mathbf{u}(\tau)) d\tau \right) ds, \quad r \in [0, 1]. \quad (2.2)$$

It is straightforward to verify that (2.1) is equivalent to the fixed-point equation

$$\mathbf{T}_\lambda \mathbf{u} = \mathbf{u} \quad \text{in } K.$$

Lemma 2.2. Assume that (H1) holds. Then $\mathbf{T}_\lambda(K) \subset K$ and $\mathbf{T}_\lambda : K \rightarrow K$ is compact and continuous.

Proof. The proof of the lemma is standard, and is omitted. \square

Lemma 2.3. Assume that (H1) holds. If $\mathbf{u} \in \partial\Omega_r$, $r > 0$, then

$$\|\mathbf{T}_\lambda \mathbf{u}\|_* \leq n\varphi^{-1}(\lambda)\varphi^{-1}(\hat{M}_r),$$

where

$$\hat{M}_r = 1 + \max\{f^i(\mathbf{u}) : \mathbf{u} \in \mathbb{R}_+^n \text{ and } \|\mathbf{u}\| \leq r, i = 1, \dots, n\} > 0.$$

Proof. From the definition of \mathbf{T}_λ , for $\mathbf{u} \in \partial\Omega_r$, we have

$$\begin{aligned} \|\mathbf{T}_\lambda \mathbf{u}\|_* &= \sum_{i=1}^n \sup_{t \in [0, 1]} |T_\lambda^i \mathbf{u}(t)| \\ &\leq \sum_{i=1}^n \int_0^1 \varphi^{-1} \left[\frac{1}{s^{N-1}} \int_0^s \tau^{N-1} \lambda \hat{M}_r d\tau \right] ds \\ &= \sum_{i=1}^n \int_0^1 \varphi^{-1} \left[\frac{1}{s^{N-1}} \int_0^s \tau^{N-1} d\tau \varphi(\varphi^{-1}(\lambda \hat{M}_r)) \right] ds \\ &\leq n\varphi^{-1}[\varphi(\varphi^{-1}(\lambda \hat{M}_r))]. \end{aligned}$$

Then the fact that $\varphi^{-1}(\varphi(t)) = t$ implies that

$$\begin{aligned}\|\mathbf{T}_\lambda \mathbf{u}\|_* &\leq n\varphi^{-1}(\lambda \hat{M}_r) \\ &= n\varphi^{-1}(\lambda)\varphi^{-1}(\hat{M}_r).\end{aligned}$$

□

3. Proof of Theorem 1.1

Proof. Fix a number $r_2 > 0$. Lemma 2.3 implies that there exists a $\lambda_0 > 0$ such that

$$\|\mathbf{T}_\lambda \mathbf{u}\|_* < \|\mathbf{u}\|_*, \quad \text{for } \mathbf{u} \in \partial\Omega_{r_2}, \quad 0 < \lambda < \lambda_0.$$

Since $f_0 = \infty$, there exists a component f^i such that $f_0^i = \infty$. Therefore, there is an $0 < r_1 < r_2$ such that

$$f^i(\mathbf{u}) \geq \varphi(\eta)\varphi(\|\mathbf{u}\|) \quad (3.1)$$

for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n$ and $\|\mathbf{u}\| \leq r_1$, where $\eta > 0$ is chosen so that

$$\frac{\eta\varphi^{-1}(\lambda)}{2}\varphi^{-1}\left(\frac{1}{N4^N}\right) \geq 1. \quad (3.2)$$

If $\mathbf{u} - \mathbf{T}_\lambda \mathbf{u} = 0$ for some $\mathbf{u} \in \partial U_{r_1}$, we find the desired solution of (1.1). Therefore, we assume that

$$\mathbf{u} - \mathbf{T}_\lambda \mathbf{u} \neq 0, \quad \text{for all } \mathbf{u} \in \partial U_{r_1}. \quad (3.3)$$

We now claim that

$$\mathbf{u} - \mathbf{T}_\lambda \mathbf{u} \neq t\mathbf{v}, \quad \text{for all } \mathbf{u} \in \partial\Omega_{r_1} \text{ and } t \geq 0, \quad (3.4)$$

where $\mathbf{v} = (\theta(r), \dots, \theta(r))$, and $\theta \in C[0, 1]$ such that $0 \leq \theta(r) \leq 1$ on $[0, 1]$, $\theta(r) \equiv 1$ on $[0, \frac{1}{4}]$ and $\theta(r) \equiv 0$ on $[\frac{1}{2}, 1]$. Thus, $\mathbf{v} \in K \setminus \{0\}$. If there exists $\mathbf{u}^* = (u_1^*, \dots, u_n^*) \in \partial\Omega_{r_1}$ and $t_0 \geq 0$ such that $\mathbf{u}^* - \mathbf{T}_\lambda \mathbf{u}^* = t_0 \mathbf{v}$, we will show that this leads to a contradiction. Since (3.3) is true, we have $t_0 > 0$. Since $\mathbf{T}_\lambda(K) \subset K$, we find that $u_i^*(r) \geq t_0 \theta(r)$ for all $r \in [0, 1]$. Let

$$t^* = \sup\{t : u_i^*(r) \geq t\theta(r) \text{ for all } r \in [0, 1]\}.$$

It follows that $t_0 \leq t^* < \infty$ and $u_i^*(r) \geq t^* \theta(r)$ for all $r \in [0, 1]$. Now, for $r \in [0, 1]$, we have

$$\begin{aligned}u_i^*(r) &= \mathbf{T}_\lambda^i \mathbf{u}^*(r) + t_0 \theta(r) \\ &= \int_r^1 \varphi^{-1}\left(\frac{1}{s^{N-1}} \int_0^s \tau^{N-1} \lambda f^i(\mathbf{u}^*(\tau)) \, d\tau\right) \, ds + t_0 \theta(r).\end{aligned}$$

Note that

$$\sum_{j=1}^n u_j^*(r) \leq r_1 \quad \text{for } r \in [0, 1].$$

Inequality (3.1) implies that, for $r \in [0, \frac{1}{2}]$,

$$\begin{aligned}
 u_i^*(r) &\geq \int_{1/2}^1 \varphi^{-1} \left(\frac{1}{s^{N-1}} \int_0^s \tau^{N-1} \lambda \varphi(\eta) \varphi \left(\sum_{j=1}^n u_j^*(\tau) \right) d\tau \right) ds + t_0 \theta(r) \\
 &\geq \int_{1/2}^1 \varphi^{-1} \left(\int_0^s \tau^{N-1} \lambda \varphi(\eta) \varphi(u_i^*(\tau)) d\tau \right) ds + t_0 \theta(r) \\
 &\geq \frac{1}{2} \varphi^{-1} \left(\int_0^{1/4} \tau^{N-1} \lambda \varphi(\eta) \varphi(t^* \theta(\tau)) d\tau \right) + t_0 \theta(r) \\
 &= \frac{1}{2} \varphi^{-1} \left(\int_0^{1/4} \tau^{N-1} d\tau \varphi(\varphi^{-1}(\lambda)) \varphi(\eta) \varphi(t^*) \right) + t_0 \theta(r) \\
 &= \frac{1}{2} \varphi^{-1} \left(\frac{1}{N4^N} \varphi(\varphi^{-1}(\lambda) \eta t^*) \right) + t_0 \theta(r).
 \end{aligned}$$

Now, in view of the fact that $\varphi^{-1}(\sigma \varphi(t)) = \varphi^{-1}(\sigma)t$, we have, for $r \in [0, \frac{1}{2}]$,

$$\begin{aligned}
 u_i^*(r) &\geq t^* \frac{\eta \varphi^{-1}(\lambda)}{2} \varphi^{-1} \left(\frac{1}{N4^N} \right) + t_0 \theta(r) \\
 &\geq t^* + t_0 \theta(r) \\
 &\geq (t^* + t_0) \theta(r),
 \end{aligned}$$

and hence

$$u_i^*(r) \geq (t^* + t_0) \theta(r), \quad r \in [0, 1],$$

which is a contradiction to the definition of t^* . Thus, in view of Lemma 2.1,

$$\begin{aligned}
 i(\mathbf{T}_\lambda, \Omega_{r_1}, K) &= 0. \\
 i(\mathbf{T}_\lambda, \Omega_{r_2}, K) &= 1.
 \end{aligned}$$

It follows from the additivity of the fixed-point index that $i(\mathbf{T}_\lambda, \Omega_{r_2} \setminus \bar{\Omega}_{r_1}, K) = 1$. Thus, \mathbf{T}_λ has a fixed point in $\Omega_{r_2} \setminus \bar{\Omega}_{r_1}$, which is the desired positive solution of (1.1). \square

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