

A MULTIPLICITY THEOREM FOR A PERTURBED SECOND-ORDER NON-AUTONOMOUS SYSTEM

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Abstract In this paper we establish a multiplicity result for a second-order non-autonomous system. Using a variational principle of Ricceri we prove that if the set of global minima of a certain function has at least k connected components, then our problem has at least k periodic solutions. Moreover, the existence of one more solution is investigated through a mountain-pass-like argument.

Keywords: multiple periodic solutions; second-order non-autonomous system; critical point theory

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1. Introduction

In this paper we consider the second-order non-autonomous system

$$\begin{cases} \ddot{u} = \alpha(t)(Au - \nabla F(u)) + \lambda \nabla_x G(t, u), & \text{a.e. in } [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \quad (P_\lambda)$$

where A is an $N \times N$ symmetric matrix satisfying

$$Ax \cdot x \geq c|x|^2 \quad \text{for all } x \in \mathbb{R}^N, \quad (1.1)$$

where c is some positive constant. Assume that $\lambda > 0$, $\alpha \in L^\infty([0, T])$, $a = \text{ess inf}_{[0, T]} \alpha > 0$, $F : \mathbb{R}^N \rightarrow \mathbb{R}$ is continuously differentiable, and that $G : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ is measurable in t for all $x \in \mathbb{R}^N$ and continuously differentiable in $x \in \mathbb{R}^N$ for a.e. $t \in [0, T]$. Moreover, assume

$$\sup_{|x| \leq s} |\nabla_x G(\cdot, x)| \in L^1([0, T]) \quad \text{for every } s > 0, \quad G(\cdot, 0) \in L^1([0, T]). \quad (1.2)$$

It is well known (see [3]) that a solution of (P_λ) is a function $u \in C^1([0, T], \mathbb{R}^N)$, with \dot{u} absolutely continuous, such that

$$\begin{cases} \ddot{u}(t) = \alpha(t)(Au(t) - \nabla F(u(t))) + \lambda \nabla_x G(t, u(t)), & \text{a.e. in } [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases}$$

For the more general problem

$$\begin{cases} \ddot{u} = \nabla_x \phi(t, u), & \text{a.e. in } [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

the existence of at least three solutions has previously been studied in [1], [6], [7] and [9] under the following assumption, firstly introduced by Brezis and Nirenberg: there exist $r > 0$ and an integer $k \geq 0$ such that

$$-\frac{1}{2}(k+1)^2 w^2 |x|^2 \leq \phi(t, x) - \phi(t, 0) \leq -\frac{1}{2}k^2 w^2 |x|^2 \quad (1.3)$$

for each $|x| \leq r$, a.e. $t \in [0, T]$, where $w = 2\pi/T$.

The perturbed problem

$$\begin{cases} \ddot{u} = \nabla_x \phi(t, u) + \lambda \psi(t), & \text{a.e. in } [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases}$$

was studied in [8], in which Tang proves the existence of at least three solutions, for $\lambda > 0$ small enough, under the stronger condition that there exist $r > 0$ and an integer $k \geq 0$ such that

$$-\mu |x|^2 \leq \phi(t, x) - \phi(t, 0) \leq -\nu |x|^2 \quad (1.4)$$

for each $|x| \leq r$, a.e. $t \in [0, T]$, where $\nu > \frac{1}{2}k^2 w^2$, $\mu < \frac{1}{2}(k+1)^2 w^2$ and $w = 2\pi/T$.

In this paper we prove a multiplicity result of the following type: for each integer $k > 1$, (P_λ) has, for λ small enough, at least k solutions.

Our main tool is a recent theorem by Ricceri [4, Theorem 6], which, for the convenience of the reader, we state here.

Theorem A. *Let X be a reflexive and separable real Banach space and let $\Psi, \Phi : X \rightarrow \mathbb{R}$ be two functionals. Assume that there exists $r > \inf_X \Psi$ such that the set $\Psi^{-1}(]-\infty, r])$ is bounded. Moreover, suppose that the functional Φ is bounded below in $(\overline{\Psi^{-1}(]-\infty, r])})_w$ and the functional $\Psi + \lambda\Phi$ is sequentially weakly lower semicontinuous for each $\lambda \geq 0$ small enough. Finally, assume that the set $\Psi^{-1}(\inf_X \Psi)$ has at least k weakly connected components.*

Then, there exists $\lambda_r > 0$ such that, for each $\lambda \in]0, \lambda_r[$, the functional $\Psi + \lambda\Phi$ has at least k τ_Ψ -local minima lying in $\Psi^{-1}(]-\infty, r])$, where τ_Ψ is the smallest topology on X which contains both the weak topology and the family of sets $\{\Psi^{-1}(]-\infty, \rho])\}_{\rho \in \mathbb{R}}$.

2. A multiplicity theorem

Let us introduce the space

$$H_T^1 = \{u : [0, T] \rightarrow \mathbb{R}^N \text{ absolutely continuous, } u(0) = u(T), \dot{u} \in L^2([0, T], \mathbb{R}^N)\}.$$

For all $u, v \in H_T^1$, define the scalar product as follows:

$$(u, v) = \int_0^T \dot{u}(t) \cdot \dot{v}(t) dt + \int_0^T \alpha(t) A u(t) \cdot v(t) dt.$$

The norm

$$\|u\| = \left(\int_0^T |\dot{u}(t)|^2 dt + \int_0^T \alpha(t) Au(t) \cdot u(t) dt \right)^{1/2}$$

in H_T^1 is equivalent to the usual one thanks to condition (1.1).

Let us observe that H_T^1 is compactly embedded in $C^0([0, T], \mathbb{R}^N)$. Define, for each $u \in H_T^1$,

$$\Psi(u) = \frac{1}{2} \|u\|^2 - \int_0^T \alpha(t) F(u(t)) dt,$$

$$\Phi(u) = \int_0^T G(t, u(t)) dt.$$

Clearly, Ψ is well defined, sequentially weakly continuous, and continuous together with its Gâteaux derivative. Moreover, from (1.2) we have

$$\sup_{|x| \leq s} |G(\cdot, x)| \in L^1([0, T]) \quad \text{for every } s > 0.$$

Thus, it is easy to prove that Φ satisfies the same properties of Ψ .

We recall that u is a solution of (P_λ) if and only if $u \in H_T^1$ and it satisfies

$$\int_0^T [\dot{u}(t) \cdot \dot{v}(t) + \alpha(t) Au(t) \cdot v(t) - \alpha(t) \nabla F(u(t)) \cdot v(t)] dt + \lambda \int_0^T \nabla_x G(t, u(t)) \cdot v(t) dt = 0$$

for all $v \in H_T^1$, that is, if u is a critical point of $\Psi + \lambda\Phi$ in H_T^1 .

Our result reads as follows.

Theorem 2.1. *Let α , A , F , G be as in § 1. Assume that*

- (i) $\limsup_{|x| \rightarrow +\infty} \frac{F(x)}{|x|^2} < \frac{1}{2}c$;
- (ii) *the set of global minima of the function $H(x) = \frac{1}{2}Ax \cdot x - F(x)$ has at least k connected components in \mathbb{R}^N ($k \geq 2$).*

Then, for every $r > \|\alpha\|_{L^1} \inf_{\mathbb{R}^N} H$, there exists $\lambda_r > 0$ such that, for every $\lambda \in]0, \lambda_r[$, (P_λ) has at least k solutions in $\Psi^{-1}(]-\infty, r])$.

Proof. Set $X = H_T^1$. Let us show that the functionals Ψ and Φ defined above satisfy the hypotheses of Theorem A. The functional $\Psi + \lambda\Phi$ is sequentially weakly continuous for each $\lambda \geq 0$. We now prove that Ψ is coercive: let σ be a positive number such that

$$\limsup_{|x| \rightarrow +\infty} \frac{F(x)}{|x|^2} < \sigma < \frac{1}{2}c.$$

Then $F(x) < \sigma|x|^2 + m$ for all $x \in \mathbb{R}^N$, for some constant m , and

$$\begin{aligned}\Psi(u) &\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \frac{1}{2} \int_0^T \alpha(t) Au(t) \cdot u(t) dt - \int_0^T \alpha(t)(\sigma|u(t)|^2 + m) dt \\ &\geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \left(\frac{1}{2} - \frac{\sigma}{c}\right) \int_0^T \alpha(t) Au(t) \cdot u(t) dt - m\|\alpha\|_{L^1} \\ &\geq \left(\frac{1}{2} - \frac{\sigma}{c}\right) \|u\|^2 - m\|\alpha\|_{L^1},\end{aligned}$$

which implies that $\Psi(u)$ tends to infinity as $\|u\|$ goes to infinity.

Specifically, from the coercivity of Ψ it follows that for every $r > \|\alpha\|_{L^1} \inf_{\mathbb{R}^N} H$ the set $\Psi^{-1}(]-\infty, r])$ is bounded. Moreover, we note that the restriction of Φ to the sequentially weakly compact set $(\Psi^{-1}(]-\infty, r]))_w$ has a global minimum.

We claim that

$$\inf_X \Psi = \|\alpha\|_{L^1} \inf_{\mathbb{R}^N} H.$$

In fact, for all $u \in X$ we have

$$\begin{aligned}\Psi(u) &\geq \frac{1}{2} \int_0^T \alpha(t) Au(t) \cdot u(t) dt - \int_0^T \alpha(t) F(u(t)) dt \\ &= \int_0^T \alpha(t) H(u(t)) dt \geq \|\alpha\|_{L^1} \inf_{\mathbb{R}^N} H.\end{aligned}$$

Let us denote by M the set of global minima of H in \mathbb{R}^N . If $x_0 \in M$, then the function defined by putting $u_0(t) = x_0$ belongs to X and

$$\Psi(u_0) = \int_0^T \alpha(t) H(u_0(t)) dt = \|\alpha\|_{L^1} \inf_{\mathbb{R}^N} H.$$

Thus, our claim is proved.

We note that, if $u \in X$ is not constant, then $|\dot{u}| > 0$ on some set of positive measure, hence it cannot be a global minimum of Ψ , and the same is true for constant functions whose value does not belong to M .

Let $\gamma : \mathbb{R}^N \rightarrow X$ be the function that maps $x \in \mathbb{R}^N$ into the constant function $u(t) = x$ in X : γ is then a homeomorphism between \mathbb{R}^N and $\gamma(\mathbb{R}^N)$ (endowed with the relativization of the weak topology). The set of global minima of Ψ is equal to $\gamma(M)$; hence it has at least k weakly connected components.

By applying Theorem A we deduce for every $r > \|\alpha\|_{L^1} \inf_{\mathbb{R}^N} H$ the existence of $\lambda_r > 0$ such that, for every $\lambda \in]0, \lambda_r[$, the functional $\Psi + \lambda\Phi$ has at least k τ_Ψ -local minima lying in $\Psi^{-1}(]-\infty, r])$.

Since Ψ is continuous, the topology τ_Ψ is weaker than the strong topology in X , and every τ_Ψ -local minimum is also a strong local minimum, and so a critical point of $\Psi + \lambda\Phi$. The proof is now complete. \square

Theorem 2.2. Let α , A , F , G be as in § 1, and let assumptions (i) and (ii) of Theorem 2.1 be satisfied. Moreover, assume that

$$(iii) \quad \liminf_{|x| \rightarrow +\infty} \frac{\operatorname{ess\,inf}_{[0,T]} G(t, x)}{|x|^2} > -\infty.$$

Then, for every $r > \inf_{\mathbb{R}^N} H \|\alpha\|_{L^1}$ there exists $\lambda_r^* > 0$ such that, for every $\lambda \in]0, \lambda_r^*[$, (P_λ) has at least $k + 1$ solutions, k of which lie in $\Psi^{-1}(]-\infty, r[)$.

Proof. Let us show that, for $\lambda > 0$ small enough, $\Psi + \lambda\Phi$ is coercive. From the proof of Theorem 2.1 we already know that

$$\Psi(u) \geq \left(\frac{1}{2} - \frac{\sigma}{c} \right) \|u\|^2 - m \|\alpha\|_{L^1}.$$

Then, there exist $b < 0$ and $s > 0$ such that

$$G(t, x) > b|x|^2$$

for $|x| > s$ and a.e. $t \in [0, T]$, while

$$g = \sup_{|x| \leq s} |G(\cdot, x)| \in L^1([0, T]).$$

Summarizing, for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$ we have

$$G(t, x) \geq b|x|^2 - g(t),$$

which implies that

$$\begin{aligned} \Phi(u) &\geq \int_0^T (b|u(t)|^2 - g(t)) \, dt \\ &\geq \frac{b}{ac} \int_0^T \alpha(t) Au(t) \cdot u(t) \, dt - \|g\|_{L^1} \\ &\geq \frac{b}{ac} \|u\|^2 - \|g\|_{L^1} \end{aligned}$$

and so

$$\Psi(u) + \lambda\Phi(u) \geq \left[\left(\frac{1}{2} - \frac{\sigma}{c} \right) + \lambda \frac{b}{ac} \right] \|u\|^2 - m \|\alpha\|_{L^1} - \lambda \|g\|_{L^1}.$$

Set

$$\lambda_r^* = \min \left\{ \lambda_r, -\frac{ac}{b} \left(\frac{1}{2} - \frac{\sigma}{c} \right) \right\},$$

where λ_r is as in Theorem 2.1. Then, for all $\lambda \in]0, \lambda_r^*[$, the functional $\Psi + \lambda\Phi$ admits at least k local minima and is coercive. Thus, $\Psi + \lambda\Phi$ satisfies the Palais–Smale condition, as it is the sum of $\frac{1}{2}\|u\|^2$, whose derivative is a homeomorphism between $H_{\mathcal{T}}^1$ and its dual, and of a functional with compact derivative. From [2] it follows that $\Psi + \lambda\Phi$ admits one more critical point. \square

Remark 2.3. As seen in the proof of Theorem 2.2, the existence of $k + 1$ solutions follows essentially from the Palais–Smale condition, and the latter is proved through the coercivity of the functional. By using another standard argument, we could assume that there exist $q > 2$ and $R > 0$ such that, for λ small enough,

$$0 < q[\alpha(t)F(x) - \lambda G(t, x)] \leq [\alpha(t)\nabla F(x) - \lambda \nabla_x G(t, x)] \cdot x \quad (2.1)$$

for a.e. $t \in [0, T]$ and for every x with $|x| > R$. This implies, as λ tends to zero, that

$$0 \leq qF(x) \leq \nabla F(x) \cdot x.$$

Now, if there is some x_1 such that $|x_1| > R$, $F(x_1) > 0$ and

$$F(x) < \frac{1}{2}c|x|^2 \quad (2.2)$$

for all $|x| > |x_1|$, then it is easy to prove that the function

$$\mu \rightarrow |\mu x_1|^{-q} F(\mu x_1)$$

is non-decreasing for $\mu \geq 1$ and so

$$F(\mu x_1) \geq |x_1|^{-q} F(x_1) |\mu x_1|^q,$$

which together with (2.2) gives a contradiction.

No contradiction arises, however, if we assume that $F(x) = 0$ for all $x \in \mathbb{R}^N$, $|x| > R$ (so condition (i) is obviously satisfied), together with (ii) and condition (2.1), which becomes

$$\nabla_x G(t, x) \cdot x \leq qG(t, x) < 0.$$

In this case we get $k + 1$ solutions for $\lambda > 0$ small enough.

In the case $N = 1$, (P_λ) becomes

$$\begin{cases} u'' = \alpha(t)(u - F'(u)) + \lambda G_x(t, u), & \text{a.e. in } [0, T], \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \quad (P'_\lambda)$$

The following result, whose proof is analogous (with minor changes) to that of Theorem 1 in [5], yields the existence of $k + 1$ solutions with no additional hypotheses on G .

Theorem 2.4. Let α , F , G be as in § 1 (with $N = 1$). Assume that

- (iv) $\lim_{|x| \rightarrow +\infty} \frac{F'(x)}{x} = 0$;
- (v) the set of global minima of the function $H(x) = \frac{1}{2}x^2 - F(x)$ has at least k connected components in \mathbb{R} ($k \geq 2$).

Then, for every $r > \|\alpha\|_{L^1} \inf_{\mathbb{R}} H$ there exists $\lambda_r > 0$ such that, for every $\lambda \in]0, \lambda_r[$, (P'_λ) has at least $k + 1$ solutions, k of which satisfy

$$\frac{1}{2} \int_0^T |u'(t)|^2 dt + \int_0^T \alpha(t) \left(\frac{1}{2} |u(t)|^2 - F(u(t)) \right) dt < r. \quad (2.3)$$

3. Examples

In the following examples α and G are as in § 1, while the function F is chosen in order to satisfy assumptions (i) and (ii).

Example 3.1. Let A be the identity matrix ($c = 1$), let $f \in C^1([0, +\infty[)$ be a periodic function such that $f(0) > b = \inf_{\mathbb{R}} f$, and let $q \in]0, 1[$ and $p \in [2, +\infty[$. Define F as follows:

$$F(x) = \begin{cases} \frac{1}{2}|x|^2 - (f(|x|^{-q}) - b)|x|^p & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Clearly, $F \in C^1(\mathbb{R}^N)$. If $p = 2$, we get

$$\limsup_{|x| \rightarrow +\infty} \frac{F(x)}{|x|^2} = \frac{1}{2} - f(0) + b < \frac{1}{2},$$

while, if $p > 2$,

$$\limsup_{|x| \rightarrow +\infty} \frac{F(x)}{|x|^2} = -\infty.$$

So (i) is satisfied.

The function H , here, is given by

$$H(x) = (f(|x|^{-q}) - b)|x|^p.$$

H is non-negative and the set of its global minima is $\{x \in \mathbb{R}^N : f(|x|^{-q}) = b\} \cup \{0\}$, which has infinitely many connected components.

Then, for every $k \geq 2$, (P_λ) has at least k solutions for λ small enough.

Remark 3.2. The function F in Example 3.1 depends only on the norm of vector x . The thesis holds if we replace $|x|^p$ with $(|x_1| + |x_2| + \dots + |x_N|)^p$ in the definition of F .

Remark 3.3. It is also clear that the problem considered in Example 3.1 does not satisfy condition (1.4). In fact, in this case, $\phi(t, x) = \alpha(t)(\frac{1}{2}|x|^2 - F(x))$. So, specifically, for all $x \in \mathbb{R}^N$, $x \neq 0$,

$$\frac{\phi(t, x) - \phi(t, 0)}{|x|^2} \geq a(f(|x|^{-q}) - b)|x|^{p-2} \geq 0.$$

Example 3.4. Let $\varphi_1, \varphi_2, \dots, \varphi_k \in C^1([0, +\infty[, \mathbb{R})$ be functions such that

- (1) for all $i \in \{1, 2, \dots, k\}$, $\varphi_i^{-1}(0) \neq \emptyset$;
- (2) for all $i, j \in \{1, 2, \dots, k\}$, $i \neq j$, $\varphi_i^{-1}(0) \cap \varphi_j^{-1}(0) = \emptyset$;
- (3) $\lim_{\rho \rightarrow \infty} \frac{1}{\rho} \prod_{i=1}^k (\varphi_i(\rho))^2 = +\infty$.

Then the problem

$$\begin{cases} \ddot{u} = 4\alpha(t) \left(\sum_{i=1}^k \varphi_i(|u|^2) \varphi'_i(|u|^2) \prod_{j \neq i} (\varphi_j(|u|^2))^2 \right) u + \lambda \nabla_x G(t, u), & \text{a.e. in } [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases} \quad (P_\lambda)$$

has at least k solutions for $\lambda > 0$ small enough. It can immediately be seen that parts (i) and (ii) of Theorem 2.1 are satisfied: specifically, $x \in \mathbb{R}^N$ is a global minimum of the function $H(x) = \prod_{i=1}^k (\varphi_i(|x|^2))^2$ if and only if $\varphi_i(|x|^2) = 0$ for some $i \in \{1, 2, \dots, k\}$.

We would like to emphasize that, for λ not sufficiently small, with all the other assumptions of our theorem fulfilled, the thesis may fail, as the following counterexample shows.

Example 3.5. Let us consider the one-dimensional problem

$$\begin{cases} u'' = u - F'(u) + \lambda G'(u), & \text{a.e. in } [0, T], \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \quad (Q_\lambda)$$

where the function F is defined by

$$F(x) = \frac{1}{2}x^2 - \frac{1}{4}d_1x^4 - \frac{1}{2}d_2x^2$$

and $d_1 > 0$, $d_2 < 0$. Clearly, F satisfies (i), as

$$\lim_{|x| \rightarrow +\infty} \frac{F(x)}{x^2} = -\infty.$$

The function

$$H(x) = \frac{1}{4}d_1x^4 + \frac{1}{2}d_2x^2$$

admits two global minima. Choose

$$G(x) = \frac{1}{2}x^2.$$

The problem (Q_λ) reads as follows:

$$\begin{cases} u'' = d_1u^3 + (d_2 + \lambda)u, & \text{a.e. in } [0, T], \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases}$$

Theorem 2.2 provides the existence of at least three solutions for λ small enough. Specifically, if $\lambda < -d_2$, (Q_λ) has at least three solutions (the constant functions corresponding to the critical points of $H + \lambda G$). If $\lambda \geq -d_2$, (Q_λ) has only the trivial solution $u = 0$, since no non-constant function u can be a solution. In fact, if there exists $t_1 \in]0, T[$ such that $u(t_1) > 0$, then we get

$$\max_{[0, T]} u = u(t^*) > 0,$$

hence

$$0 \geq u''(t^*) = u(t^*)[d_1u(t^*)^2 + (d_2 + \lambda)] > 0,$$

a contradiction (analogously, a contradiction is reached if $u(t_1) < 0$).

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