

THE RELATIVE PICARD GROUP OF A COMODULE ALGEBRA AND HARRISON COHOMOLOGY

S. CAENEPEEL AND T. GUÉDÉNON

Faculty of Engineering, Vrije Universiteit Brussel, B-1050 Brussels, Belgium
 (scaenepe@vub.ac.be; tguedeno@vub.ac.be)

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Abstract Let A be a commutative comodule algebra over a commutative bialgebra H . The group of invertible relative Hopf modules maps to the Picard group of A , and the kernel is described as a quotient group of the group of invertible group-like elements of the coring $A \otimes H$, or as a Harrison cohomology group. Our methods are based on elementary K -theory. The Hilbert 90 theorem follows as a corollary. The part of the Picard group of the coinvariants that becomes trivial after base extension embeds in the Harrison cohomology group, and the image is contained in a well-defined subgroup E . It equals E if H is a cosemisimple Hopf algebra over a field.

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1. Introduction

Let l be a cyclic Galois field extension of k . The Hilbert 90 theorem tells us that every cocycle in $Z^1(C_p, l^*)$ is a coboundary. There exist various generalizations of this result. For example, if we have a Galois extension $B \rightarrow A$ of commutative rings, with Galois group G , then the cohomology group $H^1(G, \mathbb{G}_m(A))$ is isomorphic to $\text{Pic}(A/B)$, the kernel of the natural map from the Picard group of B to the Picard group of A (see, for example, [9]). Now we can ask the following question. Suppose that G acts on A as a group of isomorphisms. Can we still give an algebraic interpretation of $H^1(G, \mathbb{G}_m(A))$? A second problem is whether there is any relation between $H^1(G, \mathbb{G}_m(A))$ and the Picard group of the ring of invariants $B = A^G$.

In this paper we will discuss these two problems in a more general situation: we will assume that A is a commutative H -comodule algebra, with H an arbitrary commutative bialgebra over a commutative ring k . We then ask for an algebraic interpretation of the first Harrison cohomology group $H_{\text{Harr}}^1(H, A, \mathbb{G}_m)$ (with notation as in [6]). If H is finitely generated and projective, then this Harrison cohomology group is isomorphic to a Sweedler cohomology group $Z_{\text{Harr}}^1(H, A, \mathbb{G}_m)$, and if $H = \mathbb{Z}G$ with G a finite group, then it reduces to the cohomology group $H^1(G, \mathbb{G}_m(A))$.

We proceed as follows: we introduce the relative Picard group $\text{Pic}^H(A)$ as the Grothendieck group of the category of invertible relative Hopf modules. The forgetful

functor to the category of invertible A -modules induces a K -theoretic exact sequence, linking the Picard group of A , the relative Picard group, and the groups of unit elements of A and the coinvariants $B = A^{\text{co } H}$; the middle term in the sequence can be computed, and it is the group of invertible group-like elements of the coring $A \otimes H$. We show also that these group-like elements are precisely the Harrison cocycles, and it follows from the exactness of the sequence that the first Harrison cohomology group is the kernel of the map $\text{Pic}^H(A) \rightarrow \text{Pic}(A)$, answering our first question.

Then we observe that there is a similar exact sequence associated with the induction functor $\underline{\text{Pic}}(B) \rightarrow \underline{\text{Pic}}(A)$, and that the two exact sequences fit into a commutative diagram. If A is a faithfully flat Hopf Galois extension of B , then the categories of B -modules and relative Hopf modules are equivalent, hence $\text{Pic}(B) \cong \text{Pic}^H(A)$, and we recover Hilbert 90. In general, we have an injection $\text{Pic}(A/B) \rightarrow H_{\text{Harr}}^1(H, A, \mathbb{G}_m)$, and we can describe a subgroup of $H_{\text{Harr}}^1(H, A, \mathbb{G}_m)$ that contains the image of $\text{Pic}(A/B)$. The image is precisely this subgroup if H is a cosemisimple Hopf algebra over a field k .

A special situation is the following: let k be an algebraically closed field, A a finitely generated commutative normal k -algebra, and G a connected algebraic group acting rationally on A . Then A is an H -comodule algebra, with H the affine coordinate ring of G . In this case, our exact sequence was given by Magid in [14], but apparently Magid was not aware of the connection to Harrison cohomology, group-like elements of corings or the generalized Hilbert 90 theorem.

In §5, we study the Harrison cocycles (or the group-like elements in $A \otimes H$) in some particular cases. First we look at the situation considered by Magid in [14], and then it turns out that the group-like elements of $G(A \otimes H)$ are induced by the group-like elements of H . In the situation where A is a \mathbb{Z} -graded commutative k -algebra, the relative Picard group turns out to be the graded Picard group $\text{Pic}_g(A)$ studied by the first author in [5]. If A is reduced, then the group-like elements of $A \otimes H$ can also be described using the group-like elements of H , according to a result in [5].

2. Preliminary results

2.1. The language of corings

Relative Hopf modules can be viewed as comodules over a coring. This will be used below, and this is why we briefly recall some properties of corings. Recall that an A -coring is a comonoid in the monoidal category ${}_A\mathcal{M}_A$ of A -bimodules. Thus an A -coring \mathfrak{C} is an A -bimodule together with two A -bimodule maps,

$$\Delta_{\mathfrak{C}} : \mathfrak{C} \rightarrow \mathfrak{C} \otimes_A \mathfrak{C} \quad \text{and} \quad \varepsilon_{\mathfrak{C}} : \mathfrak{C} \rightarrow A,$$

satisfying the usual coassociativity and counit properties. We refer to [2–4, 11, 18] for a detailed discussion of corings. The set of group-like elements of \mathfrak{C} is given by

$$G(\mathfrak{C}) = \{X \in \mathfrak{C} \mid \Delta_{\mathfrak{C}}(X) = X \otimes_A X \text{ and } \varepsilon_{\mathfrak{C}}(X) = 1\}.$$

A right \mathfrak{C} -comodule M is a right A -module together with a right A -linear map $\rho_M : M \rightarrow M \otimes_A \mathfrak{C}$ satisfying

$$(M \otimes_A \varepsilon_{\mathfrak{C}}) \circ \rho_M = M \quad \text{and} \quad (M \otimes_A \Delta_{\mathfrak{C}}) \circ \rho_M = (\rho_M \otimes_A \mathfrak{C}) \circ \rho_M.$$

A morphism of right \mathfrak{C} -comodules $f : M \rightarrow N$ is an A -linear map f such that

$$\rho_N \circ f = (f \otimes_A \mathfrak{C}) \circ \rho_M.$$

$\mathcal{M}^{\mathfrak{C}}$ will be the category of right \mathfrak{C} -comodules and comodule morphisms. We have the following interpretation of the group-like elements of \mathfrak{C} .

Lemma 2.1. *Let \mathfrak{C} be an A -coring. Then there is a bijective correspondence between $G(\mathfrak{C})$ and the set of maps $\rho : A \rightarrow A \otimes_A \mathfrak{C} = \mathfrak{C}$, making A into a right \mathfrak{C} -comodule. The coaction ρ_X corresponding to $X \in G(\mathfrak{C})$ is given by*

$$\rho_X(a) = Xa.$$

With this notation, $A^X = (A, \rho_X)$ is isomorphic to $A^Y = (A, \rho_Y)$ as a right \mathfrak{C} -comodule if and only if there exists an invertible $b \in A$ such that $\rho_Y(b) = Yb = bX$.

Proof. The first part is well known (and straightforward) (see, for example, [3]). Let $f : A^X \rightarrow A^Y$ be a right \mathfrak{C} -colinear isomorphism. Then $f(a) = ba$ for some $b \in A$, which is invertible since f is an isomorphism. The fact that f is \mathfrak{C} -colinear tells us that

$$Yb = \rho_Y(f(1)) = (f \otimes_A \mathfrak{C})(\rho_X(1)) = bX.$$

The converse property is obvious. □

2.2. Relative Hopf modules

Let H be a bialgebra over a commutative ring k , and A a right H -comodule algebra. Throughout this paper we will assume that H and A are commutative. Then A is a commutative algebra and we have a right H -coaction ρ on A such that

$$\rho(ab) = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]},$$

for all $a, b \in A$. Here we use the Sweedler–Heyneman notation for the coaction ρ : $\rho(a) = a_{[0]} \otimes a_{[1]}$, with summation implicitly understood. For the comultiplication on H , we use the notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$.

A relative Hopf module M is a k -module, together with a right A -action and a right C -coaction ρ_M such that

$$\rho_M(ma) = m_{[0]}a_{[0]} \otimes m_{[1]}a_{[1]},$$

for all $a \in A$ and $m \in M$. The category of relative Hopf modules and A -linear H -colinear maps will be denoted by \mathcal{M}_A^H . The coinvariant submodule $M^{\text{co } H}$ of $M \in \mathcal{M}_A^H$ is defined by

$$M^{\text{co } H} = \{m \in M \mid \rho_M(m) = m \otimes 1\}.$$

$A^{\text{co } H} = B$ is a k -subalgebra of A , and $M^{\text{co } H}$ is a B -module. We obtain a functor $(\cdot)^{\text{co } H} : \mathcal{M}_A^H \rightarrow \mathcal{M}_B$, which has a left adjoint $T = - \otimes_B A : \mathcal{M}_B \rightarrow \mathcal{M}_A^H$. The right H -coaction on $N \otimes_B A$ is $N \otimes_B \rho$. The unit u and counit c of the adjunction are given by the following formulae, for $N \in \mathcal{M}_B$ and $M \in \mathcal{M}_A^H$:

$$\begin{aligned} u_N : N &\rightarrow (N \otimes_B A)^{\text{co } H}, & u_N(n) &= n \otimes 1, \\ c_M : M^{\text{co } H} \otimes_B A &\rightarrow M, & c_M(m \otimes a) &= ma. \end{aligned}$$

A is called a Hopf algebra extension of $B = A^{\text{co } H}$ if the canonical map

$$\text{can} : A \otimes_B A \rightarrow A \otimes H, \quad \text{can}(a \otimes_B b) = ab_{[0]} \otimes b_{[1]}$$

is an isomorphism. If A is a faithfully flat Hopf Galois extension, then the adjunction $(- \otimes_B A, (\cdot)^{\text{co } H})$ is a pair of inverse equivalences. We refer to [10, 15, 17] for a detailed discussion of Hopf algebras and relative Hopf modules.

$\mathfrak{C} = A \otimes H$ is a coring, with structure maps

$$\begin{aligned} a'(b \otimes h)a &= a'ba_{[0]} \otimes ha_{[1]}, \\ \Delta_{\mathfrak{C}}(a \otimes h) &= (a \otimes h_{(1)}) \otimes_A (1 \otimes h_{(2)}), \\ \varepsilon_{\mathfrak{C}}(a \otimes h) &= a\varepsilon(h). \end{aligned}$$

The category $\mathcal{M}^{A \otimes H}$ is isomorphic to the category \mathcal{M}_A^H of relative Hopf modules; we refer to [2, 4] for full details. Note that $X = \sum_i a_i \otimes h_i \in G(A \otimes H)$ if and only if

$$\sum_i (a_i \otimes h_{i(1)} \otimes h_{i(2)}) = \sum_{i,j} (a_i a_{j[0]} \otimes h_i a_{j[1]} \otimes h_j) \quad \text{and} \quad \sum_i a_i \varepsilon(h_i) = 1. \quad (2.1)$$

$A \otimes H$ is also a commutative algebra, with multiplication

$$(a \otimes h)(b \otimes k) = ab \otimes hk.$$

The product of two group-like elements is a group-like element, and $1_A \otimes 1_H$ is group-like. Hence $G^i(A \otimes H)$, the set of invertible group-like elements, is an abelian group. Also observe that an invertible group-like element is precisely a normalized Harrison 1-cocycle (see, for example, [6, § 9.2] for the definition of the Harrison complex).

Let H be a finitely generated projective cocommutative Hopf algebra, and let A be a commutative left H -module algebra. Then H^* is a commutative Hopf algebra and A is a right H^* -comodule algebra. If $\sum_i a_i \otimes f_i \in A \otimes H^*$ is an invertible group-like element (or a normalized Harrison cocycle), then

$$\phi : H \rightarrow A, \quad \phi(h) = \sum_i a_i f_i(h), \quad (2.2)$$

is a normalized Sweedler 1-cocycle. This means that $\phi(1_H) = 1_A$, and the cocycle condition

$$\phi(hh') = \sum_i (h_{(1)} \cdot (\phi(h')))\phi(h_{(2)}) \quad (2.3)$$

is satisfied. This gives a bijective correspondence between Harrison and Sweedler cocycles, see [6, Proposition 9.2.3]. For the definition of the Sweedler complex, see [16] or [6, § 9.1]. In the case where $H = kG$, with G a finite group, Sweedler cohomology reduces to group cohomology.

2.3. Elementary algebraic K -theory

Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \otimes, J)$ be skeletally small symmetric monoidal categories, and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a cofinal, strong monoidal functor. Then we can consider the Grothendieck and Whitehead groups of \mathcal{C} and \mathcal{D} , and we have an exact sequence connecting them (see, for example, [1, Chapter VII]):

$$K_1\mathcal{C} \xrightarrow{K_1F} K_1\mathcal{D} \xrightarrow{d} K_1\underline{\phi F} \xrightarrow{g} K_0\mathcal{C} \xrightarrow{K_0F} K_0\mathcal{D}. \quad (2.4)$$

$C \in \mathcal{C}$ is called invertible if there exists $C' \in \mathcal{C}$ such that $C \otimes C' \cong I$. If all elements of \mathcal{C} and \mathcal{D} are invertible, then the description of the five groups in (2.4) and the connecting maps simplifies (see [6, Appendix C]). $K_0\mathcal{C}$ is the group of isomorphism classes of objects in \mathcal{C} and $K_1\mathcal{C} \cong \text{Aut}_{\mathcal{C}}(I)$ (which is then an abelian group). Let $\underline{\Psi F}$ be the following category: objects are couples (C, α) , with $C \in \mathcal{C}$ and $\alpha : F(C) \rightarrow J$ an isomorphism in \mathcal{D} . A morphism between (C, α) and (C', α') is an isomorphism $f : C \rightarrow C'$ in \mathcal{C} such that $\alpha' = F(f) \circ \alpha$. $\underline{\Psi F}$ is monoidal, every object is invertible and

$$K_1\underline{\phi F} \xrightarrow{g} \cong K_0\underline{\Psi F}.$$

The maps d and g are given as follows: $d(\alpha) = [(I, \alpha)]$ and $g[(C, \alpha)] = [C]$.

A typical example is the following: for a commutative ring A , let $\underline{\text{Pic}}(A)$ be the category of invertible A -modules. If $i : B \rightarrow A$ is a morphism of commutative rings, then we have the cofinal strongly monoidal functor

$$G = - \otimes_B A : \underline{\text{Pic}}(B) \rightarrow \underline{\text{Pic}}(A),$$

and (2.4) takes the form

$$1 \rightarrow \mathbb{G}_m(B) \rightarrow \mathbb{G}_m(A) \xrightarrow{d'} K_1\underline{\phi G} \xrightarrow{g'} \text{Pic}(B) \rightarrow \text{Pic}(A). \quad (2.5)$$

3. The relative Picard group

If $M, N \in \mathcal{M}_A^H$, then $M \otimes_A N \in \mathcal{M}_A^H$, with right H -coaction

$$\rho_{M \otimes_A N}(m \otimes_A n) = m_{[0]} \otimes_A n_{[0]} \otimes m_{[1]} n_{[1]}.$$

So we have a symmetric monoidal category $(\mathcal{M}_A^H, \otimes_A, A)$. Let $\underline{\text{Pic}}^H(A)$ be the full subcategory consisting of invertible objects. $\text{Pic}^H(A) = K_0\underline{\text{Pic}}^H(A)$, the group of isomorphism classes of relative Hopf modules, will be called the relative Picard group of A and H . The isomorphism class in $\text{Pic}^H(A)$ represented by an invertible relative Hopf module M will be denoted by $\{M\}$. This new invariant fits into an exact sequence.

Proposition 3.1. *We have an exact sequence*

$$1 \rightarrow \mathbb{G}_m(B) \rightarrow \mathbb{G}_m(A) \xrightarrow{d} G^i(A \otimes H) \xrightarrow{g} \text{Pic}^H(A) \rightarrow \text{Pic}(A). \quad (3.1)$$

Proof. This result can be proved in two ways: a first possibility is to show that (3.1) is precisely the exact sequence (2.4), associated with the functor $\underline{\text{Pic}}^H(A) \rightarrow \underline{\text{Pic}}(A)$ forgetting the H -coaction. Let us present an easy direct proof.

The map $\mathbb{G}_m(B) \rightarrow \mathbb{G}_m(A)$ is the natural inclusion. Take $a \in A$ invertible, and let $d(a) = X = a^{-1}a_{[0]} \otimes a_{[1]}$. X is group-like, since $a^{-1}a_{[0]}\varepsilon(a_{[1]}) = 1$, and

$$\begin{aligned} X \otimes_A X &= (a^{-1}a_{[0]} \otimes a_{[1]}) \otimes_A (b^{-1}b_{[0]} \otimes b_{[1]}) \\ &= a^{-1}a_{[0]}(b^{-1})_{[0]}b_{[0]} \otimes a_{[1]}(b^{-1})_{[1]}b_{[1]} \otimes b_{[2]} \\ &= a^{-1}b_{[0]} \otimes b_{[1]} \otimes b_{[2]} \\ &= (a^{-1}b_{[0]} \otimes b_{[1]}) \otimes_A (1 \otimes b_{[2]}) = \Delta(X), \end{aligned}$$

where we identified $(A \otimes H) \otimes_A (A \otimes H) = A \otimes H \otimes H$ and we wrote $a = b$. The inverse of X is $X^{-1} = a(a^{-1})_{[0]} \otimes (a^{-1})_{[1]}$, so $X \in G^i(A \otimes H)$.

If $d(a) = a^{-1}a_{[0]} \otimes a_{[1]} = 1_A \otimes 1_H$, then $a_{[0]} \otimes a_{[1]} = a \otimes 1_H$, so $a \in B$, and the sequence is exact at $\mathbb{G}_m(A)$.

For $X \in G^i(A \otimes H)$, let $g(X) = A^X$, with notation as in Lemma 2.1. g is multiplicative: take $X = \sum_i a_i \otimes h_i$ and $Y = \sum_j b_j \otimes k_j$ in $G^i(A \otimes H)$, then $A^X \otimes_A A^Y = A$ as an A -bimodule, with comultiplication given by

$$\rho_{A^X \otimes_A A^Y}(1) = \sum_{i,j} a_i \otimes_A b_j \otimes h_i k_j = XY,$$

as needed.

If $g(X) = \{A\}$ in $\text{Pic}^H(A)$, then there exists an H -colinear A -linear isomorphism $f : A^X \rightarrow A$. Then $f(1) = a$ is invertible in A , and, since f is H -colinear, $a_{[0]} \otimes a_{[1]} = \rho(a) = (f \otimes H)(X) = aX$, so $X = a^{-1}a_{[0]} \otimes a_{[1]} = d(a)$, and the sequence is also exact at $G^i(A \otimes H)$.

The exactness of the sequence at $\text{Pic}^H(A)$ follows from Lemma 2.1. \square

Remark 3.2. Let $H = k\mathbb{Z}$, and let A be a commutative \mathbb{Z} -graded k -algebra. Then $\text{Pic}^H(A) = \text{Pic}_g(A)$, the graded Picard group of A , as introduced in [5] (see also [8]). The exact sequence (3.1) reduces to the exact sequence in [5, Proposition 2.1].

The map $d : \mathbb{G}_m(A) \xrightarrow{d} G^i(A \otimes H)$ is precisely the map $\mathbb{G}_m(A) \rightarrow \mathbb{G}_m(A \otimes H)$ in the Harrison complex. From Proposition 3.1, we therefore immediately obtain the following corollary.

Corollary 3.3. *With H and A as in Proposition 3.1, we have an isomorphism of abelian groups*

$$\text{Pic}^H(A) \cong H_{\text{Harr}}^1(H, A, \mathbb{G}_m).$$

This is the promised algebraic interpretation of the first Harrison cohomology group. Note that there are no flatness or projectivity assumptions on H or A . We have Hilbert 90 as an easy consequence.

Corollary 3.4 (Hilbert 90). *Let H, A, B be as in Proposition 3.1. If A is a faithfully flat H -Galois extension of B , then we have an isomorphism of abelian groups:*

$$\mathrm{Pic}(A/B) \cong H_{\mathrm{Harr}}^1(H, A, \mathbb{G}_m).$$

Proof. From the fact that the monoidal categories \mathcal{M}_B and \mathcal{M}_A^H are equivalent, it follows that $\mathrm{Pic}(B) \cong \mathrm{Pic}^H(A)$. \square

Take the exact sequences (2.5) and (3.1), and observe that they fit into a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m(B) & \longrightarrow & \mathbb{G}_m(A) & \xrightarrow{d'} & K_1 \phi G & \xrightarrow{g'} & \mathrm{Pic}(B) & \longrightarrow & \mathrm{Pic}(A) \\ & & \downarrow = & & \downarrow = & & & & \downarrow j & & \downarrow = \\ 1 & \longrightarrow & \mathbb{G}_m(B) & \longrightarrow & \mathbb{G}_m(A) & \xrightarrow{d} & G^i(A \otimes H) & \xrightarrow{g} & \mathrm{Pic}^H(A) & \longrightarrow & \mathrm{Pic}(A) \end{array}$$

The map j maps $[N] \in \mathrm{Pic}(B)$ to $\{N \otimes_B A\} \in \mathrm{Pic}^H(A)$. Using the ‘five lemma’, we find a map $i : K_1 \phi G \rightarrow G^i(A \otimes H)$.

Lemma 3.5. *With notation as above, the maps i and j are injective.*

Proof. From the fact that u is a natural transformation between additive endofunctors of the category of B -modules, and since u_B is an isomorphism, it follows that $u_N : N \rightarrow (N \otimes_B A)^{\mathrm{co} H}$ is an isomorphism if N is finitely generated and projective as a B -module. So if $N \otimes_B A \cong A$, then $N \cong (N \otimes_B A)^{\mathrm{co} H} \cong A^{\mathrm{co} H} = B$, and j is injective. The injectivity of i then follows from an easy diagram-chasing argument. \square

Our next aim is to characterize the image of i . This will be the topic of § 4; it will turn out that we obtain nice results in the case where H is cosemisimple.

4. Coinvariantly generated relative Hopf modules

Some of our results will be more specific if we assume that H is a cosemisimple Hopf algebra over a field k . Recall that H is cosemisimple if there exists a left integral ϕ on H^* such that $\phi(1) = 1$ (see, for example, [18]). In this case, the coinvariants functor $(\cdot)^{\mathrm{co} H} : \mathcal{M}_A^H \rightarrow \mathcal{M}_B$ is exact (see [15, Lemma 2.4.3]).

A relative Hopf module M is called *coinvariantly generated* if c_M is surjective, or, equivalently, if $M = M^{\mathrm{co} H} A$. If M is coinvariantly generated, and finitely generated as an A -module, then we can find a finite set $\{m_1, \dots, m_n\} \in M^{\mathrm{co} H}$ that generates M .

It follows immediately from the properties of adjoint functors that $N \otimes_B A$ is coinvariantly generated, for every $N \in \mathcal{M}_B$; in particular, A is coinvariantly generated. We also have the following lemma.

Lemma 4.1. *Let $M \in \mathcal{M}_A^H$ and $N \in \mathcal{M}_B$. If M is an epimorphic image of $N \otimes_B A$ in \mathcal{M}_A^H , then $M^{\mathrm{co} H} = 0$ implies that $M = 0$.*

Proof. If $M^{\text{co } H} = 0$, then

$$\text{Hom}_A^H(N \otimes_B A, M) = \text{Hom}_B(N, M^{\text{co } H}) = 0.$$

But $\text{Hom}_A^H(N \otimes_B A, M)$ contains the epimorphism of relative Hopf modules $N \otimes_B A \rightarrow M$, so $M = 0$. \square

If N is an epimorphic image of M in \mathcal{M}_A^H , and if M is coinvariantly generated, then N is also coinvariantly generated.

Lemma 4.2. Assume that H is a cosemisimple Hopf algebra over a field k . If $N \in \mathcal{M}_B$ is projective, then $N \otimes_B A$ is projective in \mathcal{M}_A^H .

Proof. See [7, Proposition 2.5]. \square

Lemma 4.3. Let k be a field.

- (1) The forgetful functor $\mathcal{M}_A^H \rightarrow \mathcal{M}_A$ preserves projectives.
- (2) If H is cosemisimple, then the forgetful functor also reflects projectivity of finitely generated modules.

Proof. (1) Take $M \in \mathcal{M}_A^H$ projective, and consider the epimorphism $p : M \otimes A \rightarrow M$, $p(m \otimes a) = ma$ in \mathcal{M}_A^H . The exact sequence

$$0 \rightarrow \text{Ker } p \rightarrow M \otimes A \xrightarrow{p} M \rightarrow 0$$

splits in \mathcal{M}_A^H , since M is a projective object, and *a fortiori* in \mathcal{M}_A . Hence M is a direct factor of $M \otimes A$, which is a projective A -module, so M is also a projective A -module.

(2) Let M and N be relative Hopf modules, and assume that M is finitely generated and projective in \mathcal{M}_A . According to [7, Proposition 4.2], $\text{Hom}_A(M, N) \in \mathcal{M}_A^H$, and it is easy to show that $\text{Hom}_A(M, N)^{\text{co } H} = \text{Hom}_A^H(M, N)$. It follows that the functor $\text{Hom}_A^H(M, -) : \mathcal{M}_A^H \rightarrow \mathcal{M}$ is exact, since it is the composition of the exact functors $\text{Hom}_A(M, -) : \mathcal{M}_A^H \rightarrow \mathcal{M}^H$ ($M \in \mathcal{M}_A$ is projective) and $(\cdot)^{\text{co } H} : \mathcal{M}^H \rightarrow \mathcal{M}$ (H is cosemisimple). \square

Lemma 4.4. Let H be a cosemisimple Hopf algebra over a field k , and take $P, Q \in \mathcal{M}_A^H$ finitely generated as A -modules. Assume that Q is a projective object of \mathcal{M}_A^H . Then every epimorphism $f : P \rightarrow Q$ in \mathcal{M}_A^H has a right inverse in \mathcal{M}_A^H .

Proof. It is clear that $\text{Hom}_A(Q, P)$ and $\text{Hom}_A(Q, Q)$ are right H -comodules, and the map

$$f^* = \text{Hom}_A(Q, f) : \text{Hom}_A(Q, P) \rightarrow \text{Hom}_A(Q, Q)$$

is right H -colinear. It follows from Lemma 4.3 that Q is projective as an A -module, so f^* is surjective. Since f^* is H -colinear, f^* restricts to a surjection

$$\text{Hom}_A^H(Q, P) = \text{Hom}_A(Q, P)^{\text{co } H} \rightarrow \text{Hom}_A^H(Q, Q) = \text{Hom}_A(Q, Q)^{\text{co } H}.$$

Take a preimage $g \in \text{Hom}_A^H(Q, P)$ of the identity map id_Q on Q . Then $f \circ g = \text{id}_Q$, and the result follows. \square

For $M \in \mathcal{M}_A$, we will denote the dual module by $M^* = \text{Hom}_A(M, A)$.

Proposition 4.5. *Let H be cosemisimple, and assume that $P \in \mathcal{M}_A^H$ is coinvariantly generated and finitely generated projective as an A -module. Then*

- (1) $P^{\text{co } H}$ is a finitely generated projective B -module;
- (2) P^* is coinvariantly generated;
- (3) the map c_P is an isomorphism in \mathcal{M}_A^H .

Proof. (1) As we have seen, there exist $p_1, p_2, \dots, p_n \in P^{\text{co } H}$ such that $P = \sum_i p_i A$. Set $F = A^n$ and let $f : F \rightarrow P$ be the A -linear map given by $f(a_1, a_2, \dots, a_n) = \sum_i p_i a_i$. Then $F \in \mathcal{M}_A^H$ and f is an epimorphism in \mathcal{M}_A^H . By Lemma 4.4, there exists a monomorphism $g \in \text{Hom}_A(P, F)$ such that $f \circ g = \text{id}_P$. The restriction of g to $P^{\text{co } H}$ is then a B -linear right inverse of the restriction of f to $F^{\text{co } H}$, and $F^{\text{co } H} = B^n$, and we obtain (1).

(2) The map $g^* = \text{Hom}_A(g, A) : F^* \rightarrow P^*$ is surjective and H -colinear. The fact that F^* is coinvariantly generated then implies that P^* is also coinvariantly generated.

- (3) Consider the natural transformation $t : (\cdot)^{\text{co } H} \otimes_B A \rightarrow (\cdot)$ given by

$$t_P : P^{\text{co } H} \otimes_B A \rightarrow P, \quad t_P(p \otimes a) = pa.$$

The map t_A is an isomorphism, so t_F is an isomorphism by additivity. It follows that t_P is an isomorphism, since $F = P \oplus \text{Ker } f$ as H -comodules. \square

Let $X = \sum_i a_i \otimes h_i \in G(A \otimes H)$, and write

$$A_X = \left\{ a \in A \mid \rho(a) = aX = \sum_i aa_i \otimes h_i \right\}$$

and

$$A_X^i = \{a \in A_X \mid a \text{ is invertible}\}.$$

Observe that

$$\text{Im}(d) = \{X \in G^i(A \otimes H) \mid A_X^i \neq \emptyset\}$$

and

$$A_{1 \otimes 1} = A^{\text{co } H}.$$

Furthermore, $A_X A_Y \subset A_{XY}$: take $a \in A_X$ and $b \in A_Y$, then $\rho(a) = aX = \sum_i aa_i \otimes h_i$, $\rho(b) = bY = \sum_j bb_j \otimes k_j$ and

$$\rho(ab) = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]} = \sum_{i,j} aa_i bb_j \otimes h_i k_j = abXY.$$

Also $A_X^i \cap A_Y^i = \emptyset$ if $X \neq Y$.

Lemma 4.6. *The set*

$$E = \{X \in G^i(A \otimes H) \mid AA_X = A \text{ and } AA_{X^{-1}} = A\}$$

is a subgroup of $G^i(A \otimes H)$ containing $\text{Im}(d)$.

Proof. If $X \in \text{Im}(d)$, then there exists an invertible $a \in A_X$, and then $AA_X = A$. Since $X^{-1} \in \text{Im}(d)$, we also have $AA_{X^{-1}} = A$, hence $X \in E$. It is clear that $1 \otimes 1 \in E$. If $X, Y \in E$, then $AA_{XY} \supset AA_X A_Y = AA_Y = A$, and, in a similar way, $AA_{(XY)^{-1}} = A$, hence $XY \in E$. Finally, if $X \in E$, then obviously $X^{-1} \in E$. \square

Proposition 4.7. *Consider the injective map $j : \text{Pic}(B) \rightarrow \text{Pic}^H(A)$. If H is a cosemisimple Hopf algebra over a field k , then*

$$\text{Im}(j) = \{\{M\} \in \text{Pic}^H(A) \mid M \text{ is coinvariantly generated}\}.$$

Proof. $M \otimes_B A$ is coinvariantly generated, so $\text{Im}(j)$ is contained in the desired set. If H is cosemisimple, and $\{N\} \in \text{Pic}^H(A)$, with N coinvariantly generated, then $N = (N^{\text{co}H}) \otimes_B A \in \text{Im}(j)$, by Proposition 4.5 (3). \square

Lemma 4.8. *Take $X \in G^i(A \otimes H)$. Then A^X is coinvariantly generated if and only if $AA_{X^{-1}} = A$. If H is cosemisimple, then this is also equivalent to $X \in E$.*

Proof. The first statement follows from the fact that $(A^X)^{\text{co}H} = A_{X^{-1}}$. Indeed, $a \in (A^X)^{\text{co}H}$ if and only if $\rho_X(a) = Xa = a \otimes 1$, if and only if $\rho(a) = (1 \otimes 1)a = X^{-1}(a \otimes 1) = aX^{-1}$, which means that $a \in A_{X^{-1}}$.

Let H be cosemisimple. Note that $(A^X)^* \cong A^{X^{-1}}$ as relative Hopf modules. If A^X is coinvariantly generated, then so is $A^{X^{-1}}$, by Proposition 4.5, and then $X \in E$. \square

Now we are able to prove the main result of this section.

Theorem 4.9. *Consider the monomorphism $i : K_1 \underline{\phi} G \rightarrow G^i(A \otimes H)$ introduced in Lemma 3.5.*

Then $\text{Im}(i) \subset E$ and $\text{Im}(i) = E$ if H is a cosemisimple Hopf algebra over a field k . In this situation, $\text{Pic}(A/B) \cong E$.

Proof. Take $[(M, \alpha)] \in K_0 \underline{\psi} G$, and let $i[(M, \alpha)] = X \in G^i(A \otimes H)$. Then

$$\{A^X\} = j(g'[(M, \alpha)]) = j([M]) = \{M \otimes_B A\},$$

hence A^X is coinvariantly generated and $AA_{X^{-1}} = A$, by Lemma 4.8. In a similar way, $i([(M, \alpha)]^{-1}) = X^{-1}$, and $A^{X^{-1}} \cong M^* \otimes_B A$ is coinvariantly generated, so $AA_X = A$, again by Lemma 4.8. This proves that $X \in E$.

Assume now that H is cosemisimple, and take $X \in E$. It follows from Lemma 4.8 that A^X is coinvariantly generated, and from Proposition 4.7 that $A^X = M \otimes_B A$ for some $M \in \underline{\text{Pic}}(B)$. Since the image of M in $\text{Pic}(A)$ is trivial, $[M] = g'[(M, \alpha)]$ for some $(M, \alpha) \in \mathcal{C}$. Write $i[(M, \alpha)] = Y$. Then $X = Yd(a)$, for some $a \in \mathbb{G}_m(A)$. Consider the map $\alpha' : M \otimes_B A \rightarrow A$, $\alpha'(m \otimes b) = a^{-1}\alpha(m \otimes b)$. Then $i[(M, \alpha')] = X$. \square

5. On the group-like elements

We have an injective map $i : G(H) \rightarrow G(A \otimes H)$, $i(g) = 1_A \otimes g$. Everything simplifies if i is an isomorphism. We discuss two situations in which this is (almost) the case.

Recall that a commutative algebra which is an integral domain is called normal if it is integrally closed in its field of fractions.

Proposition 5.1. *Let k be an algebraically closed field, let A be a finitely generated commutative normal k -algebra and let G be a connected algebraic group acting rationally on A . Let H be the affine coordinate ring of G , and $\chi(G)$ be the group of characters of G . Then*

$$G(A \otimes H) = \{1 \otimes \phi \mid \phi \in G(H) = \chi(G)\}.$$

Proof. Let $x = \sum_i a_i \otimes f_i \in G(A \otimes H)$. Then we have

$$\sum_i (a_i \otimes f_{i(1)} \otimes f_{i(2)}) = \sum_{i,j} (a_i a_{j[0]} \otimes (f_i * a_{j[1]}) \otimes f_j) \quad (5.1)$$

and $\sum a_i \varepsilon(f_i) = 1$. The map

$$\alpha : A \otimes H \rightarrow \text{Hom}(kG, A), \quad \alpha(a \otimes f)(g) = af(g)$$

is injective. Let $\phi = \alpha(x)$. Using (5.1), we compute for all $g, g' \in G$ that

$$\begin{aligned} \phi(gg') &= \sum_i a_i f_i(gg') = \sum_i a_i f_{i(1)}(g) f_{i(2)}(g') \\ &= \sum_{i,j} a_i a_{j[0]} ((f_i * a_{j[1]})(g)) f_j(g') \\ &= \sum_{i,j} a_i a_{j[0]} f_i(g) a_{j[1]}(g) f_j(g') \\ &= \sum_{i,j} (g \cdot a_j) f_j(g') a_i f_i(g) \\ &= \sum_{i,j} g \cdot (a_j f_j(g')) a_i f_i(g) \\ &= (g \cdot (\phi(g'))) \phi(g). \end{aligned}$$

From the second equality, we have $1 = \sum_i a_i f_i(1_G) = \phi(1_G)$. For every $g \in G$, $\phi(g)$ is invertible in A , with inverse $g \cdot (\phi(g^{-1}))$. By the proof of [13, Proposition 1b, p. 46], $\phi(g) \in k$ for every $g \in G$, so $\phi \in \chi(G)$. Now $\chi(G) = G(H) \subset H$ (see [12, p. 25]), so it follows in particular that $\phi \in H$. For all $g \in G$ we now have that

$$\alpha(1 \otimes \phi)(g) = \phi(g) = \sum_i a_i f_i(g) = \alpha(x)(g),$$

hence $x = 1 \otimes \phi$, by the injectivity of α . □

Now consider the situation from Remark 3.2: $H = k\mathbb{Z} \cong k[X, X^{-1}]$, and A is a commutative \mathbb{Z} -graded algebra. In this situation $A \otimes H = A \otimes k[X, X^{-1}]$. Group-like elements in $A \otimes H$ can be constructed as follows. Let $1 = e_1 + \cdots + e_n$ with the e_i orthogonal idempotents, and take $d_1, \dots, d_n \in \mathbb{Z}$. Then $\sum_{i=1}^n e_i \otimes X^{d_i}$ is a group-like element in $A \otimes k[X, X^{-1}]$. In this way, we have an embedding of $\mathcal{C}(\text{Spec}(A), \mathbb{Z})$, the continuous functions from $\text{Spec}(A)$ (with the Zariski topology) to \mathbb{Z} (with the discrete topology), into $G(A \otimes k[X, X^{-1}])$. The first author was amazed to see that one of his first results, [5, Theorem 2.3], can be restated in such a way that it becomes a result about corings. Recall that a commutative ring is called reduced if it has no non-trivial nilpotents.

Proposition 5.2. *Let A be a reduced \mathbb{Z} -graded commutative k -algebra. Then the map $\mathcal{C}(\text{Spec}(A), \mathbb{Z}) \rightarrow G(A \otimes k[X, X^{-1}])$ is a bijection.*

Example 5.3 (cf. Example 2.6 in [5]). Proposition 5.2 does not hold if A contains nilpotent elements; this is related to the fact that there exist non-homogeneous units in this situation. Let $A = k[x]$, with $x^2 = 0$, and put a \mathbb{Z} -grading on A by taking $\deg(x) = 1$. Then $1 + ax \in \mathbb{G}_m(A)$, and $d(1 + ax) = (1 - ax) \otimes 1 + ax \otimes X$ is a group-like element in $G(A \otimes k[X, X^{-1}])$ which is not in the image of $\mathcal{C}(\text{Spec}(A), \mathbb{Z})$.

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